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ESTIMATING SMALL EIGENVALUES OF RIEMANN SURFACES

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0. Introduction. In this paper we will describe a unified approach to the problem of deriving geometric bounds for the small eigenvalues $\lambda_j(M)$ of the Laplace operator on a Riemann surface M. The technique combines features of [PS] and [DR], and applies to any Riemann surface having a finite-sided fundamental polygon, whether of finite or infinite area. For compact Riemann surfaces the results of this paper were obtained by a different method by Shoen, Wolpert and Yau [SWY]. The case of λ_1 for infinite-area geometrically finite surfaces is treated in [PS], and the lower bound for $\lambda_1(M)$ on a finite-volume hyperbolic manifold of any dimension is treated in [DR]. In order to bring out the essentially simple ideas before entering into their technical description, we will briefly outline the basic approach for obtaining the lower bounds, and as our model we will discuss $\lambda_1(M)$ in the compact case, since this serves as a paradigm of the general method. Since the method applies to all dimensions, and is in the compact case marginally simpler for dimension $n \geq 3$, we will, for illustrative purposes, make no initial assumption about the dimension of M.

Accordingly, assume for the moment that M is a compact n-dimensional hyperbolic manifold. The Margulis lemma (cf. [Bu], [Th], [Ra], [Be]) then implies that there exists an $\epsilon = \epsilon(n)$ such that

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M can be decomposed into a union of a part M_{thick}, at every point of which the injectivity radius is $> \epsilon$, and a possibly empty part M_{thin} which, if not empty, consists of a disjoint union of tubular neighborhoods of short closed simple geodesics. For our purposes, it is important that a neighborhood of the boundary of each tube (also called a cylinder if dim $M = 2$) in M_{thin} lies in the set M_{thick}, and that this can always be arranged. Note that this is slightly different than the usual description of M_{thick} and M_{thin} and that, if $n \geq 3$, M_{thick} is connected.

It is very easy to prove that the first Dirichlet eigenvalue of a tube is greater than $((n-1)/2)^2$ (cf. Lemma 3.2). Assume now that $\lambda_1(M)$ is very small, so that if φ is a normalized eigenfunction corresponding to $\lambda_1(M)$, its energy is by definition very small. By a simple Sobolev estimate, this means that $|\nabla\varphi|$ is small on M_{thick}. Since the injectivity radius is bounded below on M_{thick}, it follows easily that if M_{thick} is connected, which we will assume for the moment to be the case, then the oscillation of φ on M_{thick} must be small. In other words, if $\lambda_1(M)$ is very small, then φ is almost constant on M_{thick}. If we assume that φ is not uniformly small on M_{thick}, it then follows that φ must be of one sign on M_{thick}. Since $\int_M \varphi = 0$, this implies that φ must be a Dirichlet eigenfunction for a subdomain of some tube, and by the domain monotonicity of Dirichlet eigenvalues, this shows that $\lambda_1(M) > ((n-1)/2)^2$, which contradicts the presumed smallness of $\lambda_1(M)$. On the other hand, if φ is uniformly small on M_{thick}, then φ has nearly zero Dirichlet data on the boundary of each tube, and an approximate version of Lemma 3.2 (Lemma 3.3), coupled with the fact that φ must have some L^2 norm on at least one tube, shows that φ must have some energy on one of the tubes, and hence on M itself, which again contradicts the presumed smallness of $\lambda_1(M)$. In other words, it is not possible for $\lambda_1(M)$ to be too small. When properly quantified, the above argument yields a good lower bound for $\lambda_1(M)$. If it should happen to be the case that the set M_{thick} is disconnected, which can only occur if $n = 2$, the argument shows that if λ_1 is small, then φ is nearly constant on each component of M_{thick}. It then easily follows, by studying the oscillation of φ across each tube, that one recovers the results in [SWY], including those for higher eigenvalues.

This strategy carries over uneventually to the finite-volume case in all dimensions, the only new feature being that cuspidal components, as well as tubes, will be present if M is not compact. Since a version of Lemma 3.2 applies to such components, the argument is essentially unchanged. Similarly, in the 2-dimensional infinite-area case, the structure of the infinite-area ends is very simple, and a version of Lemma 3.2 is again valid for such components, which allows us to apply the argument in that case as well.

In the 2-dimensional case one can easily obtain upper bounds of eigenvalues by constructing appropriate test functions: constant on components of M_{thick} and suitably interpolated across tubes. These upper bounds turn out to be of the same order as the lower bounds obtained by the procedure outlined above. In particular, $\lambda_1(M)/L_1(M)$ is bounded above and below by constants depending only on the topology of M, where $L_1(M)$ denotes the minimum of total lengths of chains of disjoint simple closed geodesics which separate M and have lengths $< 2\epsilon$. The results concerning higher eigenvalues can be stated broadly as follows. To the surface M we associate a graph K whose edges are tubes surrounding short simple closed geodesics. The vertices of K are bounded components of M_{thick}, together with one vertex called the ground, corresponding to all expanding ends of M, if M has infinite area. Every edge of K carries a label, the length of the corresponding geodesic. In terms of these data, we define a discrete eigenvalue problem, whose eigenvalues are of the same size as the small eigenvalues of M. The number of eigenvalues which can be estimated in this way is equal to the number of vertices in the graph, i.e. depends only on the topology of M.

The paper is organized as follows. In Section 1, we describe the discretization of the eigenvalue problem and give precise statements of the estimates. Section 2 is devoted to the discrete problem. Estimates of the Dirichlet integral on doubly connected Riemann surfaces are proved in Section 3, and a detailed description of the geometry of tubes, funnels (i.e. expanding ends) and cusps is provided in Section 4. Finally, in Section 5, we prove the eigenvalue estimates.

1. Discretization and statement of the result. Let M be a Riemann surface with metric of constant curvature -1 and having finitely generated fundamental group. In this section we describe how to associate a finite graph to M and state our eigenvalue estimates in terms of this graph. The fundamental result describing the geometry of such surfaces is the "thick and thin" decomposition $M = M_{thick} \cup M_{thin}$ (cf. [Be, Theorem 11.7.1] or [Th]). We describe this decomposition in terms convenient in our context. There exists a positive number $\epsilon > 0$ independent of M (chosen once and for all to be sufficiently small) such that the set of points $M_{thick} = \{x \in M \mid inj(x) > \epsilon\}$ is non-empty, where inj(x) denotes the injectivity radius at x. The complement of M_{thick} is contained in the set M_{thin} (possibly empty) which is the union of finitely many pieces, of the following types.

a) <u>Cylinders</u>. A cylinder is a tubular neighborhood of a simple closed geodesic γ of length $\ell(\gamma) < 2\epsilon$ connecting two components of M_{thick}.
b) <u>Cusps</u>. A cusp is isometric to $[a,\infty) \times S^1$ with the metric $ds^2 = (dx^2 + dy^2)/y^2$, where $x \in S^1 = R/Z$, and $y \in [a,\infty)$. Every cusp is attached to one of the components of M_{thick}.

A cylinder might be thought of as approximately isometric to a doubled truncated cusp. A cusp is conformally equivalent to a punctured disk and a cylinder is equivalent to an annulus. Unbounded components of M_{thick} contain expanding ends, which shall be called *funnels*. The number of cusps will be denoted by p, f will stand for the number of funnels, g will be the genus of M, and the number of compact components of M_{thick} will be denoted by N. Finer aspects of this decomposition together with the choice of ϵ will be discussed in Section 4.

