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FUNCTION THEORY, RANDOM PATHS AND COVERING SPACES

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0. Introduction

For connected Riemannian manifolds M we discuss the interplay between the harmonic function theory on M, the statistical properties of random paths on M and the global geometrical structure of M.

In particular, we study the case when M is a regular or Galois cover of a smaller Riemannian manifold N. That is, there is a discrete group Γ of isometries acting on M so that $N = M/\Gamma$. M will be called an Abelian (resp. nilpotent, solvable, etc.) cover of N when Γ is an Abelian (nilpotent, solvable, etc.) discrete group of isometries.

We first illustrate the general results by an example. Let M be any Abelian cover of any compact Riemann surface N (the metric chosen for N is of no significance). Let the genus of N exceed 1 and the rank of the Abelian group exceed 2. Then by Theorems 1 and 4 below one sees that:

(i) M does not possess any nonconstant positive harmonic functions, *but*

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(ii) M has a Green function g (that is, one may find a positive function whose Laplacian is the dirac mass at point, g is the minimal function with these properties).

Next observe that $e^{-2\pi g}$ is the absolute value of a multivalued holomorphic function φ on M which becomes well defined in some Abelian cover ($\varphi = \exp - 2\pi(g + ig^*)$, $g^* =$ harmonic conjugate of g). Thus a *two-step solvable cover of a compact Riemann surface can admit nonconstant bounded harmonic functions* (even holomorphic ones).

We do not understand solvable covers in general but nilpotent ones and nonamenable ones (see below) can be dealt with.

These function theoretic properties of M have well known statistical interpretations. In particular Brownian motion on M is transient if and only if M admits a Green function (or equivalently a nonconstant bounded subharmonic function).

We now summarize the theorems; the proofs are to be found in the respective sections.

Theorem 1. *Any nilpotent covering of a compact Riemannian manifold has no nonconstant positive harmonic functions.*

The hypotheses of Theorem 1 can be relaxed if one is to conclude only that bounded harmonic functions are constant. Compactness can be generalized to recurrence of random motion (see below) and finitely generated nilpotent by ω-nilpotent ($\Gamma = \bigcup_{i=1}^{\infty} Z_i$, Z_i normal in Γ, where Z_{n+1} maps to the center of Γ/Z_n).

Theorem 2. *Any ω-nilpotent cover M of a recurrent Riemannian manifold N is Liouville.* (*That is to say if N has no Green function then any bounded harmonic function on M is constant.*)

However, compactness is required in Theorem 1 because the two-sphere with 4 points removed admits a rank 2 abelian cover with a nonconstant positive harmonic function defined on it [12].

Now we present an existence theorem for nonconstant bounded harmonic functions. A countable group Γ is called amenable (*moyennable* in French) if there is on Γ a finitely additive, translation invariant nonnegative probability measure (defined for all subsets of Γ). Amenable groups include solvable groups. This property passes to subgroups, but the free group F on two generators is nonamenable. (Here is why: Divide F into 4 disjoint sets according to starting letter of reduced word. Then observe each set is congruent to itself union two others (mod finite sets). So F can have no such measure. See Greenleaf's book *Invariant measures.*)

Theorem 3. *Any nonamenable cover M of any Riemannian manifold N possesses nonconstant bounded harmonic functions.*

Corollary. *If M admits a free nonabelian group of isometries acting discontinuously then M has nonconstant bounded harmonic functions.* (*Example*: *the universal cover of a compact negatively curved manifold.*)

The last theorem about function theory on covers concerns the Green function on M or equivalently whether random motion on M is recurrent or transient.

Theorem 4. *An Abelian cover M of a compact Riemannian manifold M has a Green function* (*equivalently random motion on M is transient*) *iff the rank of the Abelian group is at least* 3.

Guivar'ch gave a Fourier transform proof of Theorem 4.[1]

Theorem 4 is proved here using a criterion for transience of random motion on M due respectively in special cases to Kelvin, Nevanlinna, and Royden. Let M be a complete Riemannian manifold.

Theorem 4′ (*Kelvin, Nevanlinna, Royden*). *Random motion on M is transient iff there is a vector field V on M satisfying*:

(i) $\int_M |\operatorname{div} V|\, dm < \infty$,

(ii) $\int_M |V|^2\, dm < \infty$, *and*

(iii) $\int_M \operatorname{div} V\, dm \neq 0$.

In other words there is a flow on M with a net divergence and finite energy iff the random motion on M is transient.

Corollary. *Transience or recurrence* (*equivalently the existence of a Green function*) *only depends on the quasi-isometry class of the metric.*

Problem (unsolved even for Riemann surfaces). *Is the Liouville property a quasi-isometry invariant of Riemannian manifolds*?

The proof of Theorem 2 depends on a *discretization* of random motion on M. The idea is due to Furstenberg in the case of discrete subgroups of $SL(2, R)$ [7].

A discrete set $X \subset M$ is called $*$-recurrent if there are neighborhoods of the points of X with a uniform Harnack constant (see §7 for a precise definition) and so that a random path starting from any point of M hits the union of these neighborhoods with probability one. Bounded harmonic functions on M are determined in a precise way by their values on any $*$-recurrent set.

[1]For this and also an alternative proof of Theorem 1 see C. R. Acad Sci. Paris **892** (1981) 851–853.

Theorem 5. *If $X \subset M$ is a discrete set which is $*$-recurrent then there is an assignment $y \to \nu_y$ of a positive probability measure ν_y on X for each point y in M satisfying*:

(i) $\nu_y(x) > 0$ *for each* $x \in M$,

(ii) $h(y) = \sum_{x \in X} \nu_y(x) h(x)$ *for each bounded harmonic function h on M.*

Moreover, $y \to \nu_y$ may be chosen to be compatible with any symmetries of the pair (M, X).

Note that Theorem 5 says that the restriction of a bounded harmonic function on M to X is a bounded harmonic function for the discrete time Markov process on X defined by the transition probabilities: the probability a particle at $x \in X$ is next at $x' \in X$ is $\nu_x(x')$. This is so because a special case of (ii) is $h(x) = \sum_{x' \in X} \nu_x(x') h(x')$, the definition of a harmonic function for the discrete process $(X, \nu_x(x'))$.

With a stronger geometric assumption on X one can represent all positive harmonic functions, characterize recurrence or transience, and couple the dynamics of continuous random motion on M and with those of the random walk on X.

Let $P_t(x, y)$ be the unique smallest solution to the heat equation with pole at $(x, 0)$. Now for each point y in M there is a probability measure $\mathbf{P}^y$ on W, the set of all continuous paths w: $[0, \xi) \to M$ so that if $A_i \subset M$ then

$$\mathbf{P}^y(w(t_1) \in A_1, \cdots, w(t_n) \in A_n, t_1 < t_2 < \cdots < t_n)$$
$$= \int_{A_1} \int_{A_2} \cdots \int_{A_n} P_{t_1}(y, x_1) P_{t_2 - t_1}(x_1, x_2) \cdots P_{t_n - t_{n-1}}(x_{n-1}, x_n)\, dx_1 \cdots dx_n,$$

and similarly if for each y in M there is a probability measure $\mathbf{Q}^y$ defined on $M \times X^{\mathbb{N}}$ so that

$$\mathbf{Q}^y\{(Y_i) \colon Y_0 = y, Y_1 = x_1, \cdots, Y_n = x_n\} = \nu_y(x_1) \nu_{x_1}(x_2) \cdots \nu_{x_{n-1}}(x_n).$$

(In other words $\mathbf{P}^y$ makes w perform Brownian motion from y, $\mathbf{Q}^y$ makes Y into a Markov chain with transition probabilities ν_y.)

