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On the measurable dynamics of $z \mapsto e^z$

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Abstract. We study the measure theoretic properties of the complex exponential map $E(z) = e^z$. In particular, we show that the equivalence relation generated by E is recurrent and that E has no quasi-conformal deformations. This enables us to give some information concerning Devaney's semi-conjugacy between E and the shift map on sequences of integers.

0 *Introduction*

The study of iteration of entire maps from the complex numbers to themselves was begun by Fatou in 1926 [**8**] following the famous series of Memoires by Fatou and by Julia on the dynamics of rational maps of the 2-sphere [**7**], [**10**]. Fatou observed that the dynamical properties of entire maps could be analyzed by many of the same techniques as those of rational functions.

There is current interest in this subject; in 1980, Misiurewicz [**12**] solved a 60-year-old conjecture due to Fatou by proving that the set of unstable points (Julia set) of the exponential function $E(z) = e^z$ is the whole plane. More recently, Devaney and Krych [**6**] have introduced symbolic dynamics into the study of the family $E_\lambda(z) = \lambda e^z$ for a non-zero complex number λ. This has enabled them to give a detailed picture of the topological dynamics of these functions.

In this note we prove several Lebesgue measure theoretical properties of the transformation $E(z) = e^z$. We combine the measurable Riemann mapping theorem with Devaney's semi-conjugacy $S: \mathbb{C} \to \Sigma$ of E to the one-sided shift σ on the space Σ of sequences of integers. We prove (i) the fibres of Devaney's semi-conjugacy have Lebesgue measure zero; (ii) the equivalence relation generated by E is recurrent; (iii) the action of E on the tangent space to $\mathbb{C}$ is irreducible in the sense that on no set of positive measure is there an E-invariant measurable field of tangent lines.

Our third result contrasts sharply with known geometrical information about the fibres of $S: \mathbb{C} \to \Sigma$ and implies that they are geometrically rather fantastic. This point and certain further questions about E are discussed in the final paragraph.

1. Let $\mathbb{C}$ be the complex plane, $\mathbb{C}_c$, the plane punctured at c and $\hat{\mathbb{C}}$, the Riemann sphere. $E:\mathbb{C}\to\mathbb{C}$ will always denote the map $z\mapsto e^z$.

Following Devaney [6], we divide the plane into horizontal strips indexed by the integers:

$$R(n)=\begin{cases}\{z\in\mathbb{C}\mid 2\pi(n-1)<\operatorname{Im} z\le 2\pi n\} & n>0;\\ \text{real axis} & n=0;\\ \{z\in\mathbb{C}\mid \bar{z}\in R(-n)\} & n<0.\end{cases}$$

If U is any region which does not intersect the non-negative real axis, $\log_j: U\to\mathbb{C}$ is the branch of the logarithm whose image is contained in $R(j)$, and if $J=(j_0,j_1,\ldots,j_n)$ is a sequence of non-zero integers, we write $\log_J=\log_{j_0}\circ\cdots\circ\log_{j_n}$.

To each $z\in\mathbb{C}$ we associate its *itinerary* $(s_0, s_1,\ldots)$, the one-sided infinite sequence of integers such that $E^n(z)\in R(s_n)$.

Let $\Sigma=\{(s_0,s_1,\ldots)\mid s_i\in\mathbb{Z}\}$ and let $S:\mathbb{C}\to\Sigma$ be the map which associates to each $z\in\mathbb{C}$, its itinerary. If $\sigma:\Sigma\to\Sigma$ is the shift map, there is a commutative diagram:

$$\begin{array}{ccc}\mathbb{C} & \xrightarrow{E} & \mathbb{C}\\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S}\\ \Sigma & \xrightarrow{\sigma} & \Sigma\end{array}$$

and our goal is to prove:

THEOREM 8. *All the fibres of S have Lebesgue measure zero.*

2. The first step toward proving theorem 8 is to formulate a topological property which characterizes a certain finite dimensional family of holomorphic maps that includes E.

Property C. $f:\mathbb{C}\to\mathbb{C}$ *has property* C if it is not affine, and $f:\mathbb{C}\to f(\mathbb{C})$ a covering space projection. (See [13].)

PROPOSITION 1. *An entire holomorphic map has property* C *if and only if it is of the form*

$$z\mapsto e^{az+b}+c,\qquad a\in\mathbb{C}_0,\ b,\ c\in\mathbb{C}.$$

COROLLARY 2. *There is at most a* 3*-dimensional family of holomorphic maps topologically conjugate to* $z\mapsto e^z$.