A graph K is associated to M as follows. The vertices of K correspond to the bounded components of M_{thick} and there is an additional vertex, the *ground*, if M has funnels. Thus all funnels are regarded as being connected to each other at infinity. Each cylinder of M is represented by an edge joining the vertices corresponding to the components of M_{thick} connected by the cylinder. If a cylinder surrounds a short geodesic γ, we label the corresponding edge γ and attach the number $\ell(\gamma)$ to this edge. Note that there may be more than one connection between two vertices of K and the endpoints of an edge may coincide. Our main result states that the eigenvalues of the surface can be estimated in terms of these discrete data. Hence the small eigenvalues of the Laplacian Δ on M can be estimated in terms of lengths of closed geodesics.

We may think of K as an electrical circuit diagram with a resistance $1/\ell(\gamma)$ along the edge γ [PS]. If the area of M is infinite, then K contains the ground denoted v_0. In this case *all* functions defined on the set of vertices of K will be required to vanish at v_0. Some circuit diagrams which may arise when M is homeomorphic to a punctured torus are shown in Figure 1. For example c) will

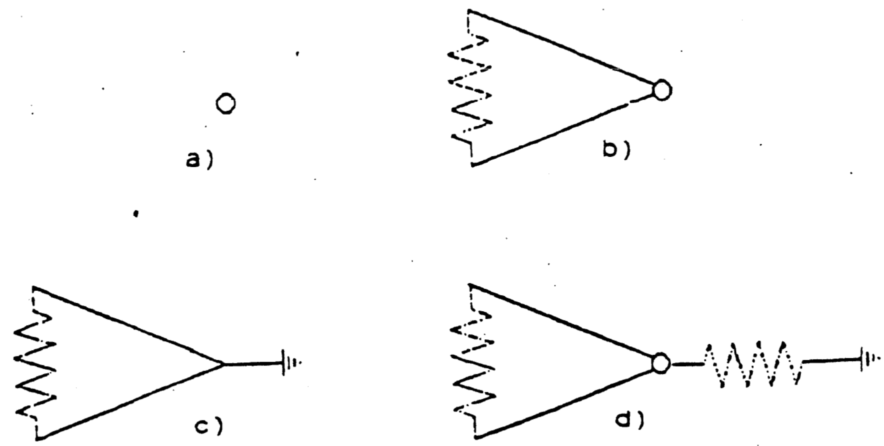

Figure 1

occur if M has a funnel bounded by a geodesic of length greater than or equal to 2ϵ and another simple closed geodesic of length less than 2ϵ.

A more complicated surface illustrated schematically in Figure 2 yields the diagram (drawn in two different ways) in Figure 3.

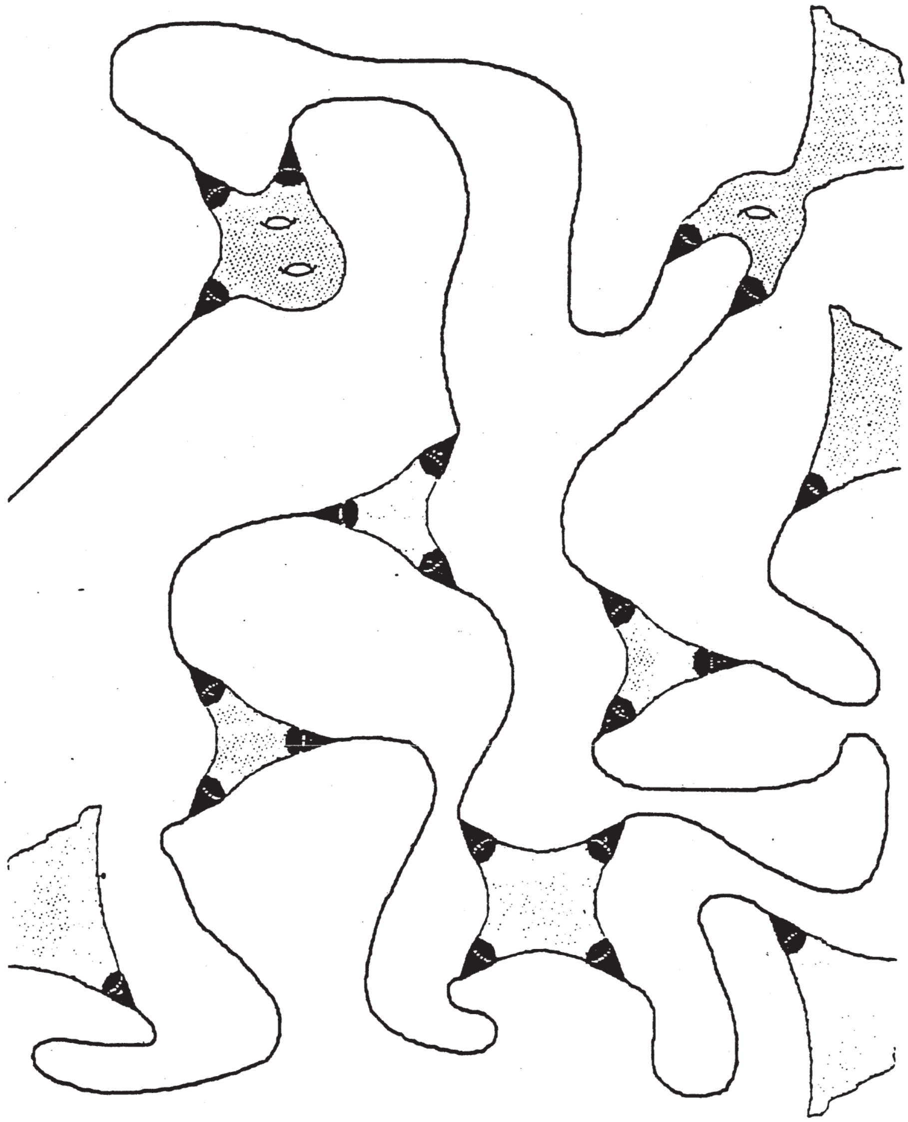

Figure 2

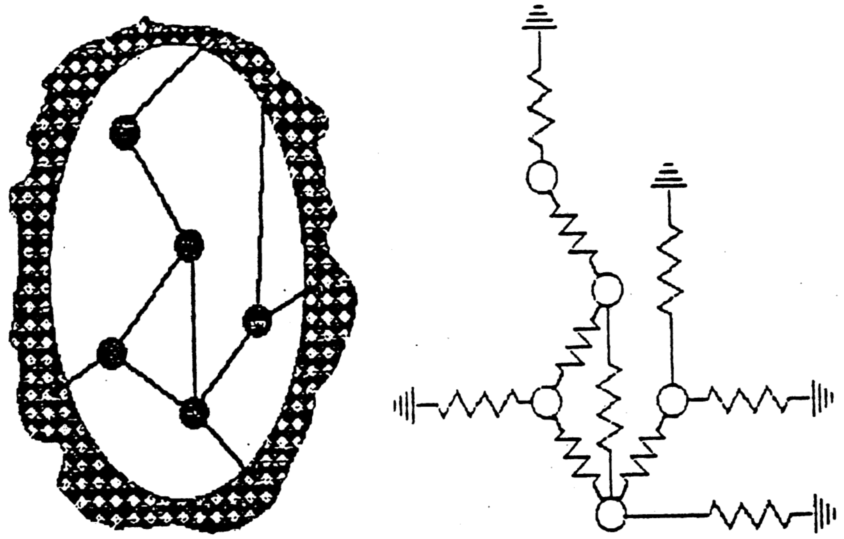

Figure 3

The analog of the Rayleigh-Ritz quotient of a vertex function h in this context is the following expression.

$$R(h) = \frac{\Sigma_\gamma \partial h(\gamma)^2\ \ell(\gamma)}{\Sigma_v h(v)^2}$$

where the summation in the numerator extends over all edges γ and if v, w are the endpoints of γ, then $\partial h(\gamma)^2 = (h(v) - h(w))^2$. The summation in the denominator is over all vertices of K. By analogy with the continuous case the critical values of the function R will be called the eigenvalues of K.