The sought for correspondence between the $\mathbf{P}^y$ and $\mathbf{Q}^y$ is realized by constructing a map from paths in W to paths in $M \times X^{\mathbb{N}}$. This map is random (in the sense that one tosses several coins and depending on their outcomes one maps the path $w \in W$ to different points of $M \times X^{\mathbb{N}}$), so more precisely it should be thought of as a map of $W \times \Omega$ into $M \times X^{\mathbb{N}}$ with probability measure $\mathbf{P}^x \times \mu$; (Ω, μ) is some suitably large probability space in which the outcomes of all the coin tosses are recorded.

Theorem 6. *Let $X \subset M$ be a discrete co-component set. For each $x \in X$ let $U_x \subset V_x$ be the associated neighbourhoods satisfying the uniform Harnack estimates. Then there is*

(a) *an assignment of a probability measure ν_y on X for each y in M,*

(b) *an increasing sequence of stopping times T_n for the Brownian paths w from y with the property that $w(T_n)$ is on the boundary of V_{x_n} for some n where x_n is known as the centre f of w at T_n, and*

(c) *an associated map of paths π: $w \to (y, x_1, x_2, \ldots)$ where $x_1, x_2 \ldots$ are defined in* (b). *These satisfy*:

(i) *For each positive harmonic function h on M, $h(y) = \nu_y(h)$.*

(ii) *π maps $\mathbf{P}^y$ onto $\mathbf{Q}^y$.*

(iii) *There is a distance d on M for which the probability that*

$$\max_{T_n < t < T_{n+1}} d(x_n(\omega), w(t)) > k$$

is at most e^{-kc} for some c.

(iv) *If the cover of M by the $(V_x)_{x \in X}$ is locally finite, then the random walk on $M \times X^{\mathbf{N}}$ is recurrent if and only if Brownian motion on M is.*

Remark. Our notion of distance in (iii) only comes close to being the usual notion of distance if the (V_x) are a locally finite cover.

1. Background (potential and ergodic theory)

In the thirties Myrberg [14] studied these questions for Riemann surfaces M such as the sphere S^2 (or any compact surface) minus a compact set X: $M = S^2 - X$. He found that for such surfaces M *the existence of a Green function was equivalent to the existence of a nonconstant bounded harmonic function on M*. Myrberg proved that either property was equivalent to a thinness property of X, namely whether or not the logarithmic capacity of X was zero or not. Thus for X the ordinary middle third Cantor set or any set of Hausdorf dimension > 0, then there is a Green function and uncountably many coconstant bounded harmonic functions on $M = S^2 - X$ while for X a countable closed set there are no such functions on $M = S^2 - X$.

Up to the early fifties it was a problem in Riemann surface theory to find a surface with a Green function but no nonconstant bounded harmonic functions.

This problem was settled by Ahlfors and Toki [2], [18] in the early fifties. Their example was ingenious and quite different from the one described above in the first part of the introduction.

The example above arose from another point of view—the ergodic theory of geodesic flows on negatively curved manifolds. Eberlein [6] asked whether an infinite volume manifold existed with ergodic foliations of the asymptotic geodesics while the flow of geodesics was not ergodic.

In studying these questions one of the authors found the first question (ergodicity of the asymptotic fibration) was equivalent in the constant negative curvature case to the existence or not of bounded harmonic functions on the associated Riemannian manifold. The second question (ergodicity of the geodesic flow) was found to be equivalent to the recurrence of random motion on the Riemannian manifold and thus equivalent to the nonexistence or existence of a Green function (Sullivan [17]). So the Myrberg question is equivalent to the Eberlein question, and either example answers either question.

As another example of connections between ergodic and function theory we remark that the existence of a bounded holomorphic function (in dimension two) implies no ergodic component of the asymptotic foliation or the corresponding Fuchsian group has positive measure.

2. Problems: Solvable groups and exponential growth groups

These function theory questions for solvable groups are not settled. We record here some information currently available and some outstanding problems.

In the Introduction we have constructed a (2-step) finitely generated solvable cover of a compact manifold with a nonconstant bounded harmonic function. In this example the commutator subgroup was not finitely generated.

Problem. Is every finite type solvable cover of a compact manifold Liouville? Namely if $M = N/\Gamma$, where every subgroup of Γ is a finitely generated solvable group, is every bounded harmonic function on M constant?

Evidence. Let Γ be the semidirect product $(Z + Z) \underline{\times} Z$ with action matrix $\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$. Then Γ is a cocompact subgroup of the 3-dimensional unimodular solvable Lie group $\mathscr{S} = R^2 \underline{\times} R$ with action

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Then $\mathscr{S}/\Gamma$ is a compact 3-manifold N. The Abelian cover N_1 of N is a Z cover so is recurrent by Theorem 4. Then $\mathscr{S}$ is a $Z + Z$ cover of N_1. So by Theorem 2, $\mathscr{S}$ has no nonconstant bounded harmonic functions. This is a nontrivial example where the problem has an affirmative answer.

Concerning positive harmonic functions we ask the following

Problem. Let M be a regular cover of compact manifold with a group of exponential growth. (If $w(n) = \text{card}\{\text{part of } \Gamma \text{ represented by words of length} \leqslant n \text{ in some fixed set of generators}\}$, then $\underline{\lim}_n 1/n \log w(n) > 0$.) Then does M admit nonconstant positive harmonic functions? Conversely, if M has nonconstant positive harmonic functions does Γ have exponential growth?

Evidence. For the first part of the conjecture we look at the first case of an amenable Γ of exponential growth that comes to mind. Nonamenable groups are handled by Theorem 3. Let $\Gamma \subset \mathscr{S}$ be the pair of solvable groups put in evidence above. We claim $\mathscr{S}$ admits nonconstant positive harmonic functions. In fact the Martin boundary contains two circles.

Problem. What is the rest of the Martin boundary of $\mathscr{S}$?

Sketch of proof. $\mathscr{S}$ has two transversal foliations by hyperbolic planes intersecting in the cosets of $\mathscr{S}/[\mathscr{S}, \mathscr{S}]$. Two leaves are at constant distance along horocycles in one concentric family with this distance *expanding* exponentially as we approach the center of the family.

Take a function h on one plane H_0 and extend it to $\mathscr{S}$ by defining h to be $h(x_0)$ on the horocycle of the transversal leaf piercing H_0 at x_0. If $\text{grad}_{\mathscr{S}} h$ is to be volume preserving (i.e. $\Delta h = \text{div}\,\text{grad}\, h = 0$) then $(\text{grad}_H h) \cdot v$ must be equal to $-\text{div}_H \text{grad}_H h$, where v is the unit vector field pointing in normal to canonical horocycle family in H.

But we can find many such functions on H. Let y denote the function so that $\text{grad} \log y = v$. Then $y^{1/2}$ satisfies $\Delta_H y^{1/2} = -\frac{1}{4} y^{1/2}$. Let φ be any other positive eigenfunction of Δ_H of eigenvalue $-\frac{1}{4}$ (any probability measure on $\partial H = S'$ gives rise to one integrating it against the square root of the usual Poisson-kernel). Then the ratio $h = \varphi / y^{1/2}$ satisfies $\Delta_H h = -(\text{grad}_H h) \cdot \text{grad}_H \log y^{1/2}$. Using the other foliation gives another S^1 in the Martin boundary.

More evidence. If Γ is finitely presented and has less than exponential growth, as far as anybody knows it may well have polynomial growth (problem). But in this case Gromov [9] has shown Γ contains a nilpotent group of finite index. By our Theorem 1 there are then no nonconstant positive harmonic functions on a Γ cover of a compact manifold.

Thus a counterexample to the second part involves a finitely presented group of more than polynomial growth but less than exponential growth.

3. Nilpotent covers of compact manifolds (Theorem 1)

Proof of Theorem 1. *Reduction to the Abelian case.* (i) The translation of M by an element t in the center only moves points a bounded amount. By

Harnack if h is a positive harmonic function $h(tx) \leqslant c \cdot h(x)$ for some constant c. (The Harnack inequality is uniform because the geometry of M is quasihomogeneous since N is compact.)

If h were a minimal harmonic function we would have $h(tx) = c(t) \cdot h(x)$. Moreover, $t \to c(t)$ is a multiplicative character on the center of Γ.