Proof. Property C is an invariant of topological conjugacy. □

COROLLARY 3 (Rigidity). *An entire map which has property* C *is affine conjugate to a unique map of the form* $z\mapsto\lambda e^z$, $\lambda\in\mathbb{C}_0$.

Proof. Conjugate by $z\mapsto(z/a)+c$, then $\lambda=ae^{b+ac}$. □

Proof of proposition 1. Let f be an entire map which has property C. It follows from the Picard theorem that image $(f)=\mathbb{C}$ or $\mathbb{C}_c$. Since any holomorphic covering map from $\mathbb{C}$ to $\mathbb{C}$ must be affine [1], the former case does not occur.

In the latter case, $f:\mathbb{C}\to\mathbb{C}_c$ is a holomorphic universal covering map, since $\Pi_1(\mathbb{C})$ is trivial. This map is unique up to affine change of coordinates in the domain, therefore f is of the form

$$z \mapsto e^{az+b} + c, \qquad a \neq 0,\ b \in \mathbb{C}.$$

3. In this section, we begin to discuss some of the measure theoretic properties of $z \mapsto e^z$. ('Measure' always refers to 'Lebesgue measure' unless otherwise stated.)

Let $g : \mathbb{C} \to \mathbb{C}$ be any measurable map. Two points z_1 and z_2 are *related under* g if there exists $n, m \geq 0$ such that $g^n(z_1) = g^m(z_2)$. This is an equivalence relation on $\mathbb{C}$ called the *g-relation*, and an equivalence class of points is called a *large orbit of* g. A measurable set $X \subset \mathbb{C}$ is a *cross section* to the *g-relation* if X contains at most one point in each large orbit. We say the g-relation is *recurrent* (or *conservative*) if there are no positive measure cross sections to g. The g-relation is called *ergodic* if any measurable set made up of large orbits has zero or full measure.

LEMMA 4. *If the g-relation is ergodic, the g-relation is conservative.*

Proof. Let X be a positive measure cross section to g and let $X = A \cup B$ where $\mu(A) > 0$, $\mu(B) > 0$ and $A \cap B = \varnothing$. Then $\bigcup_{z \in A} \{\text{orbit}\,(z)\}$ and $\bigcup_{z \in B} \{\text{orbit}\,(z)\}$ are disjoint, positive measure, completely g-invariant sets. Therefore, g is not ergodic. □

Remark. There is another notion of 'recurrent' or 'conservative' for an endomorphism which is: g is *recurrent* if $g^{-1}(A) \subset A$ implies measure $(A - g^{-1}A) = 0$. (See **[9]**.)

Conservativity for an endomorphism g implies conservativity for the g-relation, but it is *not* implied by ergodicity of the g-relation. To summarize:

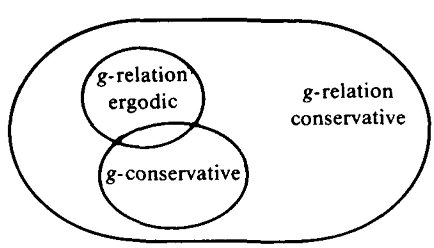

FIGURE 1.

Our proof that $z \mapsto e^z$ is recurrent uses the theory of quasiconformal mappings and the measurable Riemann mapping theorem of Ahlfors and Bers **[4]** which we briefly describe below. Refer to **[2]**, **[11]** for basic definitions.

A *conformal structure* μ on $\mathbb{C}$ is a conformal equivalence class of Riemannian metrics. Equivalently, we can define μ as a measurable tangent field of ellipses. The conformal distortion $|\mu|$ of μ is the eccentricity of the ellipses, and the standard conformal structure μ_0 is the tangent field of circles.

Given any conformal structure μ on $\mathbb{C}$ whose conformal distortion is bounded a.e., the measurable Riemann mapping theorem asserts that there is a unique quasiconformal (q.c.) homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ fixing 0 and 1 such that the pullback $\phi^* \mu = \mu_0$. We will need a slightly more general version of this theorem.

MEASURABLE RIEMANN MAPPING THEOREM WITH PARAMETERS. *Given any finite dimensional real analytic family* $\{\mu_\alpha\}$ *of conformal structures on* $\mathbb{C}$ *whose conformal distortions are bounded a.e. there is a unique real analytic family* $\{\phi_\alpha\}$ *of q.c. homeomorphisms fixing* 0 *and* 1 *such that* $\phi_\alpha^* \mu_\alpha = \mu_0$. $\{\phi_\alpha\}$ *is real analytic in the sense that for each* $z \in \mathbb{C}$, $\phi_\alpha(z)$ *varies real analytically; and the map* $\alpha \mapsto \phi_\alpha$ *is a continuous map of* $\{\alpha\}$ *into the space of homeomorphisms of* $\mathbb{C}$ *with the compact open topology.*