We are now ready to state our result. Let n be the number of nonzero eigenvalues of K. Since the trivial eigenvalue $\lambda_0 = 0$ occurs only when the area A(M) is finite, n = N if the area A(M) = ∞ and n = N − 1 otherwise. Broadly speaking, we claim that the first n eigenvalues $\{\lambda_j\}$ of the Laplacian on M are of the same order as the eigenvalues $\nu_1, \nu_2, \ldots \nu_n$ of the graph K associated to M. The statement is complicated somewhat by the fact that M might have continuous spectrum in the interval $[1/4,\infty)$ (see the remark following Lemma 3.2).

Theorem 1.1. There exist positive constants α_1, α_2, α_3 which depend only on ϵ and the topology of M (i.e. on g, f and p) such that, for every j = 1,2, ... n

a) If $\lambda_j < 1/4$, then

$$\alpha_1 \leqslant \lambda_j/\nu_j \leqslant \alpha_2 .$$

b) If $\alpha_2\nu_j < 1/4$, then the spectrum of Δ contains at least j eigenvalues in $(0,1/4)$ and a) holds.

c) $\inf \operatorname{spec}(\Delta) \cap (\lambda_n,\infty) \geqslant \alpha_3$.

If M is compact, the requirement that $\lambda_j < 1/4$ in a) is unnecessary. Moreover, when $A(M) < \infty$, the constants α_1, α_2, α_3 can be made independent of ϵ using [B].

To translate our estimates of λ_j into geometric terms we introduce a notion of a cut and its length.

Definition 1.2. A j-cut C is a collection of edges of the graph K whose removal disconnects K into j+1 components. Equivalently, it is a collection of simple closed geodesics of length $< 2\epsilon$, whose removal disconnects M into j+1 pieces, where we regard the union of all components containing funnels as a single piece. The length L(C) of the j-cut C is the sum of the numbers $\ell(\gamma)$ over the edges γ constituting the cut and $L_j(K) = L_j(M)$ is defined to be the minimal length of a j-cut. If there are no j-cuts we take $L_j(K) = \infty$.

Using results of the next section, Theorem 1.1 can be restated in the following equivalent form.

Theorem 1.1'. There exist constants β_1, β_2, β_3, depending only on ϵ and the topology of M, so that for j = 1, 2, ... n,

a) If $\lambda_j < 1/4$, then

$$\beta_1 \leq \lambda_j / L_j(M) \leq \beta_2$$

b) If $\beta_2 L_j(M) < 1/4$, then the spectrum of Δ contains at least j eigenvalues in (0,1/4) and a) holds.

c) $\inf \operatorname{spec}(\Delta) \cap (\lambda_n, \infty) \geq \beta_3$.

If M is compact, the requirement that $\lambda_j < 1/4$ in a) is unnecessary. Moreover, when $A(M) < \infty$, the constants β_1, β_2, β_3 can be made independent of ϵ using [B].

Remark. This result was proved for compact surfaces in [SWY] (see also [Bu]). The estimate for λ_1, in case $f > 0$, was given in [PS], and the lower bounds for surfaces of finite area were derived in [DR].

Recall that $L_j(K)$ is the minimum sum of lengths of simple closed geodesics of length less than 2ϵ separating M into $j + 1$ components, where we regard the union of all pieces containing a funnel as a single component. The maximum number of disjoint, simple, closed geodesics on M is $3g + p + 2f - 3$, and their complement in M has $2g + p + f - 2$ bounded components. If $\gamma_1, \ldots, \gamma_{3g+p+2f-3}$ are such geodesics and all their lengths tend to zero, then λ_j approaches zero for $j = 1, 2, \ldots n$, but λ_{n+1} is bounded away from zero. Here, $n = N - 1 = 2g + f + p - 3$ if $f = 0$, and $n = N = 2g + f + p - 2$ otherwise. Thus we obtain

Corollary 1.3. Let M be as above. There exists a constant $\beta > 0$, depending only on $2g + p + f$ such that $\operatorname{spec}(\Delta) \cap (0,\beta)$ is discrete and contains at most n eigenvalues, where $n = 2g + f + p - 2$ if $A(M) = \infty$ and $n = 2g + p - 3$ for $A(M) < \infty$.

Remark. Since the lengths of disjoint simple closed geodesics on M can be prescribed arbitrarily, one can produce examples of surfaces with a maximal number of arbitrarily small eigenvalues.

On the other hand, if the number of disjoint simple closed geodesics of length $< 2\epsilon$ is smaller than $3g + p + 2f - 3$, one can still obtain estimates of $2g + f + p - 2$ eigenvalues. However, the constants in Theorem 1.1' would depend in this case not only on the topology, but also on lengths of "boundary curves", i.e. geodesics which bound funnels.

2. The discrete model. In this section we consider the discrete model of our eigenvalue problem. Thus, let K be a finite connected graph with possibly more than one connection (edge) between a pair of nodes (vertices) and possibly with edges whose endpoints coincide. The graph K is equipped with an additional structure: a number $\ell(\gamma) > 0$ is assigned to every edge γ. We shall estimate the smallest positive eigenvalue ν_1 of K in terms of the numbers $(\ell(\gamma))$. This, in turn, will allow us to estimate the higher eigenvalues. Clearly, if there is no ground,

$$\nu_1 = \inf \Sigma_\gamma \, \partial h^2 \, \ell(\gamma) \tag{2.1}$$

over the set of all vertex functions satisfying $\Sigma_v h(v)^2 = 1$, $\Sigma_v h(v) = 0$ (as in the continuous case ν_1 is the infimum of the Rayleigh-Ritz quotient defined in Section 1 over functions perpendicular to constants). Similarly, if the ground, v_0, is present (2.1) is still valid if the infimum is taken instead over all functions h satisfying $\Sigma_n h(v)^2 = 1$, $h(v_0) = 0$.

The notion of the j-cut and its length was introduced in Section 1. Recall that $N = N(K)$ is the number of vertices of K different from the ground. In addition, let $E = E(K)$ be the number of edges.

Theorem 2.2. Suppose $N(K) \leq q$. There exists a constant α, depending only on q, so that

$$\alpha \leq \nu_1/L_1(K) \leq 2.$$

Proof: Let M_h and m_h denote the maximum and the minimum value of the function h, respectively. Replacing h by its negative if necessary, we can assume that $M_h > 0$ in case the ground is present, and $M_h \geq |m_h|$ if K does not contain a ground. Since $\Sigma_v h(v)^2 = 1$, $M_h \geq 1/\sqrt{q}$. Consider the sequence of all values $m_h = a_1 \leq a_2 \leq a_3 \leq ... \leq a_k = M_h$. Clearly, $M_h - m_h \geq 1/\sqrt{q}$ and $k \leq q$ so there will exist an integer j, $1 \leq j \leq k-1$ for which $a_{j+1} - a_j \geq 1/(q-1)\sqrt{q}$. Split the set of vertices of K into two disjoint subsets such that $h \geq a_{j+1}$ on one and $h \leq a_j$ on the other, and define a cut consisting of all edges connecting vertices in different sets. It is now obvious that

$$\nu_1 = \inf \Sigma_\gamma \, \partial h^2 \, \ell(\gamma) \geq \alpha L_1(K),$$

with $\alpha = (1/(q-1)\sqrt{q})^2$.

To prove the upper bound consider a minimal 1-cut C and a function h which is constant on each of the two components of K − C. The constants have to be chosen so that h is an appropriate test function for (2.1) to hold. Since h is normalized, $|\partial h| \leq \sqrt{2}$ for every edge. It follows easily that $\nu_1 \leq R(h) \leq 2L_1(K)$.