Proposition. *The homomorphism $Z \to R^*$ defined by the character $t \to c(t)$ (where $Z =$ center of Γ) extends to all of Γ. (We need only to assume Γ amenable.)*

Proof. The 1-form $\omega = dh/h$ is bounded together with its first 2 derivatives say, again by Harnack. (The value of positive harmonic function controls the values and those of a finite number of derivatives on a neighborhood. Since $(|f''|, |f'|) \leqslant k|f|$, and $(f'/f)' = f''/f - (f')^2/f^2$ so $|(f'/f)'| \leqslant 2k^2$.)

For each point x in M replace $\omega(x)$ by $\bar{\omega}(x)$ by forming a mean over the amenable group Γ of the set of bounded-correctors $\{\omega(\gamma x)\}\gamma$ in Γ.

The form $\bar{\omega}(x)$ has Lipschitz first derivatives and these derivatives are the means of the derivatives of ω. This is true because ω bounded in C^2 means the difference quotients converge uniformly to the first derivatives of ω, and the mean is linear and continuous in the superior norm topology. In particular $d\bar{\omega} = \text{mean}(d\omega) = \text{mean}(0) = 0$.

Let l be an arc between x_0 and tx_0 in M.

$$\int_l \omega = \int_l \frac{dh}{h} = \log h(tx_0) - \log h(x_0) = \log c(t).$$

For $\gamma \in \Gamma$, γl is an arc between γx_0 and $\gamma(tx_0)$ or γx_0 and $t(\gamma x_0)$ because t is in the center. So we also have $\int_{\gamma l} \omega = \log c(t)$.

We claim that $\int_l \bar{\omega} = \log c(t)$. This is so because all the arcs γ^l are related by isometries of M—they are congruent. On one such the Riemann integral $\int_{\gamma^l} \omega$ is approximated by a finite sum with an error estimated by the Lipschitz constant of ω which is uniform. Again by linearity and continuity of the mean we get $\int_l \bar{\omega} = \log c(t)$.

But $\bar{\omega}$ is Γ invariant and closed. It then defines a closed form on $N = M/\Gamma$. The periods around closed loops of ω give the desired extension. This proves the proposition.

(ii) The character $t \to c(t)$ is defined on the entire center. We can form the quotient by the kernel of this character because h is invariant under translation by this subgroup. Thus we may assume $t \to c(t)$ is injective on the center. *But then Γ must be Abelian.* For let K be the kernel of the homomorphism $\Gamma \to R^*$ given by the proposition. Then K is normal so must (in a nilpotent group) intersect the center. This intersection contradicts our assumption that t is injective. Thus the extended character is injective and Γ is *Abelian*.

We are now in the situation where we have a minimal positive harmonic function h on a cocompact abelian covering space. The following theorem shows that h must be constant.

Theorem. *If M is a regular covering space of a recurrent manifold N with covering group of isometries Γ, then if h is a minimal positive harmonic function, either $\gamma h/h$ is unbounded for some γ in Γ or h is constant.*

Proof. First observe that we may as well treat the case $\gamma h = c(\gamma)h$ for all γ and prove that this implies that h is constant. Quotienting by the kernel of c, we may, in studying h, assume Γ is Abelian. If h is bounded then one sees that h must be constant; thus we suppose h is unbounded.

Now because N is recurrent we know that if $\delta > 0$ then the union of the balls of radius δ about each element of the fibre Γ_x will be hit almost surely by Brownian motion on M. On the other hand because h is an unbounded function we know that the measure expressing the constant function 1 as a convex combination of minimal positive harmonic functions has no atom at h. It follows that $h(b_t)$ tends to zero almost surely. (In general $\mathbf{E}^x(\lim_{t\to\infty} h(b_t))$ is the integral of the Radon-Nikodym derivative of the measure for h over the measure for 1.) We know therefore that with probability one Brownian motion will not hit the set

$$\bigcup_{c(\gamma)>1} \gamma B_x(\delta)$$

infinitely often, where $B_x(\delta)$ is the ball of radius δ centered at x. On the other hand it will hit

$$\bigcup_{c(\gamma)<1} \gamma B_x(\delta)$$

infinitely often. The following lemma shows this to be impossible.

Lemma. *Let E be a subset of Γ (Abelian) such that for every δ and some fixed $y \notin \Gamma x$ we have*

$$\mathbf{P}^x\Big(b_t \textit{ hits } \bigcup_{\gamma\in E} \gamma B_x(\delta)\Big) = 1.$$

Then $E^{-1} = \{\gamma\colon \gamma^{-1} \in E\}$ has the same property.

Proof. It is enough to prove that

$$\lim_{\delta\to 0} \mathbf{P}^y\Big(b_t \text{ hits } \bigcup_{\gamma\in E} \gamma^{-1} B_x(\delta)\Big) > 0$$

for some $y \notin E^{-1}x$. In particular we will assume E does not contain x and construct a measure μ on $E^{-1}B_x(\delta)$ such that $G\mu(x) > \frac{1}{2}$ for each δ and also so that $|G\mu(z)| < 1$ for all z in $E^{-1}B_x(\delta)$, where G is the Green kernel. The domination principle tells us that since $G\mu$ is dominated by the hitting probability of $E^{-1}B_x(\delta)$ on the support of μ this inequality extends to the whole of M. Therefore we obtain the inequality

$$\lim_{\delta\to 0} \mathbf{P}^x\big(b_t \text{ hits } E^{-1}B_x(\delta)\big) \geq \tfrac{1}{2}$$

as we require.

We choose a potential q supported on $EB_x(\delta)$ which satisfies $\|q\|_\infty \leq 1$ and $q(x) > \frac{3}{4}$. We may do this by considering the potential obtained by considering a finite but large subset E_n of E and looking at the hitting probability of $E_nB_x(\delta)$. Let $q = G\nu$ for some measure ν on $EB_x(\delta)$. We now produce a new measure μ on $E^{-1}B_x(\delta)$ by translating the part of ν supported on $\gamma B_x(\delta)$ by γ^{-2} so that it is now supported on $\gamma^{-1}B_x(\delta)$.

We must now estimate $G\mu$. Let $y \in \gamma^{-1}B_x(\delta)$. Then

$$G\mu(y) = \sum_{\rho\in E} \int_{\rho^{-1}B_x(\delta)} g(y,z)\mu(dz).$$

We will prove that if δ is small enough then $G\mu(y)$ is very close to $G\nu(\gamma^2 y)$.

The fundamental fact is

$$g(\gamma x, \rho x) = g(\gamma^{-1}x, \rho^{-1}x).$$

To prove this observe that

$$g(\gamma x, \rho x) = g(\rho^{-1}\gamma^{-1}\gamma x, \rho^{-1}\gamma^{-1}\rho x) = g(\rho^{-1}x, \gamma^{-1}x) = g(\gamma^{-1}x, \rho^{-1}x),$$

and note how fundamental use is made of the fact that ρ and γ commute and the fact that g is symmetric.

On the other hand Harnack's estimate tells us that if y and z are in $\gamma B_x(\delta)$ and $\rho B_x(\delta)$ respectively, where $\gamma \neq \rho$, then

$$1 - \varepsilon < \frac{g(y,z)}{g(\gamma x, \rho x)} < 1 + \varepsilon,$$

where ε is independent of x, y, ρ, γ and can be made as small as one likes by choosing δ small.