LEMMA 5. *Suppose* $f:\mathbb{C}\to\mathbb{C}$ *is analytic,* μ *is an a.e. f-invariant conformal structure on* $\mathbb{C}$ *and* $\phi:\mathbb{C}\to\mathbb{C}$ *is a q.c. homeomorphism such that* $\phi^*\mu = \mu_0$. *Then* $g = \phi \circ f \circ \phi^{-1}$ *is analytic.*

Proof. An easy verification shows that g preserves μ_0 a.e. and so is a.e. conformal. Therefore g, being quasiconformal, is conformal everywhere [**2**] and conformal is equivalent to complex analytic in dimension one. □

LEMMA 6. *The group of homeomorphisms which commute with E contains no non-trivial arcs in the compact open topology.*

Proof. Let $\phi:[0, 1]\to H$, where H consists of the homeomorphisms of $\mathbb{C}$, be an arc such that $\phi(t)\circ E = E\circ\phi(t)$ for all t. Let $P_n = \{z\in\mathbb{C} \mid z$ is a periodic point of E of period exactly $n\}$. P_n is a discrete subset of $\mathbb{C}$ which is invariant under each $\phi(t)$. Thus continuity implies $\phi(t)|_{P_n} = \phi(0)|_{P_n}$ for all t and all n.

The lemma follows since [**12**] implies $\bigcup_n P_n$ is dense in $\mathbb{C}$. □

THEOREM 7. *The E relation of* $E(z) = e^z$ *is recurrent.*

Proof. We use as a basis for our proof, an argument due to Sullivan [**14**]. The idea is to assume that E is not recurrent, and use the measurable Riemann mapping theorem to construct a family of analytic maps of dimension greater than 3 which are q.c. conjugate to E. This contradicts corollary 2.

Let X be a positive measure cross-section to the orbits of E and let $g: X\to\mathbb{C}$, $g(z) = (r(z), \theta(z))$ be any L^∞ map. Extend g to $\tilde{X} = \bigcup_{z\in X}\{\text{orbit}\,(z)\}$ by

$$g(E^n(z)) = (e^{r(z)}, \theta(z) + \arg dE^n(z)),$$
$$g(\log_J(z) = (e^{r(z)}, \theta(z) + \arg d\log_J(z)).$$

g extends to an L^∞ map on all of $\mathbb{C}$ if we define $g(z) = 0$ for $z\in\mathbb{C}-\tilde{X}$.

g determines a unique measurable conformal structure μ_g on $\mathbb{C}$; $\mu_g(z)$ is the ellipse with major axis $\theta(z)$ and eccentricity $e^{r(z)}$, $|\mu_g(z)|$ is bounded a.e. and μ_g is g-invariant by construction.

According to the measurable Riemann mapping theorem, there is a unique normalized q.c. homeomorphism ϕ_g such that $\phi_g^*(\mu_g) = \mu_0$. Then $\phi_g\circ E\circ\phi_g^{-1}$ is an analytic map, by lemma 5, which is q.c. conjugate to E.

Let $L\subset L^\infty(X)$ be a real analytic family of L^∞ maps of (complex) dimension at least 4 and define $\Phi: L\to Q(E) = \{$holomorphic maps q.c. conjugate to $E\}$ by $\phi_g\circ E\circ\phi_g^{-1}$. Since the dimension of $Q(E)$ is at most 3, the measurable Riemann mapping theorem with parameters implies that some fibre $\Phi^{-1}(F)$ contains a non-trivial arc $\rho:[0, 1]\to\Phi^{-1}(F)\subset L$.

$\phi_{\rho(t)} \circ E \circ \phi_{\rho(t)}^{-1} = F$ for all t, in other words, $\psi_t \circ E \circ \psi_t^{-1} = E$ for all t where $\psi_t = \phi_{\rho(0)}^{-1} \circ \phi_{\rho(t)}$. Therefore, ψ_t is an arc of homeomorphisms commuting with E and so must be trivial by lemma 6. Hence, ρ must also be a trivial arc, contradicting our assumption. □

4. Theorem 8 will now follow fairly quickly from theorem 7. We begin with a summary of results from [**6**] concerning the Devaney semi-conjugacy $S: \mathbb{C} \to \Sigma$.

THEOREM A [**6**]. *A sequence $\underline{s} = (s_0, s_1, \ldots) \in \Sigma$ is in the image of S if and only if it satisfies the following conditions*:

(a) *For some $x \in \mathbb{R}$, $2\pi|s_n| < E^n(x)$ for all n.*

(b) *If some $s_i = 0$, $s_{i+j} = 0$ for all $j \geq 1$.*

Notice that the image of s contains all possible periodic sequences of non-zero integers. In fact, there is a complete description of the periodic points of E.

