Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups

by

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Given a (closed) set Λ contained in the plane and a positive function $\psi(r)$ (for example r^δ, $r^\delta(\log 1/r)^{\delta'}$, etc.) one defines the (covering) Hausdorff measure of Λ relative to $\psi(r)$ by considering *coverings* of a subset $A\subset\Lambda$ by balls $B_1, B_2, \ldots$ of radii $r_1, r_2, \ldots$ all less than $\varepsilon\geqslant 0$. The (covering) Hausdorff ψ-measure of A is the limit as $\varepsilon\to 0$ of the *infimum over such coverings* of the sums $\Sigma_i \psi(r_i)$.

We have been led by the study of limit sets of Kleinian groups to a dual construction based on *packings* rather than coverings. Here one considers for U *open* in Λ collections of disjoint(¹) balls $B_1, B_2, \ldots$ contained in U of radii $r_1, r_2, \ldots$ all less than $\varepsilon\geqslant 0$. Then the *packing* Hausdorff ψ-measure of U is by definition the limit as $\varepsilon\to 0$ of the *supremum over such packings* of the sums $\Sigma_i \psi(r_i)$. If two open sets are disjoint these limits add. It follows (§ 7) that the outer measure defined by this function of open sets defines a countably additive Borel measure.

The covering and packing Hausdorff measures can be distinguished rather dramatically for the limit sets of Kleinian groups which are geometrically finite but have cusps.

For example the "Apollonian packing" limit set (Fig. 1) *has for a certain* δ *a positive (locally finite)* r^δ *Hausdorff measure for the covering definition* ($\delta\sim 1.3$). *The* r^δ *packing measure is locally infinite.*

For a second example consider the packing obtained by the rectangular array of unit diameter circles (Fig. 2). Inverting these in the dotted circle of unit diameter packs them into one quadrilateral interstice. Now translate in both directions and repeat the procedure ad infinitum to get the second example. *This limit set has locally finite* r^δ

(¹) Balls are disjoint if the distance between their centers is greater than the sum of the radii.

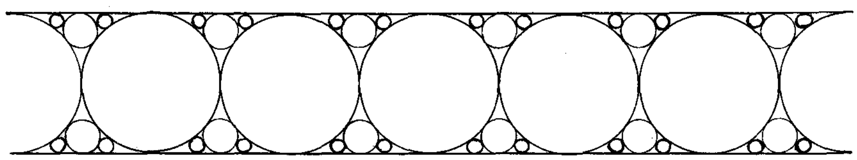

Fig. 1

Hausdorff measures with the packing definition, for some $1<\delta<2$. *The* r^δ *covering measure is zero.*

These examples are limit sets of subgroups of the Picard group, consisting of 2×2 matrices of Gaussian integers.

More generally, let Γ be a *discrete* subgroup of linear fractional transformations $\{z\to(az+b)/(cz+d)\}$ which is *geometrically finite*, that is Γ has a finite sided fundamental domain for its action on hyperbolic 3-space (thought of as the upper half space with boundary $\mathbf{C}\cup\infty$ or the unit 3-ball with boundary the sphere). The limit set of Γ denoted $\Lambda=\Lambda(\Gamma)$, is the set of cluster points in $\mathbf{C}\cup\infty$ of any orbit of Γ. In this paper we study the geometry of Λ using the ergodic nature of the action of Γ on Λ.

If $n(R)$ is the cardinality of the intersection of a fixed orbit of Γ in hyperbolic 3-space with balls of hyperbolic radius R with fixed center, the *critical exponent of* Γ is by definition the exponential growth rate of $n(R)$,

$$\delta=\delta(\Gamma)=\overline{\lim_{R\to\infty}}\ \frac{1}{R}\log n(R).$$

The number $\delta(\Gamma)$ is the critical exponent for absolute convergence of Poincaré series associated to Γ (see § 1).

Now let $D=D(\Gamma)$ denote the *Hausdorff dimension* of the limit set $\Lambda(\Gamma)$. Thus by definition, D=infimum of the set of α so that the (covering) Hausdorff r^α measure of $\Lambda(\Gamma)$ is zero (=supremum of the set of α where the covering Hausdorff r^α measure of $\Lambda(\Gamma)$ is infinity).

Let $|\gamma'|$ denote the linear distortion of any linear fractional transformation γ in a fixed spherical metric on $\mathbf{C}\cup\infty$.

We say that a *finite measure* μ *on the limit set is geometric if*

$$\gamma^*\mu=|\gamma'|^D\mu,\quad \gamma\in\Gamma.$$

(Here $\gamma^*\mu(A)=\mu(\gamma A)$.)

THEOREM 1. *For a geometrically finite group* Γ, *the critical exponent* $\delta(\Gamma)$ *equals the Hausdorff dimension* $D(\Gamma)$ *of the limit set* $\Lambda(\Gamma)$; *and* $D(\Gamma)<2$ *if* $\Lambda(\Gamma)$ *is a proper* *subset of* $\mathbf{C} \cup \infty$. *Moreover, there exists one and only one geometric measure* μ *of total mass one supported on* $\Lambda(\Gamma)$.

We have tried for some time to describe the canonical geometric measure in terms of the metric structure of the limit set alone (without using the group of conformal symmetries Λ). The cusps present interesting difficulties.