Putting this together we obtain that if y is in $\rho B_x(\delta)$ where $\rho \in E^{-1}$, then

$$\begin{aligned} G_\mu(y) &= \sum_{\gamma\in E^{-1}} \int_{\gamma^{-1}B_x(\delta)} g(y,\gamma^2 z)\nu(dz) \\ &= \sum_{\gamma\in E} \int_{\gamma B_x(\delta)} g(y,\gamma^{-2}z)\nu(dz), \end{aligned}$$

and by Harnack's estimate this

$$\sim \int_{\rho^{-1}B_x(\delta)} g(y, \rho^2 z)\nu(dz) + \sum_{\substack{\gamma \neq \rho^{-1} \\ \gamma \in E}} \int_{\gamma B_x(\delta)} g(\rho x, \gamma^{-2}x)\nu(dz).$$

Using the Green function identity we obtain that this expression equals

$$\int_{\rho^{-1}B_x(\delta)} g(\rho^{-2}y, z)\nu(dz) + \sum_{\substack{\gamma \neq \rho^{-1} \\ \gamma \in E}} \int_{\gamma B_x(\delta)} g(\rho^{-1}x, \gamma^2 x)\nu(dz)$$

and again by Harnack this expression $\sim G\nu(\rho^{-2}y)$ proving all we require.

In the proof we refer only to the canonical random walk on M. With care the argument can also be applied to the case of a certain singular diffusions on the Heisenberg group (as studied in [8], [10]) or to any other situation with the three essential ingredients: a nilpotent group, a symmetric Green function, and a Harnack principle.

4. ω-nilpotent covers of recurrent manifolds (Theorem 2)

Let M be an ω-nilpotent cover of a manifold N where the random motion on N is recurrent. Any fibre Γ over a point of N is $*$-recurrent so we may apply Theorem 5 and discretize. The bounded harmonic functions on M inject into the bounded ν-harmonic functions for the random walk determined by ν on Γ. Moreover we may think of Γ as the ω-nilpotent covering group. The construction of ν in Theorem 5 allows us to assume for each γ, γ' in Γ, $\nu_\gamma(\gamma) > 0$. Then Theorem 2 follows from the

Proposition (*Choquet-Deny-Dynkin* [3]). *If $\nu_e(\gamma) > 0$ for all $\gamma \in \Gamma$, where Γ is an ω-nilpotent group, then all bounded ν-harmonic functions on Γ are constant.*

Proof. If t is in the center of Γ and h is a positive harmonic function on Γ, then $h(\gamma) \geqslant \nu_\gamma(t\gamma)h(t\gamma)$. But the Γ-homogeneity of $\nu_\gamma(\gamma')$ means that $\nu_\gamma(t\gamma) = \nu_e(\gamma^{-1}t\gamma)$. Since t is central this quantity is independent of γ. Thus $h(t\gamma)/h(\gamma)$ is bounded above by $\nu_e(t)$ independently of γ. If h is minimal we have $h(t\gamma) = c_h(t)h(\gamma)$, where $t \to c_h(t)$ is a character on the center of Γ.

This character function is continuous in h as h varies over the extreme rays of the cone of positive harmonic functions (since $c_h(t) = h(t\gamma)/h(\gamma)$). On the support of m, where $1 = \int h_\xi \, dm(\xi)$, we must have $c_h \equiv 1$. This is so because $1 = t^{\pm 1}(1) = t^{\pm 2}(1) = \cdots$ implies $\int c_h^k \, dm = 1$ for k any integer.

Thus the center of Γ fixes minimal positive harmonic functions which make up the constant harmonic function. But any bounded harmonic is a convex combination of these. We divide by the center, consider the center of the quotient and continue by induction.

Corollary of Proof. The center of any group Γ acts trivially on the minimal positive harmonic functions for a (positive) random walk which are in the support of the constant harmonic function.

Note that our arguments extend to any Γ-invariant diffusion with a Harnack principle. We do not exploit symmetry of the Green function.

5. A natural projection of bounded functions onto bounded harmonic functions (Theorem 3)

Theorem 3′. *There is a projection of $L^\infty(M)$ into the bounded harmonic functions on M. If Γ is a group of isometries of M then this projection commutes with the action of Γ. In particular if M admits no nonconstant bounded harmonic functions and Γ acts discontinuously this projection induces an invariant mean on $l^\infty(\Gamma)$ and so Γ must be amenable.*

Proof. (1) The idea is to choose an invariant mean φ for the abelian semigroup $(t \geqslant 0, t)$ and apply it to the function $\mathbf{P}_t f(x)$ thought of as a bounded continuous function on $\mathbf{R}_+$ for each x. Then one might hope that $\varphi[\mathbf{P}_t f] = \varphi(\mathbf{P}_{t_0+t} f) = \mathbf{P}_{t_0}\varphi(\mathbf{P}_t f)$ so $\varphi(\mathbf{P}_t f(x))$ is a harmonic function of x. Now for the rigour.

(2) The locally compact additive semigroup $\mathbf{R}_+$ is abelian and hence amenable. Therefore there exists a continuous linear functional φ on the bounded continuous functions on $\mathbf{R}_+$ (written $C_b(\mathbf{R})$) which is invariant under translation. That is, let $g(t) = f(t_0 + t)$, then $\varphi(f) = \varphi(g)$.

We wish to interchange the order of operation of φ and the semigroup action. To do this we need an analytical fact. Let $P(t, x, y)$ denote the transition density of Brownian motion on M. We assume for now that $\int_M P(t, x, y)\,dy = 1$ for all x in M. Then for fixed t we have

$$\lim_{x \to x_0} \int_M |P(t, x, y) - P(t, x_0, y)|\,dy = 0.$$

This follows from the joint-continuity of $P(t, x, y)$ for all x, y and $t > 0$. Similarly,

$$\lim_{t \to t_0 > 0} \int_M |P(t, x, y) - P(t, x, y)|\,dy = 0.$$

Let f be in $L^\infty(M)$. Then the second estimate says that the function

$$\mathbf{P}_t^x f = \int_M P(t, x, y) f(y)\, dy$$

is a bounded continuous function of t for each x. Moreover,

$$\sup_{t>0}\left(\mathbf{P}_{t_0+t}^x f - \mathbf{P}_{t_0+t}^y f\right) = \sup_{t>0}\left(\mathbf{P}_{t_0}^x - \mathbf{P}_{t_0}^y\right)(\mathbf{P}_t f)$$
$$\leqslant \|f\|_\infty \int_M |P(t_0, x, z) - P(t_0, y, z)|\, dz,$$

and so the second inequality tells us that the map $g_x(t)$ defined by $g_x(t) = \mathbf{P}_{t_0+t}^x f$ is a continuous map from M into $C_b(\mathbf{R}_+)$, where $C_b(\mathbf{R}_+)$ is given the norm topology.

We now define $\tilde{\varphi}$: $L^\infty(M) \to$ bounded harmonic functions on M. Define $h(x) = \varphi(\mathbf{P}_t^x f)$. Then

$$h(x) = \varphi(\mathbf{P}_t^x f) = \varphi\left(\mathbf{P}_{t+2t_0}^x f\right)$$
$$= \varphi\left(\int_M P(t_0, x, y)\left(\mathbf{P}_{t_0+t}^y f\right) dy\right)$$
$$= \varphi\left(\int_M P(t_0, x, y) g_y\, dy\right).$$

But $y \to g_y$ is a continuous bounded function from M to $C_b(\mathbf{R}_+)$. It follows that we can approximate to the integral uniformly by Riemann sums and hence we can interchange the integration and φ to obtain

$$h(x) = \int_M P(t_0, x, y)\varphi(g_y)\, dy = \int_M P(t_0, x, y) h(y)\, dy.$$

So $\mathbf{P}_{t_0} h = h$, and since t_0 is arbitrary h is harmonic.

It is clear that the construction of $\tilde{\varphi}$ described above will commute with any isometric group actions. If Γ is a discontinuous group, then it is easy to construct a Γ invariant injection of $l^\infty(\Gamma)$ to $L^\infty(M)$, which maps 1 to 1: simply extend the l^∞ function to M by making it constant on fundamental domains.

This completes the proof under the assumption that $\mathbf{P}_t 1 = 1$. If $\mathbf{P}_t 1 \neq 1$, then one simply replaces Δ by $\rho\Delta$, where ρ is a smooth scalar function which decays at infinity at a rate sufficient to ensure that the associated diffusion (which is just a time changed version of the old one) does not leave M in finite time. By general considerations (see for example the excellent discussion in [20]), this diffusion will exist and will also have a C^∞ transition density $p_t(x, y)$.