THEOREM B. (1) [**6**]. *Given any periodic sequence $\underline{s} = (\overline{s_0, s_1, \ldots, s_{n-1}}, \ldots)$ of non-zero integers, there is a unique periodic cycle $p(\underline{s}) = (p_0, p_1, \ldots, p_{n-1})$ such that the itinerary of p_0 is $\underline{s}$.*

(2) *All the periodic points of $z \to e^z$ are expanding.*

(To see (2) note that [**12**, lemma 1] states that $|(E^n)'(z)| \geq |\mathrm{Im}\,(E^n(z))|$ for all $z \in \mathbb{C} \setminus \mathbb{R}$ and all $n \geq 0$, but according to [**12**, lemma 2], given any $z \in \mathbb{C} \setminus \mathbb{R}$, there is some $N > 0$ such that $|\mathrm{Im}\, E^N(z)| \geq \pi/3$. Therefore, if $\gamma = \{z_0, z_1, \ldots, z_{n-1}\}$ is a periodic cycle, $|(E^n)'(\gamma)| = |(E^n)'(z_i)|$ for any $i \geq \sup |\mathrm{Im}\,(E^n)'(z_i)| \geq \pi/3$.)

Furthermore, Misiurewicz's result that the Julia set of $z \to e^z$ is $\mathbb{C}$ implies that the periodic points are dense in the plane.

We briefly recall the proof of theorem B(1). Let $J = (s_0, s_1, \ldots, s_{n-1})$, Devaney shows that $\log_j : R(s_0) \to R(s_0)$ is a map that takes a compact subset K of $R(s_0)$ into itself. Since $R(s_0)$ is conformally equivalent to the disk, the Schwartz-Pick lemma [**3**] implies that there is a unique attracting fixed point p_0 for $\log_j$. Evidently p_0 is a period n periodic point for E with itinerary $S(p_0) = (\overline{s_0, s_1, \ldots, s_{n-1}}, \ldots)$.

Note that the proof of B(1) actually shows the following:

THEOREM C. *For any periodic sequence $\underline{s} = (\overline{s_0, s_1, \ldots, s_{n-1}}, \ldots)$,*

$$\lim_{N \to \infty} \log_J^N (z) = p_0(\underline{s}) \qquad \textit{for all } z \in S^{-1}(\underline{s}), J = (s_0, \ldots, s_{n-1}).$$

We can now prove theorem 8.

THEOREM 8. *Every fibre of $S: \mathbb{C} \to \Sigma$ has Lebesgue measure zero.*

Proof. Let $\underline{s} \in \mathrm{image}\,(S)$. If $\underline{s}$ is not eventually repeating, $S^{-1}(\underline{s})$ is a cross section to the large orbits of E and thus has measure zero by theorem 7.

Suppose that $\underline{s}$ is eventually periodic, then $\sigma^n(\underline{s})$ is repeating for some $n \geq 0$ and $S^{-1}(\underline{s}) = \log_J (S^{-1}(\sigma^n(\underline{s})))$ for some J. It follows that $S^{-1}(\underline{s})$ has measure zero if and only if $S^{-1}(\sigma^n(\underline{s}))$ does, and we are reduced to the case of periodic sequences.

Assume $\underline{s} = (\overline{s_0, s_1, \ldots, s_{n-1}}, \ldots)$ is periodic of period n, let $p(s) = (p_0, \ldots, p_{n-1})$ be the periodic cycle described in theorem B(1). By theorem B(2), $p(s)$ is repelling, therefore we can find a neighbourhood U of p_0 such that $E^n|_U$ is analytically conjugate to the map $z \mapsto ((E^n)')(p_0) \cdot z$.

Let $\nu : S^1 \to \log_J(R(s_0)) \cap U$, $J = (s_0, \ldots, s_{n-1})$. Then $E^n \circ \nu(S^1)$ and $\nu(S^1)$ bound a doubly connected region $A \subset R(s_0)$ such that p_0 lies in the bounded component of the complement of A.