As a consequence, we can estimate the higher eigenvalues.

Theorem 2.3. Let $\nu_1 \leq \nu_2 \leq \nu_3 \leq ... \leq \nu_n$ be the sequence of all positive eigenvalues (if there is no ground, $n = N(K) - 1$, otherwise $n = N(K)$). There exists a constant α, depending only on upper bounds for N(K) and E(K), such that, for j = 1,2, ... n,

$$\alpha \leq \nu_j/L_j(K) \leq 2.$$

Proof: Suppose K is cut into j + 1 pieces $K_1, K_2, ..., K_{j+1}$ by removing a cut C. Assume also that $L(C) = L_j(K)$. Consider the space F of all functions which are constant on every component of K − C and vanish on the component containing the ground. As above we see that $\partial g(\gamma) = 0$ unless $\gamma \in C$, and $|\partial g| \leq \sqrt{2}$ for $\gamma \in C$ if $\Sigma_v g(v)^2 = 1$. Therefore, using the mini-max principle [Ch], $\nu_j \leq$

$\sup \{R(g) \mid g \in F\} \leqslant 2L_j(K)$.

To prove the lower bound suppose that a (j−1)-cut $C = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ decomposes K into j pieces $K_1, K_2, \ldots, K_j$, that $L(C) = L_{j-1}(K)$, and that K_j contains the ground. The obvious inequality

$$\sum_{\gamma \in K-C} \partial h(\gamma)^2 \ell(\gamma) \leqslant \sum_{\gamma \in K} \partial h(\gamma)^2 \ell(\gamma) \tag{2.4}$$

implies that

$$\min_{1 \leqslant i \leqslant j} \nu_1(K_i) \leqslant \nu_j(K). \tag{2.5}$$

The two quadratic forms in (2.4) are defined on the space of all vertex functions on K. The eigenvalues of the quadratic form on the left are simply the eigenvalues of all the pieces $K_1, K_2, \ldots, K_j$. Among these eigenvalues of the pieces, 0 occurs $j - 1$ times (corresponding to constants on components other than the ground). The first non-zero eigenvalue occupies the j-th slot in the sequence and is equal to $\min_i \nu_1(K_i)$. Therefore, by the mini-max principle, the inequality (2.4) implies (2.5). It remains to estimate $\nu_1(K_i)$. By Theorem 2.3

$$\nu_1(K_i) \geqslant \alpha L_1(K_i). \tag{2.6}$$

Recall that $C = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$. We claim that $L_1(K_i) \geqslant \ell(\gamma)$ for all $\gamma \in C$. Otherwise, by cutting K_i along a cut C' realizing $L_1(K_i)$ and reattaching along γ, we would obtain a (j−1)-cut of K of total length smaller than $L(C) = L_{j-1}(K)$. Therefore $L_1(K_i) \geqslant \ell(\gamma)$. It follows that $kL_1(K_i) \geqslant L_{j-1}(K)$ and $(k+1)L_1(K_i) \geqslant L_{j-1}(K) + L(C')$. The number on the right is the length of the cut $C \cup C'$ separating K into $j + 1$ components. Therefore $L_1(K_i) \geqslant L_j(K)/(k+1) \geqslant L_j(K)/(E(K) + 1)$. This, in view of (2.6), completes the proof in the case when the ground is present. A very similar proof in the case of no ground is omitted.

Remark. Theorem 2.3 and Lemma 4.2 below imply that Theorem 1.1 and Theorem 1.1' are equivalent.

3. Energy integral for doubly connected Riemann surfaces. This section contains technical lemmas needed to prove the results of Section 2. We begin with the lower bound for the bottom of the spectrum of a doubly connected complete surface with constant Gaussian curvature -1.

Lemma 3.1. Let M be one of the following Riemann surfaces.

a) $(-\infty,\infty) \times S^1$ with the metric $dr^2 + \exp(-2r)d\theta^2$

b) $(-\infty,\infty) \times S^1$ with the metric $dr^2 + \ell^2\cosh^2 r d\theta^2$, where $r \in (-\infty,\infty)$, $\theta \in S^1 = R/Z$ and $\ell > 0$ is a constant. Then

$$\int_M |\nabla h|^2 \geq (1/4)\int_M |h|^2.$$

for every smooth function h with compact support.

Proof: The two cases are very similar and we shall give the proof only in the case a) (cf. [DR] for details in case b)). Note that the volume element $dV = \exp(-r)drd\theta$ and $|\nabla h| \geq |h_r|$. We write $h' = h_r$ and consider the integral

$$\int_{-\infty}^{\infty} h^2e^{-r}dr = \int_{-\infty}^{\infty} 2hh'e^{-r}dr.$$

Applying the Schwartz inequality with respect to the measure $\exp(-r)dr$ we obtain

$$\int_{-\infty}^{\infty} h^2e^{-r}dr \leq 2\left[\int_{-\infty}^{\infty} h^2e^{-r}dr\right]^{1/2}\left[\int_{-\infty}^{\infty}(h')^2e^{-r}dr\right]^{1/2}.$$

Hence

$$\frac{1}{4}\int_{-\infty}^{\infty} h^2e^{-r}dr \leq \int_{-\infty}^{\infty}(h')^2e^{-r}dr.$$

Integration with respect to θ yields the desired estimate.

In the sequel we shall have to apply the inequality of the lemma above to some functions without compact support. The result we need is

Lemma 3.2. Let T be one of the following:

a) a cusp $[a,\infty) \times S^1$ with the metric $dr^2 + \exp(-2r)d\theta^2$

b) a cylinder $[a,b] \times S^1$ with the metric $dr^2 + \ell^2\cosh^2 r d\theta^2$

c) a funnel $[a,\infty) \times S^1$ with the metric $dr^2 + \ell^2\cosh^2 r d\theta^2$

Suppose h is a continuously differentiable function which vanishes on ∂T and in case a) and c) satisfies $\int_T h^2 < \infty$, $\int_T |\nabla h|^2 < \infty$. Then

$$\frac{1}{4}\int_T h^2 \leq \int_T |\nabla h|^2.$$

Proof: Case b) is covered in Lemma 3.1. To reduce the remaining cases to Lemma 3.1 we use a standard cut-off argument (cf. [DR], page 187). Consider the function μ_R defines as follows.

$$\mu_R(r) = \begin{cases} 1 & \text{if } r \leq R-1 \\ R-r & \text{if } R-1 \leq r \leq R \\ 0 & \text{if } r \geq R. \end{cases}$$

Let $h_R = h\mu_R$. Lemma 3.1 applies to h_R and when R tends to infinity the integrals of $|\nabla h_R|^2$ and $|h_R|^2$ converge to the integrals of $|\nabla h|^2$ and $|h|^2$ respectively.

Remark. For a complete Riemannian manifold the Laplacian Δ on $C_0^\infty(M)$ has a unique extension to an unbounded self-adjoint operator (also denoted by Δ) on $L^2(M)$ [Che]. The lemma implies that the essential spectrum of Δ on a Riemann surface with finitely generated fundamental group is contained in $[1/4,\infty)$. Indeed, by a result of Donnelly and Li [DL], the essential spectrum of Δ on M is equal to the essential spectrum of Δ on $M - F$, for an arbitrary compact subset F with smooth boundary, if Dirichlet boundary conditions are imposed on ∂F. If F is sufficiently large $M - F$ consists of cusps and funnels only and their entire spectrum is contained in $[1/4,\infty)$ by the lemma.

The next lemma is an approximate version of the preceding one. It says that a function h with substantial L^2 norm on T which is nearly zero on ∂T must have substantial energy $\int_T |\nabla h|^2$.