Since ρ may be chosen in terms of the local geometry it may be assumed to be compatible with a discrete group of isometries. And so we have a proof of Theorem 3. Any Riemannian manifold with a discontinuous nonamenable group action has nonconstant bounded harmonic functions.

A simpler argument of the same kind can be applied to discrete time processes on countable state spaces, but we have no continuity problems. The approximation by Riemann sums is immediate.

6. The Kelvin-Nevanlinna-Royden recurrence criterion (Theorems 4, 4′)

Theorem (*Kelvin, Nevanlinna, Royden*). *M is transient if and only if there is a vector field ψ on M with*

$$\int \psi \cdot \psi \, dm < \infty, \quad \int |\operatorname{div} \psi| dm < \infty,$$

and

$$\int \operatorname{div} \psi \, dm \neq 0.$$

In other words it should be possible to have a net source or sink in a vector field with finite energy. Before sketching a proof of this we should make some remarks.

The idea of constructing a flow as a way of proving transience seems, in various parts, to have occurred to many people at different times in history. Kelvin noticed that of all flows in a domain with prescribed normal flow at the boundary there is a unique one with minimal energy. This flow is irrotational and hence the gradient of a harmonic function. Nevanlinna [15] proved that if the length $A(r)$ of the circle of radius r in a Riemann surface satisfies $\int_1^\infty 1/A(r)\, dr = \infty$, then the surface is recurrent. It is an easy application of the Cauchy-Schwartz inequality to prove that if this integral diverges then there cannot be a flow satisfying the hypothesis of the theorem. Let S_r be the sphere of radius r and $B(r)$ the associated ball. Suppose v is a vector field satisfying the hypothesis of the theorem; then

$$\int_M \|v\|^2 \, dx = \int_0^\infty \left(\int_{S_r} \|v\|^2 \, dx \right) dr \geqslant \int_0^\infty \left(\int_{S_r} (\underline{v} \cdot \underline{n})^2 \, dx \right) dr,$$

where n is the outward normal of S_r at x. An application of Cauchy-Schwartz shows that

$$\int_0^\infty \left(\int_{S_r} (\underline{v} \cdot \underline{n})^2 \, dx \right) dr \geqslant \int_0^\infty \frac{1}{A_r} \left(\int_{S_r} (v \cdot n) \, dx \right)^2 dr,$$

and by the divergence theorem this is equal to

$$\int_0^\infty \frac{1}{A_r}\left(\int_{B_r} \operatorname{div} v \, dx\right)^2 dr.$$

Because $\lim_{r\to\infty}\int_{B_r}\operatorname{div} v\,dx$ exists and is nonzero we see that this integral diverges if $\int_1(1/A_r)\,dr$ diverges. This proves that v does not exist.

Royden [16] may have been the first person to put both ideas together. That is if there is a flow then one has transience, if there is no flow there is recurrence. Tsuji has a clear proof of Roydens results [20]. One of the authors [11] and P. Doyle [5] have considered the analogous ideas for reversible Markov chains. D. R. Debaum [3] has the same idea stated in modern form in terms of L^2-cohomology.

Sketch of proof. Suppose M is transient and so admits a Green function. Let $f \geqslant 0$ be a smooth function of compact support on M. Let $u(x) = \int g(x, y) f(y)\,dy$. Then put $v = \nabla u$. We claim v satisfies all the hypotheses. First the field v has finite energy:

$$\int_M (\nabla u \cdot \nabla u)\,dx = -\int_M u\Delta u\,dx = -\int_M uf\,dx \leqslant \infty.$$

Of course $\int_M \Delta u\,dx = \int_M f\,dx \neq 0$ and so the other hypothesis on ∇u is satisfied.

The converse direction involves more work and justification, and owes much to Kelvin. We will deal first with the case where the divergence of the field is nonnegative and of compact support. By a theorem in De Rham [4] there is a smoothing operator which commutes with d and so we may assume the field and its divergence are both smooth. Let ψ be a vector field on M with finite L^2 integral and a smooth divergence φ which is nonnegative, of compact support, and not identically zero. We will use ψ to obtain a contradiction if a Brownian motion on M is recurrent.

The idea is to consider a second square integrable field ψ' with $\operatorname{div}\psi' = \operatorname{div}\psi = \varphi$, but chosen to minimise the L^2 norm. Let E be the Hilbert space of square integrable vector fields with zero divergence. Then because E is complete there is a vector field ψ' in $\psi + E$ which minimizes $\int_M \psi' \cdot \psi'\,dx$. By a standard argument in Hilbert space theory, $\int_M \psi' \cdot e\,dx = 0$ for all e in E. Therefore ψ' is orthogonal to all the smooth cycles and so must have no curl. In other words $\psi' = \nabla u$ for some function u. But $\operatorname{div}\nabla u = \Delta u$ is φ and so u is in fact smooth because Δ is elliptic. If we now show that u is bounded our argument is complete, for we will have constructed a nonconstant bounded subharmonic function and hence a Green function.

To prove this consider a disc D containing the support of $\varphi = \Delta u$. Off D, u is harmonic and $\int_{M-D}(\nabla u)^2\,dx$ is finite. Now it is a fact that a harmonic function with finite Dirichlet integral is determined by its boundary values (because it will yield an L^2 bounded martingale when composed with Brownian motion). Because Brownian motion on M is recurrent we know that harmonic measure for any point in $M - D$ relative to $M - D$ is supported on ∂D and not at ∞. But u is bounded on D and hence on M. This leads to our contradiction.

Suppose we have a vector field φ with

$$\int |\varphi|^2 + |\operatorname{div}\varphi| < \infty, \quad \int \operatorname{div}\varphi \neq 0.$$

Then there is a vector field with nonnegative divergence of compact support and the listed properties. To see this observe that we may as well assume φ is smooth. Now find a relatively compact open subset E of M with piecewise smooth boundary such that $\operatorname{div}\varphi \geqslant 0$ on E and

$$\int_E \operatorname{div}\varphi \geqslant \int_{M\setminus E} |\operatorname{div}\varphi|.$$

The set $M \setminus E$ admits a Green function $G_{M\setminus E}$, and if

$$u(x) = \int_{M\setminus E} G_{M\setminus E}(x, y)(\operatorname{div}\varphi)(y)\,dy,$$

then ∇u minimizes the energy among all vector fields ψ on $M \setminus E$ with

$$\operatorname{div}\psi - \operatorname{div}\phi = 0 \quad \text{on } M\setminus E.$$

Now

$$\int_{\partial E} |\partial u/\partial \underline{n}| \leqslant \int_{M\setminus E} |\operatorname{div}\varphi|.$$

Define c by $c\int_E \operatorname{div}\varphi = \int_{\partial E} \partial u/\partial \underline{n}$. Then because

$$\int_{\partial E} |\partial u/\partial \underline{n}| \leqslant \int_{M\setminus E} |\operatorname{div}\varphi|$$

we have $|c| < 1$. Let v solve the following Neumann problem in E. $\Delta v = +c\operatorname{div}\varphi$; $\partial v/\partial \underline{n} = \partial u/\partial \underline{n}$ on ∂E. Let $\psi = \nabla v$ on E and ∇u on $M \setminus E$. Then

$$\operatorname{div}\psi = +c\operatorname{div}\varphi \quad \text{on } E, \qquad \operatorname{div}\psi = \operatorname{div}\varphi \quad \text{off } E,$$

so

$$\operatorname{div}(\varphi - \psi) = (1 - c)\operatorname{div}\varphi \quad \text{on } E,$$

and is zero off E.

However, because $\operatorname{div} \varphi$ is smooth, $\partial u / \partial \underline{n}$ is piecewise continuous and bounded on E and this is certainly sufficient to imply that v has bounded Dirichlet norm. It follows that $\varphi - \psi$ has finite L^2-norm and the reduction is complete.