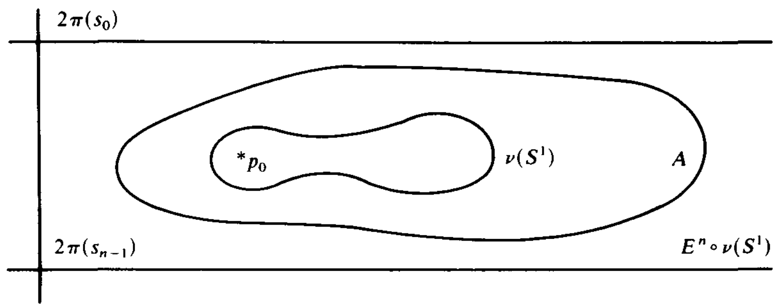

FIGURE 2.

Let $X = (A \cup \nu(S^1)) \cap S^{-1}(\underline{s})$. Evidently, $X \cap E(X) = \varnothing$, therefore X is a cross section to the large orbits of $z \mapsto e^z$. Furthermore, theorem C implies

$$S^{-1}(\underline{s}) = p_0 \cup \bigcup_{n \geq 0} E^n(X) \cup \bigcup_{n \geq 0} (\log_J)^n(X),$$

from which we conclude that $S^{-1}(\underline{s})$ has Lebesgue measure zero. □

5. We conclude with some observations concerning the space of line elements $T_1\mathbb{C}$ of $\mathbb{C}$, which we identify with $\mathbb{C} \times S^1$. E induces a map E_* from $T_1\mathbb{C}$ to itself defined by

$$E_*(z, \theta) = (e^z, \theta + \arg dE(z))$$

whose properties are related to the dynamics of E.

Suppose $X \subset \mathbb{C}$ has positive measure and $\tau : X \to T_1\mathbb{C}$ is an E-invariant section of the projection $\pi_1 : T_1\mathbb{C}|_X \to X$. We can use τ to construct a non-trivial one dimensional family of E-invariant conformal structures.

Define L^∞ maps $g_\lambda : \mathbb{C} \to \mathbb{C}$, $\lambda \in \mathbb{C}_0$ by

$$g_\lambda(z) = \begin{cases} (e^{|\lambda|}, \arg \lambda \cdot (\pi_2 \circ \tau(z))) & z \in X, \\ 0 & z \in \mathbb{C} - X. \end{cases}$$

As in the proof of theorem 7, g_λ is equivalent to a one parameter family of E-invariant L^∞ conformal structures μ_λ, and the measurable Riemann mapping theorem gives a non-trivial q.c. deformation $\phi_\lambda \circ E \circ \phi_\lambda^{-1}$ of E, where $\phi_\lambda^*(\mu_\lambda) = \mu_0$.

Conversely, if $\phi \circ E \circ \phi^{-1}$ is entire and ϕ is q.c. but not conformal, the direction of the major axes of the ellipses of $\phi^*(\mu_0)$ defines an E-invariant section of $T_1\mathbb{C}|_X$ where $X = \text{support } (\phi^*(\mu_0))$.

THEOREM 9. *There are no non-trivial q.c. deformations of $z \mapsto e^z$.*

THEOREM 9′. *There are no E-invariant sections* $\tau: X \to T_1\mathbb{C}|_X$ on any positive measure subset $X \subset \mathbb{C}$.

Proof. We use a result due to Devaney [5] that there is a sequence of distinct complex numbers $\lambda_i \to 1$ such that the map $z \mapsto \lambda_i e^z$ is not topologically conjugate to $z \mapsto e^z$.

Our rigidity statement (corollary 3) implies that there is a (complex) one dimensional family of entire maps q.c. conjugate to $z \mapsto e^z$, but not conformally conjugate to $z \mapsto e^z$, if and only if there is a neighbourhood U of 1 in $\mathbb{C}$ such that $z \mapsto \lambda e^z$ is q.c. conjugate to $z \mapsto e^z$ for all $\lambda \in U$. The existence of such a U contradicts Devaney's result. □

At this point, we would like to remark that theorem 9′ presents an interesting contrast to Devaney's description of $S: \mathbb{C} \to \Sigma$. On one hand, the semi-conjugacy gives a decomposition of the plane into measure zero fibres, all but countably many of which have an infinite tail which is a Lipschitz curve. Lipschitz curves have tangents a.e., and these tangents define an E-invariant line field on the union of the tails. Therefore theorem 9′ asserts that this union must have measure zero, and so the full measure of the plane is concentrated on the 'initial segments' of these fibres. Consequently, most of these initial segments cannot be Lipschitz curves.

Further study of the structure of the fibres of $S: \mathbb{C} \to \Sigma$ is required in order to understand better the measurable dynamics of $z \mapsto e^z$. Along with this project, we pose some related open questions:

(1) Is there an E-invariant measure in the same measure class as Lebesgue measure?

(2) Study the projection by S of Lebesgue measure on to the space of sequences Σ. Is the shift map ergodic for this measure?

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