Lemma 3.3. Let T be as in Lemma 3.2 and let $S \subset T$ be the shell i.e. the set of points of T at distances less than or equal to 1 from the boundary of T. If T is a cylinder (case b) in Lemma 3.2) we require that $b - a > 2$. There exists $\eta > 0$ such that if h is a function on T satisfying

a) $\infty > \int_T h^2 = c > 0$

b) $\int_S h^2 < \eta c$

c) $\int_S |\nabla h|^2 < \eta c$,

then

$$\int_T |\nabla h|^2 > c/8.$$

Proof: Let μ be a function on T depending on r only, equal to 1 on $T - S$, vanishing on ∂T and varying linearly in r on S. Set $H = \mu h$. By b), if η is small enough, most of the contribution to the integral in a) comes from $T - S$. Thus, for small η, $\int_T H^2 > (3/4)c$, and it follows from Lemma 3.2 that $\int_T |\nabla H|^2 > (3/16)c$. On the other hand

$$\int_S |\nabla H|^2 = \int_S |\mu \nabla h + h \nabla \mu|^2$$
$$\leq \left[\left(\int_S |\nabla h|^2 \right)^{1/2} + \left(\int_S h^2 \right)^{1/2} \right]^2 = O(\eta c),$$

by b) and c), since $|\mu| \leq 1$ and $|\nabla \mu| = 1$ on S. Thus, for sufficiently small η,

$$\int_{T-S} |\nabla H|^2 = \int_{T-S} |\nabla h|^2 > c/8,$$

which proves the lemma.

The preceding lemma gives a lower bound for the energy of a function which is very small near the boundary of a doubly connected surface T. The next result will be used to estimate the energy of functions which are not nearly zero on ∂T. Since the energy integral is a conformal invariant in two dimensions, we replace T by a right circular cylinder.

Lemma 3.4. Let $T = [a,b] \times S^1$ with the flat metric $d\rho^2 + d\theta^2$. Suppose h is a continuous function smooth in the interior of T. Then

$$\int_T |\nabla h|^2 \geqslant \frac{1}{b-a} \int_{S^1} (h(b,\theta) - h(a,\theta))^2 d\theta.$$

Proof: Write $h(b,\theta) - h(a,\theta)$ as the integral of h_ρ with respect to ρ, apply Schwartz inequality and integrate in θ.

Remark 3.5. Lemma 3.4 will be applied to cylinders and funnels with the metric $dr^2 + \ell^2\cosh^2 r d\theta^2$. An explicit conformal mapping of such a surface to a right circular cylinder of circumference 1 is given by $(r,\theta) \to (\rho,\theta)$, with $\rho = (2/\ell)\tan^{-1}(\exp(r))$. This will allow us to estimate the energy in terms of the length ℓ of the simple closed geodesic defining the cylinder.

4. **Thick and thin decomposition.** Before proving Theorem 2.2 we have to discuss the decomposition $M = M_{thick} \cup M_{thin}$ of a surface with finitely generated fundamental group and explain the choice of the number ϵ used to define it. A reference for most of this material is [Be, Section 11.7].

The result that we need states that each simple closed geodesic lies in an open "collar," called a cylinder whose size can be estimated in terms of the length of the geodesic; every cusp, i.e. contracting end, is of a definite size; two collars corresponding to disjoint geodesics do not intersect; different cusps are disjoint; and cusps do not itersect collars. We list below relevant properties of these sets and of funnels.

Cylinders. Let γ be a simple closed geodesic of length $\ell(\gamma) = \ell$

in M and let $r(\gamma)$ be defined by

$$\sinh(r(\gamma))\sinh(\ell/2) = 1.$$

The set of points of distance $\leq r(\gamma)$ from γ is an embedded annulus. In terms of Fermi coordinates, the metric on this set is given by $dr^2 + \ell^2\cosh^2 r d\theta^2$, where $\ell d\theta$ is the arc length element along γ and r denotes the signed distance from γ. When $\ell(\gamma) < 2\epsilon$, we call the set a cylinder and denote it by T_γ. If ϵ is chosen small enough and $\ell(\gamma) < 2\epsilon$, then $r(\gamma) > 1$ and the shell $S = \{p \in T_\gamma \mid r(\gamma) - 1 < |r| < r(\gamma)\}$ consists of points with injectivity radius $\geq \epsilon$. This is

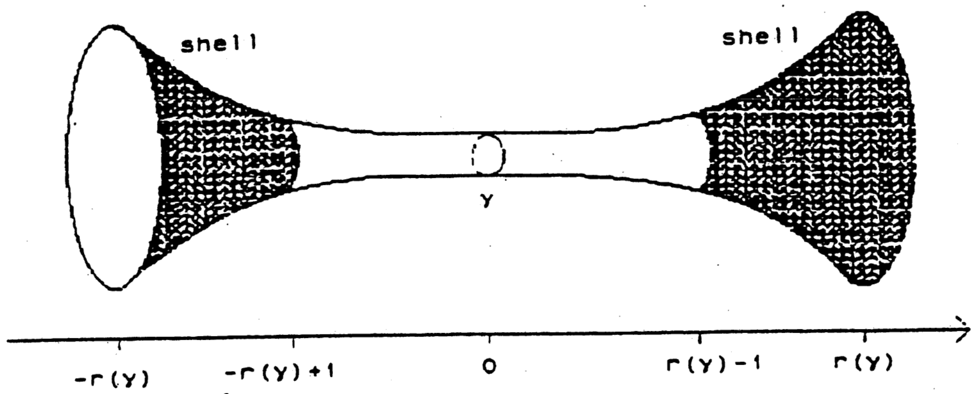

Figure 4

one of the conditions ϵ has to satisfy. For further reference, note the diameter of each component of S is bounded independently of $\ell(\gamma)$, when $\ell(\gamma) < 2\epsilon$.

Cusps. In every cusp we can choose coordinates near infinity so that the metric takes form $dr^2 + \exp(-2r)d\theta^2$, with $\theta \in S^1 = R/Z$. Then, the set

$$T = \{(r,\theta) \mid r \geq -\ln 2\}$$

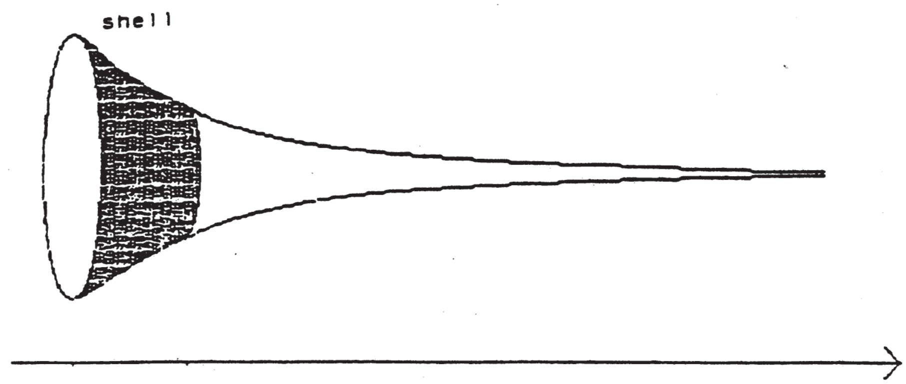

Figure 5

embeds in M. Clearly, if ϵ is sufficiently small, then $S = \{(r,\theta) \mid -\ln 2 \leq r \leq -\ln 2 + 1\}$ is contained in M_{thick}. N.B. From now on ϵ is fixed so that every cylinder T_γ with $\ell(\gamma) < 2\epsilon$ has radius $r(\gamma) > 1$ and its shell contained in M_{thick}, and so that the shell of every cusp is contained in M_{thick}. This is a universal choice independent of M.