Let us give an application to covering manifolds of compact manifolds, which for Riemann surfaces is due to Mori [13]. Namely we prove a $\mathbf{Z} \times \mathbf{Z}$ cover M of a compact Riemannian manifold N is always recurrent; a $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$ cover M of a compact Riemannian manifold N is always transient. This is in contrast to finite volume manifolds. It is shown in [12] that a $\mathbf{Z} \times \mathbf{Z}$ cover of $\mathbf{C} \setminus \{0, 1\}$ is transient.

Suppose N is our compact manifold. We can induce the Abelian cover in question from a smooth map $N \to T^k$, $k = 2$, or $k = 3$. By transversality, we pull back the square (or cubic) decomposition of R^2 (or R^3) to get a similar decomposition of M over N.

Letting $A(r)$ denote for the $\mathbf{Z} + \mathbf{Z}$ cover the length of the boundary of the union of those cells whose images in R^2 touch the disk of radius r, one sees easily that $\int 1/A(r)\, dr = \infty$. By Nevanlinna's criterion M is recurrent.

Similarly for the $\mathbf{Z} + \mathbf{Z} + \mathbf{Z}$ cover $M \to N$ consider a dual 1-skeleton K_1 to the cubical decomposition. We think of K_1 as mapping to the corresponding dual 1-skeleton L_1 of the cubical decomposition of R^3. Associate to each oriented edge e of L_1 the flux $f(e)$ of w, the gradient of the Green function across the corresponding cubical face. Since w varies slowly in norm and $\int_{x \in R^3, \|x\| > \delta} |w|^2 < \infty$ it is clear that $\Sigma_{e \in L_1} (f(e))^2 < \infty$. Also at each vertex v (except for the one nearest the pole of the Green function) the sum of the fluxes of those edges touching v is zero (by the divergence theorem).

Now we transport these fluxes on the 1-dimensional complex K_1. This defines a 1-chain with the desired properties. Now diffuse this 1-chain using an equivariant smoothing operator on M to obtain an $n - 1$ form ω so that $d\omega$ is positive with compact support and $\int |\omega|^2 < \infty$.

7. Discretization of the random motion in the $*$-recurrent case (Theorem 5)

In this section we will prove Theorem 5. Let X be a discrete subset of a Riemannian manifold M. We say that X is $*$-recurrent if for each x in X there are a relatively compact open set U_x containing x and a second such set $V_x \supset \overline{U}_x$ chosen so that:

(i) $\bigcup_{x \in X} U_x$ is a recurrent set,

(ii) the supremum of the Harnack constants of the pairs (V_x, U_x) is $c < \infty$,

(iii) $\bigcup_{x \in X} \overline{U}_x = \overline{(\bigcup_{x \in X} U_x)}$.

We say that X is cocompact if for each x in X there are a relatively compact open set U_x containing x and a second such set $V_x \supset \overline{U}_x$ chosen so that:

(i) $\bigcup_{x \in X} U_x = M$,

(ii) the supremum of the Harnack constants of the pairs (V_x, U_x) is $C < \infty$.

The Harnack constant C of a pair (V, E), where V is open and $E \subset V$, is defined by

$$C = \sup\left\{ \frac{h(x)}{h(y)} \middle| x, y \in E, h \text{ positive and harmonic on } V \right\}.$$

Clearly if E is relatively compact in V this is finite. For given E it gets smaller as V increases. This is the point of the definitions above—V might not get large enough to make C uniformly small.

It is an easy observation that any Riemannian manifold M admits a set X which is cocompact simply because M is locally compact.

Sometimes it will be significant that the cover V_x is chosen to be locally finite. It is clear that one can do this in the case of a cocompact covering, but less clear that it can be done in general.

The other idea we will need in the proof of Theorem 5 is that of harmonic measure and balayage. Let $U \subset M$ be any open set and $x \in U$. Let ε_x^U denote the measure obtained by allowing Brownian motion to start at x and run until it leaves U and then setting ε_x^U eqaul to its exit distribution. This will always be a probability measure if U is relatively compact in M, and will frequently be one anyway. To say that a compact set K is recurrent is precisely the same as saying that ε_x^{M-K} is a probability measure for each x. The main points which we will use in the following are that if f is bounded and harmonic on U and if $\varepsilon_x^U(1) = 1$, then f has fine boundary values at ε_x^U almost every Martin boundary point and $f(x) = \varepsilon_x^U(f)$. This follows from the martingale convergence theorem. If f is a bounded Borel function on ∂U, its extension to U given by $x \to \varepsilon_x^U(f)$ is harmonic. The balayage of a measure μ onto ∂U is simply defined to be $\mu^U = \int \varepsilon_x^U \mu(dx)$, so $\mu(h) = \mu^U(h)$ for every bounded harmonic function on U providing $M - U$ is recurrent. We can now prove Theorem 5.

Proof. Let X be our $*$-recurrent set, U_x, V_x our associated family of open sets and suppose c is the supremum of the Harnack constants of the pairs (V_x, U_x). Let us suppose for convenience that the $\overline{U}_i$ are disjoint (otherwise one must decompose $\bigcup_{x \in X} \overline{U}_x$ into countably many disjoint Borel sets $E_x \subset \overline{U}_x$) and locally finite.

Here is the inductive step. Suppose μ_n is any measure on M, we may find two new measures μ_{n+1} and τ_{n+1} such that:

(i) τ_{n+1} is supported on X,
(ii) $\mu_n(h) = \mu_{n+1}(h) + \tau_{n+1}(h)$, and
(iii) $\tau_{n+1}(h) \geqslant \mu_n(h)/C^2$,
whenever h is a bounded positive harmonic function.

Start by moving all mass off X so put

$$\mu = \mu_n - \mu_n(x)\left(\varepsilon_x - \varepsilon_x^{V_x}\right).$$

If h is bounded and harmonic we certainly have

$$\mu(h) = \mu_n(h) - \mu_n(x)(h(x) - h(x)) = \mu_n(h).$$

Now balayage this measure μ back onto $\bigcup_{x\in X} U_x$ to obtain a new measure $\tilde{\mu}$. (Leave any mass which is already in $\bigcup_{x\in X} U_x$ alone.) Because $\bigcup_{x\in X} U_x$ is recurrent we get that $\tilde{\mu}$ is also a probability measure and $\tilde{\mu}(h) = \mu(h)$ for all bounded harmonic functions h. Now let ρ_x be the restriction of $\tilde{\mu}$ to $\overline{U}_x$ for each $x \in X$. Then consider $\rho_x^{M-U_x}$ and $\varepsilon_x^{V_x}$. By Harnack's inequality these are both mutually absolutely continuous and

$$\|\rho_x\|\frac{1}{C} \leqslant \frac{\partial\rho_x^{V_x}}{\partial\varepsilon_x^{V_x}} \leqslant C\|\rho_x\|,$$

for if not we could choose a bounded continuous function on ∂V_x which extended to a harmonic function on V_x violating the Harnack condition on U_x. Observe that

$$\begin{aligned}\mu_n(h) &= \sum_{x\in X} \rho_x^{V_x}(h)\\ &= \sum_{x\in X}\left[\left(\rho_x^{V_x} - \frac{\|\rho_x\|}{C}\varepsilon_x^{V_x}\right)h + \frac{\|\rho_x\|}{C}h(x)\right].\end{aligned}$$

Put

$$\mu_{n+1} = \sum_{x\in X}\left(\rho_x^{V_x} - \frac{\|\rho_x\|}{C}\varepsilon_x^{V_x}\right) \quad \text{and} \quad \tau_{n+1} = \frac{\|\rho_x\|}{C}\varepsilon_x.$$

Both μ_{n+1} and τ_{n+1} are positive measures and the required property $\tau_{n+1}(h) \geqslant \mu_n(h)/C^2$ is obtained by using the other half of the Harnack inequality:

$$\partial\varepsilon_x^{V_x}/\partial\rho_x^{V_x} \leqslant \frac{C}{\|\rho_x\|}.$$

The construction of ν_y is now very straightforward. Let $\mu_0 = \varepsilon_y$, the unit mass at y. Then if h is a bounded harmonic function we have

$$h(y) = \mu_0(h) = \mu_1(h) + \tau_1(h) = \mu_n(h) + \sum_1^n \tau_k(h),$$

and since $\mu_n(h) \leqslant (1 - 1/C^2)^n h(y)$ we have $h(y) = \sum_1^\infty \tau_k(h)$. Put $\nu_y = \sum_1^\infty \tau_k$. It has all the required properties.

