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A RELATION BETWEEN PROBABILITY, GEOMETRY AND DYNAMICS - THE RANDOM PATHS OF THE HEAT EQUATION ON A RIEMANNIAN MANIFOLD

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The purpose of this lecture is to examine several results concerning relations between Markov chain properties and harmonic functions defined on Riemannian manifolds. Our starting point is a classical result of Kakutani:

Let $D \subset \mathbb{R}^2$ be a domain with nice boundary; $h: D \to \mathbb{R}$ is a harmonic function $(\Delta h = 0)$ with a continuous extension to ∂D.

If $\varphi = h|_{\partial D}$ then

$$h(x) = \int \varphi(\xi)\, d\mu_x(\xi)$$

where $\mu_x(A)$ is the probability that a random path starting at x first hits ∂D in A. Let us discuss the notion of random path.

- Random Path on the Plane.

There is one and only one probability measure W_x (Wiener measure) on the space of continuous paths starting at x (i.e. continuous maps $w: [0,\infty) \to \mathbb{R}^2$ such that $w(0) = x$) so that the probability

Harmonic functions remain harmonic; Wiener measure W_x is altered by reparametrization of w. $[w_\rho(t) = w(t_w(\rho))$ where $t_w(\rho) = \int_w \rho]$. In particular the measure defined by the random path's 1st hitting on the boundary is unchanged.

3.2 - Calculation of Hitting Measures on Disk in Terms of non-Euclidean Geometry.

G := conformal transformation of $D = \{z; |z| < 1\}$

$$G: z \mapsto e^{i\theta} \frac{z-a}{1-\bar{a}z}$$

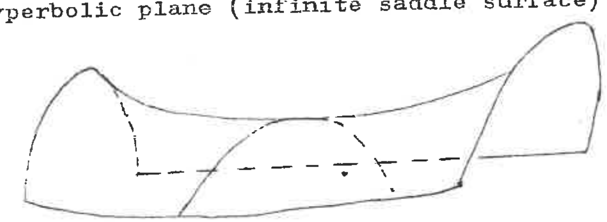
$$ds \text{ (non euclidean)} = \rho(r) \; ds \text{ (euclidean)}$$

Near ∂D, $\rho(r) \sim \frac{1}{\text{distance to } \partial D}$ and G acts as isometries of ds (non-euclidean).

Consider random motion on the hyperbolic or non-Euclidean plane. Then the probability of hitting the boundary in A is $\mu_x(A)$ where $\mu_x(A)$ = the non-Euclidean viewing angle from x of A.

This result follows from the measure's symmetry properties.

Hyperbolic plane (infinite saddle surface).



Corollary: In the non-Euclidean plane, a random path wanders away from it's starting point and heads towards infinity with a definite angle.

This can be seen in two ways:

1st. In Euclidean geometry, the angle is determined by the Euclidean random motion's first hit at the boundary.

2nd. The random motion on a hyperbolic plane behaves similarly to a random walk on a trivalent tree. The $\left(\frac{1}{3}, \frac{2}{3}\right)$ argument is replaced by a convexity argument.

4. Applications of Corollary:

4.1 - Application to Dynamics (statement and reference).

Let $S = D|_\Gamma$ be a hyperbolic surface. Then the geodesic flow on S is ergodic iff random motion on S is recurrent. (Poincaré: iff a certain series $\sum_{\gamma \in \Gamma} \exp - (x, \gamma x)$ diverges).

The idea is that a random path on S approximates a geodesic on S. (See Sullivan IHES publ. 1980).

4.2 - Application to Geometry (statement and reference).

Given a simply connected, complete Riemannian manifold with a negative curvature $(-b^2 \le k \le -a^2)$ it is possible to find many non-trivial bounded harmonic functions:

$$h(x) = \int \varphi(\xi) d\mu_x(\xi)$$

(see Sullivan, Journal of Diff. Geometry, Nov. 1983).

4.3 - Application to Geometric Analysis (statement and explaination).

Removable singularities of harmonic functions.

Riemann's theorem:

"If f is a bounded harmonic function on $D \setminus pt$ then f is harmonic across the point."

In other words, a point is a removable singularity of bounded harmonic functions.

This is also true for: a countable set

but not for an arc:

Also a standard cantor set is not a removable singularity. We well explain this geometrically.

PROPOSITION: X is a removable singularity iff X is invisible to random paths.

For example, here is the universal harmonic function with a non removable singularity $X \subset$ interior of disc:

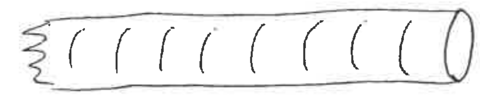
$$h_X(x) = \text{probability } (w(t) \text{ hits } X \text{ before leaving disc})$$

<u>Example</u>: All positive measure sets are visible.

The standard cantor set has measure zero but it is still visible. To see this, make a conformal change of metric in $D^2 - X$:

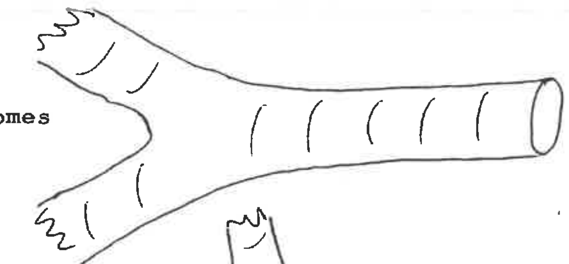
$$ds^2 \to \frac{1}{\text{dist } X} ds^2$$

$D^2 - 1(pt)$ becomes



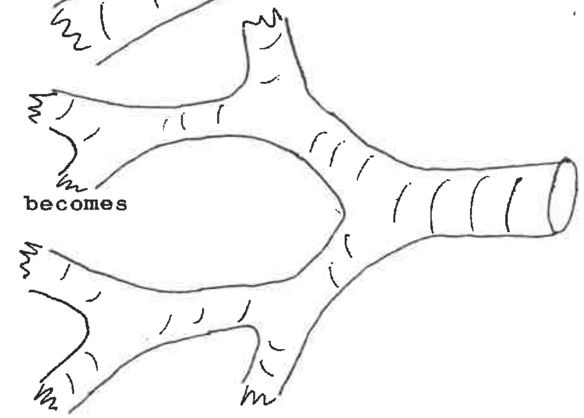
cylinder

$D^2 - 2(pts)$ becomes



tree cylinder

$D^2 -$ standard Cantor set becomes



infinite tree of cylinders

A random path has a positive probability to get lost out in the tree ($\frac{1}{3}$, $\frac{2}{3}$ argument). Thus random paths in D^2 see the standard Cantor set because this probability statement is unchanged by the reparametrization corresponding to the conformal change in metric. Q.E.D.

REFERENCE: Dynkin-Yushkevich - "Problems in Markov Processes".