Funnels. Every expanding end has a simple closed geodesic γ at its base. If $\ell(\gamma) < 2\epsilon$ we shall call this a thin end. Otherwise the end shall be "thick". In terms of the Fermi coordinates the metric on an expanding end is given by $dr^2 + \ell^2\cosh^2 r d\theta^2$, with $\ell = \ell(\gamma)$. Part of a thin end is contained in the cylinder T_γ. The set $\{(r,\theta) \mid r > r(\gamma) - 1\}$ will be referred to as a thin funnel, and we define its shell to be $\{(r,\theta) \mid r(\gamma) - 1 < r < r(\gamma)\}$. Thus the shell of every thin funnel is just one of the components of the shell of the cylinder adjacent to it. For a thick end $\ell(\gamma) \geq 2\epsilon$. In this case the term thick funnel will be used to refer to the whole end cut off by γ and the shell of such a funnel will be the set of points in the funnel at distance ≤ 1 from the base geodesic.

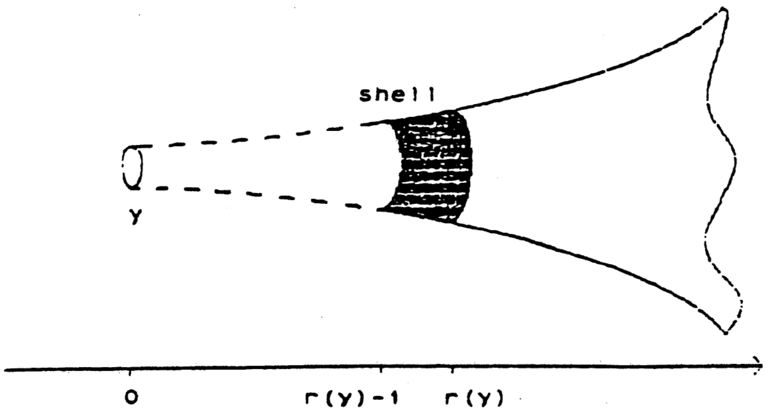

Figure 6

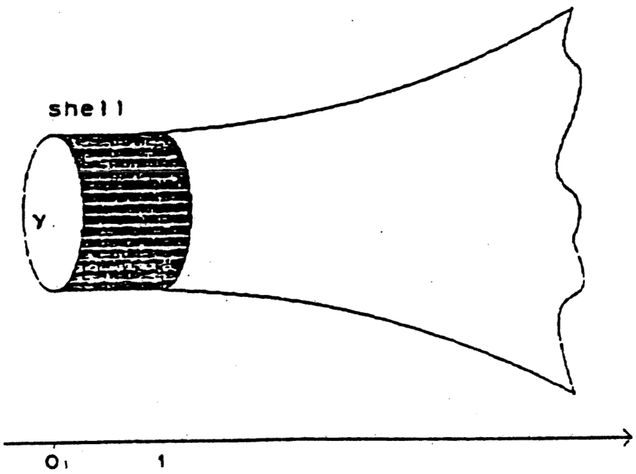

Figure 7

In the next section, we shall have to estimate the energy of eigenfunctions on cylinders and funnels using Lemma 3.4. To do this we need to know the length of the right circular cylinder of unit circumference conformally diffeomorphic to a cylinder or funnel. The diffemorphism is given explicitly by $(r,\theta) \to (\rho,\theta)$, where $\rho = (2/\ell)\tan^{-1}(\exp(r))$. The following is an easy consequence.

Lemma 4.1. a) The length d of the right circular cylinder conformally equivalent to a cylinder T_γ with $\ell(\gamma) < 2\epsilon$ satisfies

$c(\epsilon)/\ell(\gamma) \leq d \leq \pi/\ell(\gamma)$, where $c(\epsilon) = \tan^{-1}(\exp(\sinh^{-1}(2/\sinh\epsilon))) - \tan^{-1}(\exp(-\sinh^{-1}(1/\sinh\epsilon)))$.

b) A thick funnel bounded by a geodesic γ of length $\ell(\gamma) \geq 2\epsilon$ is conformally equivalent to the cylinder of length $\pi/\ell(\gamma)$. Moreover for every $r_0 \in [0,1]$, the subset $\{(r,\theta) \mid r > r_0\}$ is conformally equivalent to a cylinder of length d, $c/\ell(\gamma) < d < \pi/\ell(\gamma)$. The constant c is independent of r_0 and $\ell(\gamma)$, in fact one can take $c = 0.6\pi$.

c) The subset $\{(r,\theta) \mid r > r_0\}$, for $r(\gamma) - 1 \leq r_0 \leq r(\gamma)$, of a thin funnel is conformally equivalent to a cylinder of length d, $c_1 \leq d \leq c_2$, where $c_1 > 0$ and c_2 depend only on ϵ. c_2 can be taken to be $\pi/2\epsilon$ while c_1 is more complicated and is equal to

$$\min_{0<\ell<2\epsilon}\{(\pi - 2\tan^{-1}(\exp(\sinh^{-1}(1/\sinh(\ell/2)))))/\ell\}.$$

In Section 1 we described how to associate a finite graph K to the surface M. We now state an estimate of the number of edges and vertices of this graph in terms of the topology of M. Recall that N(K) is the number of vertices of K different from the ground, and E(K) is the number of edges.

Lemma 4.2. If g denotes the genus of M, p equals the number of cusps, and f is the number of funnels, then $N(K) \leq 2g + p + f-2$ and $E(K) \leq 3g + 2f + 2p - 3$.

Proof: Every funnel is bounded by a simple closed geodesic. Cut off all funnels to form a surface M_0 with f holes bounded by closed geodesics. The Gauss-Bonnet formula for surfaces with boundary implies that the area $A(M_0) = 2\pi(2g + p + f - 2)$. On the other hand if we cut M_0 along a maximal set of disjoint, simple, closed geodesics, the number of pieces obtained is at least N(K). By Gauss-Bonnet again, every component has area greater than or equal to 2π. Hence

$$2\pi N(K) \leq A(M_0) = 2\pi(2g + f + p - 2).$$

In the decomposition of M_0 by a maximal set of disjoint simple closed geodesics, every piece is homeomorphic to a three-holed sphere. An easy count shows that the maximal number of disjoint simple closed geodesics is equal to $3g + 2f + p - 3$. Thus $E(K) \leqslant 3g + 2f + 2p - 3$, a quantity depending only on the topology.

5. **The proof.** In this section we prove the eigenvalue estimates. We adopt the following convention. The letter c, with or without subscripts, will stand for a constant independent of the surface M; α or β will denote a constant depending only on the topology of M, i.e. on the genus g, the number of cusps p, and the number of funnels f. These constants may depend on the choice of ϵ, but ϵ is fixed once and for all. Moreover constants represented by the same symbol in different inequalities need not be equal.

We begin by proving the upper bounds in Theorem 1.1. Note that, by using a cut-off argument as in the proof of Lemma 3.2, one can prove that the space of piecewise C^1 functions with compact support is dense in the Sobolev space of L^2 functions with L^2 gradient. Therefore, to prove the upper bounds it suffices to construct test functions in this space.