8. Discretization of the random motion in the cocompact case (Theorem 6)

We now treat the cocompact case. First we will construct the measure ν_y on X in much the same way as we did in the recurrent case. Then we will explain how to think of ν_y as coming from a stopped Brownian motion. As before we proceed by induction. Let $(U_x)_{x \in X}$ be the relatively compact open cover of M and $V_x \supset \overline{U}_x$ a second relatively compact cover of M such that the pairs (V_x, U_x) have Harnack constant bounded above by a fixed constant C. Such a family of U_x, V_x exists by definition. Now let $(E_x)_{x \in X}$ be a Borel partition of M subordinate to the U_x's. For any point y in M denote by $x(y)$ the unique point x in M with $y \in E_x$.

Now let μ_n be any positive measure on M, we will construct two positive measures μ_{n+1} and τ_{n+1} so that τ_{n+1} is supported on X,

$$\mu_n(h) = \mu_{n+1}(h) + \tau_{n+1}(h), \qquad \tau_{n+1}(h) \geqslant \frac{1}{C^2}\mu_n(h).$$

Because all the expressions are linear we may as well assume μ_n is supported on E_x. Now because V_x is relatively compact any positive harmonic function h is bounded on V_x and therefore

$$\begin{aligned}\mu_n(h) = \mu_n^{V_x}(h) &= \left(\mu_n^{V_x} - \frac{\|\mu_n\|}{C}\varepsilon_x^{V_x}\right)(h) + \frac{\|\mu_n\|}{C}\varepsilon_x^{V_x}(h) \\ &= \left(\mu_n^{V_x} - \frac{\|\mu_n\|}{C}\varepsilon_x^{V_x}\right)(h) + \frac{\|\mu_n\|}{C}h(x).\end{aligned}$$

Put $\mu_{n+1} = \mu^{V_x} - \|\mu_n\|\varepsilon_x^{V_x}/C$ and $\tau_{n+1} = \|\mu_n\|\varepsilon_x/C$, because μ_n is supported entirely on U_x we know that μ_{n+1} is positive; moreover we know that $h(y) > h(x)/C$ for all y in U_x, and therefore $\tau_1(h) \geqslant \mu(h)/C^2$.

This is the inductive step; to obtain ν_y if $y \notin X$ we simply start off with $\mu_0 = \varepsilon_y$, then $\nu_y' = \sum_1^\infty \tau_i$ as in the $*$-recurrent case. If y is an element of X then we start off with $\mu_0 = \varepsilon_y^{V_y}$. The reason for this slightly different treatment of points of X will become apparent as we try to connect the Markov chain on ν induced by the ν_y with the diffusion on M.

Let us think of how ν_y was constructed. We start off with unit mass at y, look for the point $x(y)$ which is the 'closest' to y in X, and balayage our mass at y onto the boundary of $V_x(y)$. Because the mass originated near the centre of $V_x(y)$ (i.e., from $U_{x(y)}$) the balayaged measure dominates $\varepsilon_{x(y)}^{V_{x(y)}}/C$. We leave that measure behind on the boundary of V_x, and reapply the procedure to the residual mass; so for each $\tilde{x}$ in X balayage that bit of the remaining mass which is nearest to $\tilde{x}$ onto the boundary of $V_{\tilde{x}}$, leave the appropriate multiple of $\varepsilon_{\tilde{x}}^{V_{\tilde{x}}}$ behind, and then iterate the procedure. Ultimately nothing is left iterate and we have a new measure,

$$\sum_{x \in X} \lambda_y(x) \varepsilon_x^{V_x},$$

supported entirely on the boundaries of the V_x's with the property that

$$h(y) = \sum_{x \in X} \lambda_y(x) \varepsilon_x^{V_x}(h).$$

But $\varepsilon_x^{V_x}(h) = h(x)$, and so

$$h(y) = \sum_{x \in X} \lambda_y(X) h(y).$$

In other words $\lambda_y = \nu_y$. In the special case where y is in X we change the first step slightly by replacing the first measure ε_y by $\varepsilon_y^{V_y}$.

We wish to do this whole iteration procedure using Brownian motion, and in fact go quite a lot further. Let (Z_n) be the Markov chain with transition probabilities $(\nu_y)_{y \in M}$. The paths of (Z_n) are elements of $M \times X^{\mathbf{N}}$; let $\mathbf{Q}^y$ be the measure on $M \times X^{\mathbf{N}}$ obtained by starting (Z_n) with $Z_0 = y$. Let $\mathbf{P}^y$ be the measure on our Brownian event space Ω corresponding to starting Brownian motion at y. We will explain how to construct from any Brownian path $w(t)$ and the results of infinitely many independent coin tosses a sequence of points $(Y_n)_{n=0}^{\infty}$ with $Y_0 = y$ in M and Y_n in X for all $n > 0$ and such that the map

$$w \to (Y_n): \Omega \to M \times X^{\mathbf{N}}$$

takes $\mathbf{P}^y$ onto $\mathbf{Q}^y$. In fact the Markov chain is almost a skeleton of the Brownian motion.

We must do two things. First we must describe how to obtain the Y_n, then we must justify our claim that $\mathbf{P}^y$ is taken onto $\mathbf{Q}^y$. Both are rather formal but also rather technical.

To simplify the discussion change the V_x's slightly so that $\partial V_x \cap X$ is empty for every x in X. Let W be the set of all continuous paths on M, and let $w(t)$ be a path in W. We wish to define some stopping times. Suppose first that $y = w(0)$ is not in X, then put $S(w) = \inf\{t: w(t) \notin V_{x(y)}\}$, in other words w starts near $x(y)$, S is the first time w leaves the ball $V_{x(y)}$ centered at $x(y)$. If $w(0)$ is in X put $S' = \inf\{t: w(t) \notin V_{w(0)}\}$, then $S(w) = S(w(t + S'(w))) + S'(w)$; in other words if w starts at $y \in X$ run it until it leaves V_y, look to see which $x \in X$ is the nearest neighbor in X at this point and then run to the boundary of V_x.

We may define $S(n)$ inductively from $S(1) = S$ by either of the following relations:

$$S(n, w) = S(n-1, w(t+S(w))) + S(w),$$
$$S(n, w) = S(w(t+S_{n-1}(w))) + S_{n-1}(w).$$

The times S_n break the journey of $w(t)$ up into discrete pieces. Between S_n and S_{n+1} the path w is travelling from the middle to the boundary of some V_x. It will be convenient to call $w(S(n, w))$ the nth exit point and $x(w(S(n-1, w)))$ the nth center point of w except that if $w(0) = y \in X$ then we will use the notation the zeroth exit point of w to mean $x(w(S'(w)))$, otherwise it will be $x(w(0))$. Observe that the nth center of $w(t+S)$ is the $n+1$st center of $w(t)$.

Now, keeping the construction of ν_y in mind, let w be in W. At $S(n)$ flip a coin with probability

$$\frac{1}{C}\frac{d\varepsilon_x^{V_x}}{d\varepsilon_z^{V_x}}(w(S_n))$$

of coming up heads, where x is the nth center and z is the $(n-1)$th exit point of w. We know that

$$\frac{1}{C^2} \leqslant \frac{1}{C}\frac{d\varepsilon_x^{V_x}}{d\varepsilon_z^{V_x}} \leqslant 1 \qquad (z \in U_x),$$

and so $(1/C)d\varepsilon_x^{V_x}/d\varepsilon_z^{V_x}$ is a legitimate probability.