Let K be the graph associated to the surface M and let h be a function on K which vanishes on the ground. We associate to h a function ψ on M as follows. Let B_v be the component of M_{thick} corresponding to a vertex v. Make ψ constant and equal to h(v) on the complement of shells in B_v. If a cylinder T_γ connects B_v and B_w, we interpolate ψ linearly in the variable ρ on the associated flat cylinder. On a cusp attached to a component B_v of M_{thick}, ψ is also constant and equal to h(v). Clearly

$$\int_M \psi^2 \geqslant c \sum_v h(v)^2,$$

since $A(B_v) \geqslant A(D_\epsilon)$, where D_ϵ is the hyperbolic disk of radius ϵ. On the other hand, a simple calculation using Lemma 4.1 yields

$$\int_M |\nabla\psi|^2 = \sum_\gamma \int_{T_\gamma} |\nabla\psi|^2 \leq c \sum_\gamma \ell(\gamma)\partial h(\gamma)^2.$$

It follows that the discrete Rayleigh-Ritz quotient of h is greater than a universal constant c_1 times the Rayleigh-Ritz quotient of ψ. In particular, let $0 < \nu_1 \leq \nu_2 \dots \leq \nu_n$ be the positive eigenvalues of K. Denote by $h_1, h_2, \dots h_n$ the corresponding eigenfunctions, and let $\psi_1, \psi_2, \dots \psi_n$ be the associated functions on M. Let j be the largest integer such that $\nu_j/c_1 \leq 1/4$ if M is noncompact, and $j = n$ otherwise. (This restriction is necessary since the essential spectrum of is contained in $[1/4,\infty)$ when M is noncompact.) The functions $h_1, h_2, \dots h_j$ are linearly independent and the inequality between the Rayleigh-Ritz quotients implies that the Laplacian Δ has at least j positive eigenvalues in $(0,\nu_j/c_1]$ and $\lambda_k \leq \nu_k/c_1$ for $k = 1, 2 \dots j$.

As in the discrete model, the lower bounds for higher eigenvalues follow easily from the lower bound on λ_1. We give the proof of this estimate for the case of infinite area. The proof for finite area surfaces is very similar and, in fact, somewhat easier. We shall only indicate necessary changes in the argument for that case.

Thus let $A(M) = \infty$, and let φ be a normalized eigenfunction belonging to an eigenvalue $-\lambda < 1/4$. The following lemma describes the behavior of φ on M_{thick}.

Lemma 5.1. Suppose $\lambda < \eta^2 < 1/4$, $\eta > 0$. For every $x \in M_{thick}$,

$$|\nabla\varphi(x)| \leq c\eta$$

and

$$|\varphi(x)|^2 \leq c\int_{D_\epsilon(x)} \varphi^2,$$

where $D_\epsilon(x)$ is the disk of radius ϵ centered at x.

Proof: By the Sobolev and Gårding inequalities (cf. [A])

$$|\varphi(x)|^2 \leqslant c\int_{D_\epsilon(x)}\varphi^2 + c\int_{D_\epsilon(x)}|\Delta\varphi|^2 \leqslant c(1+\lambda^2)\int_{D_\epsilon(x)}\varphi^2 .$$

Similarly, since $\Delta_1 d\varphi = d\Delta\varphi = -\lambda d\varphi$, where Δ_1 is the Laplacian on forms of degree one,

$$|\nabla\varphi(x)|^2 \leqslant c\int_{D_\epsilon(x)}|\nabla\varphi|^2 + c\int_{D_\epsilon(x)}|d\Delta\varphi|^2$$

$$\leqslant c(1+\lambda^2)\int_M|\nabla\varphi|^2 = c(1+\lambda^2)\lambda .$$

The lemma follows.

The lemma above implies that an eigenfunction φ belonging to a small eigenvalue λ is almost constant on the bounded components of M_{thick} and on the shells of thin funnels since these sets are bounded and $|\nabla\varphi|$ is uniformly small on them. Next, we investigate the behavior of funnels.

Lemma 5.2. Suppose N is a funnel with shell S. Then

$$\int_S\varphi^2 \leqslant c_1\lambda \quad\text{and}\quad \varphi^2 \leqslant c_2\lambda \quad\text{on } S.$$

Proof: We give a proof for a thick funnel only. Let γ be the geodesic at the base of N, $\ell = \ell(\gamma)$. We use the Fermi coordinates. Since φ is in L^2, it follows from Lemma 5.1 that $\varphi(r,\theta)$ approaches zero uniformly in θ as r tends to infinity. Therefore

$$\int_S\varphi^2 = \int_0^1\int_0^1 \varphi^2(r,\theta)\ell\cosh r\,dr\,d\theta = \int_0^1 \ell\cosh r\,dr \int_0^1 \varphi^2(r,\theta)d\theta$$

$$\leqslant \cosh(1)\ell \int_0^1 d(r)\,dr \int_V|\nabla\varphi|^2,$$

where the inner integral is estimated using Lemma 3.4 and d(r) is the length of the flat cylinder of waist one conformally equivalent to $\{x \in V \mid \mathrm{dist}(x,y) \geqslant r\}$. Since $d(r) < \pi/\ell(\gamma)$ by Lemma 4.1, the first inequality follows. The second inequality is a consequence of Lemma 5.1.

Let φ be as above, but assume that $\lambda = \lambda_1$. We will show that there exists constants μ and α, depending only on the topology, such that if $\lambda_1 < \mu^2$ then $\lambda_1 \geq \alpha L_1(M)$. Hence $\lambda_1 \geq \min(\mu^2, \alpha L_1(M))$. Since all geodesics contributing to $L_1(M)$ are shorter than 2ϵ and their number is bounded in terms of the topology, $L_1(M)$ itself is bounded. Therefore $\lambda_1 \geq \min(\mu^2, \alpha L_1(M)) \geq \beta L_1(M)$.

Thus suppose that $\eta > 0$ is a small constant (to be specified later) and that $\lambda_1 < \eta^2$. We saw above that the oscillation of φ is very small (of size $c_1\eta$) on every bounded component of M_{thick}, and that φ is small (at most $c_2\eta$) on the shell of every funnel. If $T_\gamma = T$ is a cylinder and $x \in \partial T$, denote by x^* the reflection of x in γ. The quantity $\sup_{x \in \partial T}|\varphi(x) - \varphi(x^*)|$ will be called the variation of φ on T. We will show that if η is sufficiently small then φ has substantial variation on one of the cylinders. Suppose the variation of φ is smaller than $c_2\eta$ on every cylinder. Since φ is near zero on the shells of funnels, it follows that $|\varphi| < \alpha\eta$ on M_{thick}. Therefore the integral of φ^2 is very small on every bounded component of M_{thick}. By Lemma 5.2 these integrals are also small on the shells of funnels. Moreover, every thick funnel N is attached to a bounded piece of $M_{thick} - N$ and the integral of φ^2 over such a set is also very small. The L^2 norm of φ on the union of all shells and bounded components of M_{thick} is $\leq \alpha\eta$.

Remark. This leads to a contradiction if M_{thick} is connected (i.e. $L_1(M) = \infty$) and η is small. Therefore spec(Δ) is bounded below for all surfaces whose thick part is connected.

Since $\int_M \varphi^2 = 1$, there exists a cylinder, a cusp, or a funnel on which the L^2 norm of φ is substantial (i.e. greater than $(1 - \alpha\eta)/m$, where m is the number of all cusps, cylinders and funnels). Denote such a doubly connected surface by T. The assumptions of Lemma 3.3 are satisfied and it follows that $\lambda_1 = \int_M |\nabla\varphi|^2 \geq \beta$. However λ_1 was assumed to be smaller than η^2. For small η, this is a contradiction, which proves that, if $\lambda_1 \leq \eta^2$ and η is small, then the eigenfunction φ has variation $\geq c\eta$ on at least one cylinder.

We now suppose that $\lambda_1 < \mu^2 < \eta^2$, where μ is a small fraction of η (how small depends only on the number of cylinders and components of M_{thick}). By the above discussion, sup $\varphi|M_{thick}$ is at least $c\eta$, and the oscillation of φ on every bounded component of M_{thick} is at most $c\mu$. In addition, $\varphi \leq c\mu$ on the shells of funnels. The argument used in the proof of Theorem 2.2 shows that there exists a collection $\gamma_1, \gamma_2, \dots \gamma_k$, of short, disjoint, simple closed geodesics whose removal disconnects M into two components such that the variation of φ on each of the associated cylinders is at least $\beta > 0$, where β depends on the topology only. Recall that the union of all unbounded pieces of M_{thick} is regarded as a single component. Using Lemma 3.4 and Lemma 4.1 we obtain

$$\lambda_1 \geq \sum_{i=1}^{k} \int_{T\gamma_i} |\nabla\varphi|^2 \geq \beta \sum_{i=1}^{k} \ell(\gamma_i) \geq \beta L_1,$$

the desired lower bound.