Flip the coin in a way that is completely independent of w once z and x are known. We have a sequence of heads and tails at the times S_n. Let $T = T_1$ be the time at which the first head occurs, and T_2 the time the second head occurs, etc. Clearly $T_n(w) = T_{n-1}(w(t+T)) + T(w)$. For each i there is a $k(w)$ such that $T_i(w) = S_k(w)$. Let $Y_n(w)$ be the $k(w)$th center of w. Then ν_y is the law of Y_1 given that the Brownian motion starts at y. In other words

$$\nu_y(f) = \mathbf{E}^y(f(Y_1)) \quad \text{whenever } f \in l^\infty(X).$$

By construction, the $\mathbf{P}^y$ law of $w(T)$ (in other words the measure on M which satisfies $\mathbf{E}^y(f(w(T))) = \mu(f)$ for all bounded continuous functions on M) is just

$$\sum_{x \in X} \nu_y(x)\varepsilon_x^{V_x}.$$

The point is the following. If $\mu_0 = \varepsilon_y$ (or $\varepsilon_y^{V_y}$ if $y \in X$), and μ_n, τ_n are defined by the inductive process described at the beginning of the proof, then

$$\mathbf{E}^y(f(w(T \wedge S_r))) = \left[\sum_{x \in X}\sum_{R=1}^{r} \tau_R(x)\varepsilon_x^{V_x} + \mu_r\right] f,$$

for all bounded continuous functions f. To see this we use the strong Markov property. It suffices to see that if μ_n is any measure then

$$\mathbf{E}^{\mu_n}(f(w(T \wedge S_1))\chi(S_1 = T)) = \left[\sum_{x \in X} \tau_{n+1}(x)\varepsilon_x^{V_x}\right] f,$$

and

$$\mathbf{E}^{\mu_n}(f(w(T \wedge S_1)) \cdot \chi(S_1 < T)) = \mu_{n+1}(f).$$

This was the point of stopping the paths with density

$$d\varepsilon^{V_x}_{x(w(S_1))}/d\varepsilon^{V_x}_y,$$

if they started at $y \in E_x$.

We now prove that the map $w \to (Y_i)_{i=0}^\infty$ does map $\mathbf{P}^y$ onto $\mathbf{Q}^y$. Because $\mathbf{Q}^y$ is determined by its finite dimensional distributions it is enough to prove that

$$\mathbf{P}^y(Y_0 = y, Y_1 = x_1, \cdots, Y_n = x_n) = \nu_y(x_1)\nu_{x_1}(x_2) \cdots \nu_{x_{k-1}}(x_k)$$

for any $y, x_1, \cdots, x_n \in M \times X^n$. Again we use induction and the strong Markov property: suppose that for each $k < n$ we have

$$\begin{aligned}\mathbf{E}^y(\chi(Y_0 = y_0 = y, y_1 = x_1, \cdots, y_k = x_k) f(w(T_k)))\\ = \nu_y(x_i)\nu_{x_1}(x_2) \cdots \nu_{x_{k-1}}(x_k)\varepsilon^{V_{x_k}}_{x_k}(f).\end{aligned}$$

Then we have the same relation for $k = n$. Let $m_{k-1} = \nu_y(x_1)\nu_{x_1}(x_2) \cdots \nu_{x_{k-1}}(x_k)$. Then by the strong Markov property and the result for $k = n - 1$,

$$\begin{aligned}\mathbf{E}^y(\chi(Y_0 = y, Y_1 = x_1, \cdots, Y_n = x_n) f(w(T_n)))\\ = m_{n-1}\int \mathbf{E}^{\tilde{y}}((Y_1 = x_n) f(W(T_1)))\varepsilon^{V_{x_{n-1}}}_{x_{n-1}}(d\tilde{y})\\ = m_{n-1}\mathbf{E}^{x_{n-1}}((Y_1 = x_n) f(w(T_1))),\end{aligned}$$

because of the definition of Y_1, T_1 when $w(0) \in X$. Using the result in the case $k = 1$ we have that this

$$= m_{n-1}\nu_{x_{n-1}}(x_n)\varepsilon^{V_x}_{x_n}(f) = m_n\varepsilon^{V_x}_x(f).$$

This establishes the identity for all n. Putting $f = 1$ we obtain

$$\mathbf{P}^y(Y_0 = y, \cdots, Y_n = x_n) = m_n = \nu_y(x) \cdots \nu_{x_{n-1}}(x_n).$$

The distance between the Brownian and discrete paths. Let us define a notion of distance from a point of X. We will say that y is a distance at most n from x if there are points $(x_k)_1^n \in X$ with $x_1 = x$ such that $E_{x_n} \cap \partial V_{x_{n-1}}$ is always nonempty and $y \in V_{x_n}$. Put $d(x, y)$ equal to the smallest possible of these n.

Now consider a Brownian path $w(t)$, and ask how close it stays to the path $(Y_n)_{n-0}^{\infty}$ which it projects to in $M \times X^N$. Clearly, at T_n, $d(w(T_n), Y_n) = 1$ and $d(W(t), x) \leqslant d(w(S_k), x) + 1$ for any choice of $S_k < t \leqslant S_{k+1}$. It follows that the maximum distance that w gets from Y_n between T_n and T_{n+1} is at most one more than the number of S_k's that occur between the head which determined T_n and that which determined T_{n+1}. In other words if w is to get a distance $k + 1$ away from Y_n between $T_n \cdot T_{n+1}$ we must have a string of k tails. The probability of a tail is at most $(1 - 1/C^2)$. If we define

$$d_n(w, Y) = \sup_{k \leqslant n} \{ d(w(t), Y_k) \colon T_k < t < T_{k+1} \},$$

then we have the following easy estimate.

Theorem. *The probability that $d_n(w, Y)$ exceeds λ is at most $1 - (1 - (1 - 1/C^2)^\lambda)^n$. In particular if λ is large, $1 - (1 - (1 - 1/C^2)^\lambda)^n \sim n(1 - 1/C^2)^\lambda$.*

Proof. The sets where $d_n(w, Y)$ exceed λ, and the set of w where the first n heads occur with no more than $(\lambda - 1)$ tails interspersed between any two of them, are mutually disjoint. It follows that the probability that the first event occurs is at most 1 minus the probability that the second event occurs. At each coin toss the probability is at least $1/C^2$ that it will be a head, this inequality holding independently of the outcomes of the other tosses. The probability that the first head comes on or before the λth toss is at most $1 - (1 - 1/C^2)^\lambda$. Repeating this estimate n times, and using the fact that the estimate holds independently of the outcomes of the other tosses we obtain the result.

Most of the time the Brownian path and the discrete Markov chain stay close together in this ball metric. It would not be surprising that the two mirrored each other in other ways.

Theorem 6′. *Providing the V_x's are a locally finite cover of M, the ν-random walk on X will be recurrent if and only if Brownian motion on M is recurrent.*

Proof. Suppose first that M is recurrent. Fix a compact recurrent set K in M for Brownian motion on M. Let $F = \{x \in X \colon V_x \cap K \neq \varnothing\}$. By hypothesis F is finite. Now for some x in F there must be infinitely many k such that $w(S_{k-1}) \in E_x$, $w(S_k) \in \partial V_x$. But at each of these k there is a probability $1/C^2$ that $S_k(w) = T_j(w)$ for some j, this lower bound on the probability being independent of everything else. It follows that there are infinitely many $k(j)$ with $w(S_{k-1}) \in E_x$, $S_k = T_{j(k)}$. But therefore $Y_j = x$ infinitely often and the Markov chain is recurrent.

Suppose now that M is transient. Clearly, for each x in X, w will almost surely leave V_x for the last time $\tau(w)$. For all n such that $T_n(w) > \tau(w)$ it follows that $Y_n(w) \neq x$. In other words Y_n eventually leaves all finite sets with probability one. Y_n is transient.

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