Now, if $A(M) < \infty$, M has no funnels. The normalized eigenfunction φ corresponding to λ_1 is orthogonal to constants and changes sign. Lemma 5.1 is still valid and implies that φ is almost constant on every component of M_{thick}. As in the previous case, we want to show that φ has a substantial variation on one of the cylinders, provided $\lambda_1 < \eta^2$ and η is a small constant depending on the topology of M. We only sketch the argument. If the variation on each cylinder is small, then φ is almost constant on all of M_{thick}, $\varphi|M_{thick} \approx \bar{\varphi}$, its average. Either $\bar{\varphi}$ is very small, or it is substantial and can therefore be assumed positive. If it is small, then the integral of φ^2 over M_{thick} is small and Lemma 3.3 leads to a contradiction, exactly as in the case of infinite area. If, on the other hand, $\bar{\varphi}$ is large, then φ has a constant sign on M_{thick}. However φ changes sign on M. Therefore there exists an open set D contained in a cusp or a cylinder such that $\varphi < 0$ on D, and $\varphi|\partial D \equiv 0$. Thus φ is an eigenfunction for Δ on D with Dirichlet boundary conditions and λ_1 is the first eigenvalue. By Green's formula and by Lemma 3.1

$$\lambda_1 \int_D \varphi^2 = \int_D |\nabla\varphi|^2 \geq \frac{1}{4} \int_D \varphi^2 ,$$

so that $\lambda_1 \geq 1/4$. This is a contradiction.

We have thus proved that, for a surface with finite area and small λ_1, the first eigenfunction is almost constant on every component of M_{thick} and has a substantial variation on at least one cylinder. The proof is completed now exactly as in the case of infinite area.

It remains to prove the lower bounds for higher eigenvalues. The argument is very similar to the proof of the analogous result in Section 1. We cut M into pieces by removing a number of geodesic loops and relate the Neumann eigenvalues of the pieces to the eigenvalues of M. Thus, suppose M is decomposed into $j + 1$ components $M_1, M_2 \ldots M_{j+1}$ by removing disjoint geodesics $\gamma_1, \gamma_2 \ldots \gamma_m$ of length $< 2\epsilon$. Recall that the union of all funnels (if present) is regarded as a single component.

Lemma 5.3. If M is noncompact assume that $\lambda_j < 1/4$. Let $\lambda_1(M_i)$ be the first positive eigenvalue of Δ on M_i with Neumann boundary conditions on ∂M_i, then

$$\lambda_j \geq \min_i \lambda_1(M_i).$$

This follows from the mini-max characterization of the eigenvalues which remains valid in the noncompact case below the bottom of the essential spectrum, cf. [Ch].

To relate $\lambda_1(M_i)$ to the geometry define $L_1(M_i)$ the same way as $L_1(M)$ was defined, allowing only geodesics contained in the interior of M_i.* Note that if K is the graph of M and $K_1, K_2 \ldots K_{j+1}$ are suitably numbered graphs obtained by removing the edges $\gamma_1, \gamma_2 \ldots \gamma_m$, then $L_1(K_i) = L_1(M_i)$.

Lemma 5.4. There exists a constant β, depending only on the topology of M, such that

$$\lambda_1(M_i) \geq \beta L_1(M_i).$$

Proof: Let φ be the eigenfunction on M_i belonging to $\lambda_1(M_i)$. Since φ satisfies the Neumann boundary conditions on ∂M_i and since ∂M_i consists of geodesics, φ extends by reflection to a function φ^* on the double M^* of M_i. Repeating the argument in the proof of the lower bound for λ_1, we see that if $\lambda_1(M_i)$ is small then φ^* must have a substantial variation on one of the cylinders of M^*. By symmetry, such a cylinder cannot correspond to a boundary geodesic of M_i and we can assume that it is contained in M_i. Hence φ has a substantial variation of one of the cylinders of M_i. The proof is completed exactly as in the case of the first eigenvalue of M.

We proved in Section 1 that $L_1(M_i) = L_1(K_i) \geq \alpha L_j(K)$. Since $L_j(K) = L_j(M)$ the desired inequality follows from Lemma 5.3 and Lemma 5.4.

Finally, we outline the argument proving that $\lambda_{n+1}(M)$ is bounded below by a constant depending only on the topology. Again this is very similar to the proof of the lower bound for λ_1. Thus suppose that $\lambda_{n+1} \leq \eta^2$. We will get a contradiction if η is very small. Note that Lemma 5.1 and Lemma 5.2 remain valid for linear combinations of eigenfunctions. The linear span of the eigenfunctions $\varphi_1, \varphi_2, \dots \varphi_{n+1}$ belonging to $\lambda_1, \lambda_2, \dots \lambda_{n+1}$ contains a function ψ whose mean value on every bounded component of M_{thick} is zero because the number of these components is n. We can assume that $\int_M \psi^2 = 1$. As in the proof of the lower bound for λ_1, we conclude that a) ψ and $\nabla\psi$ are small in a suitable sense on M_{thick}, b) the L^2 norm of ψ is substantial on at least one funnel, cylinder or cusp, and c) the energy of ψ is substantial on that set. However,

$$\int_M |\nabla\psi|^2 < \eta^2 .$$

For sufficiently small η this is a contradiction, which concludes the proof.

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REFERENCES

[A] S. Agmon, *Lectures on elliptic boundary value problems*, Van Nostrand, Princeton, NJ, 1965.

[B] L. Bers, An inequality for Riemann surfaces, in *Differential Geometry and Complex Analysis*, ed. I. Chavel and H. M. Farkas, Springer-Verlag, 1985.

[Be] A. F. Beardon, *The geometry of discrete groups*, Springer-Verlag, Graduate Texts in Mathematics, vol. 91.

[Bu] P. Buser, On Cheeger's inequality $\lambda_1 \geq h^2/4$, in *Geometry of the Laplace operator*, Amer. Math. Soc. (1980), 29-77.

[Ch] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic Press, 1984.

[Che] P. R. Chernoff, Essential self-adjointness of powers of elliptic operators, J. Functional Analysis 12 (1973), 401-414.

[D] H. Donnelly and P. Li, Pure point spectrum and negative curvature for noncompact manifolds, Duke Math. J., 46 (1979), 497-503.

[dR] G. de Rham, *Variétés différentiables*, Hermann, Paris, 1973.

[DR] J. Dodziuk and B. Randol, Lower bounds for λ_1 on a finite-volume hyperbolic manifold, J. Differential Geometry, 24 (1986), 133-139.

[PS] T. Pignataro and D. Sullivan, Ground state and lowest eigenvalue of the Laplacian for non-compact hyperbolic surfaces, Comm. Math. Physics, 104 (1986), 529-535.

[Ra] B. Randol, Cylinders in Riemann surfaces, Comment. Math. Helv. 54 (1979), 1-5.

[SWY] R. Schoen, S. Wolpert and S. T. Yau, Geometric bounds on low eigenvalues of a compact surface, in *Geometry of the Laplace operator*, Amer. Math. Soc. (1980), 279-285.

[Th] W. Thurston, *The geometry and topology of 3-manifolds*, Lecture Notes, Department of Mathematics, Princeton University, Princeton, NJ, 1980.

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