By a *cusp of* Γ we mean a conjugacy class in Γ of maximal parabolic subgroups fixing points of $\mathbf{C} \cup \infty$ (necessarily in $\Lambda(\Gamma)$). Such a subgroup either contains $\mathbf{Z}$ or $\mathbf{Z}+\mathbf{Z}$ of finite index and these are called *rank one cusps* or *rank two cusps* accordingly.

Let μ denote the canonical geometric measure (Theorem 1) on the limit set of a geometrically finite group Γ. Let ν_p and ν_c denote the packing and covering Hausdorff measures on $\Lambda(\Gamma)$ using the gauge function r^D. (These measures may a priori be either zero or not even σ-finite.)

THEOREM 2. *Unless there are cusps of rank* 1 *and of rank* 2, *the canonical geometric measure* μ *equals one or both of the metrically defined measures* ν_p *or* ν_c. *The situation is described by the table:*

	No cusps	Only rank 1 cusps	Only rank 2 cusps	Cusps of both ranks
$D=2$	$\mu=\nu_p=\nu_c$	Not possible	$\mu=\nu_p=\nu_c$ =spherical measure	Not possible
$1<D<2$	$\mu=\nu_p=\nu_c$	$\mu=\nu_c\neq\nu_p=\infty$	$\mu=\nu_p\neq\nu_c=c=0$	$\mu=?$ $0=\nu_c\neq\nu_p=\infty$
$D=1$	$\mu=\nu_p=\nu_c$	$\mu=\nu_p=\nu_c$	Not possible	Not possible
$D<1$	$\mu=\nu_p=\nu_c$	$\mu=\nu_p\neq\nu_c=0$	Not possible	Not possible

Remark. (1) An interesting case is finitely generated Fuchsian groups of the second kind with cusps ($\Lambda(\Gamma)$ is a Cantor subset of a round circle contained in the Riemann sphere). Looking at the table ($1/2<D<1$, only rank one cusps) shows the canonical geometric measure is the packing measure, but the covering measure is zero.

(2) A deeper analysis ([S1]) shows that if μ is not equal to ν_p (or ν_c) then μ is not equivalent to a packing (or covering) measure relative to any gauge function. Thus limit sets provide natural examples where packing measures are finite and positive but no covering measure is (for any gauge function) and vice versa.

The canonical geometric measure μ determines a Γ invariant measure $\mu\times\mu/|x-y|^{2D}$

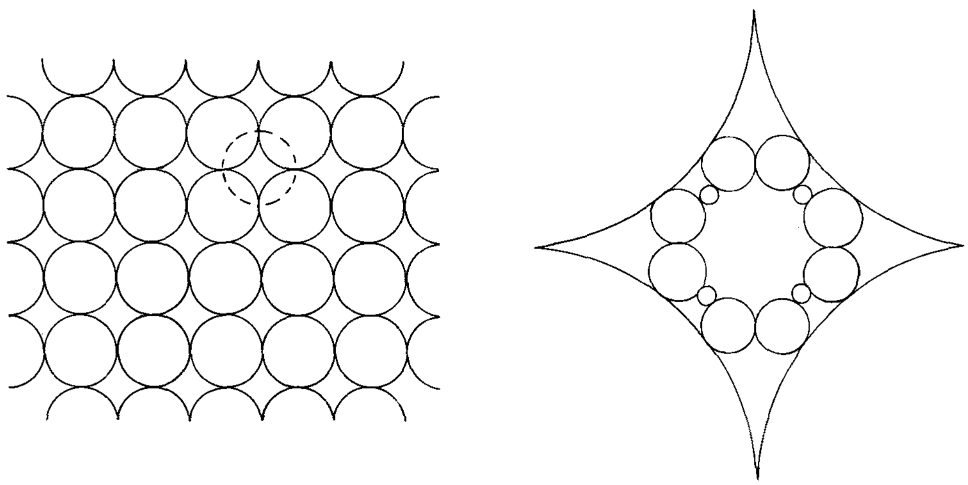

Fig. 2

on pairs of points on the sphere.(2) Combining this measure with arc length determines an invariant measure dm_μ for the geodesic flow associated to $\mathbf{H}^3/\Gamma$. This measure plays a key role in the proofs of Theorems 1 and 2 using part (i) of

THEOREM 3. (i) *The measure dm_μ has finite total mass and is ergodic for the geodesic flow. Moreover,*

(ii) *The measure theoretical entropy of the geodesic flow relative to dm_μ is D, the Hausdorff dimension of $\Lambda(\Gamma)$.*

(iii) *In case there are no cusps the geodesics with both endpoints in $\Lambda(\Gamma)$ form a compact invariant set, and the topological entropy is again D. Thus in this case dm_μ is a measure that maximizes entropy.*

In summary we have defined the same real number in terms of the Poincaré series, the Hausdorff dimension of the limit set, and the entropy of the geodesic flow.

Historical note and acknowledgements. Most of Theorem 1 was proven for most Fuchsian groups by Patterson in [P]. In particular, he constructed measures satisfying $\gamma^*\mu=|\gamma'|^\alpha\mu$ for any discrete group and estimated an associated eigenfunction φ_μ (cf. §4).

Rufus Bowen [B] constructed such measures using Markov partitions and showed $D>1$ for quasi-Fuchsian compact surface groups.

(2) This follows from $\gamma^*\mu=|\gamma'|^D\mu$ and $|\gamma x-\gamma y|^2=|\gamma' x|\,|\gamma' y|\,|x-y|^2$ where $|a-b|$ is the Euclidean distance between a and b on the unit sphere.

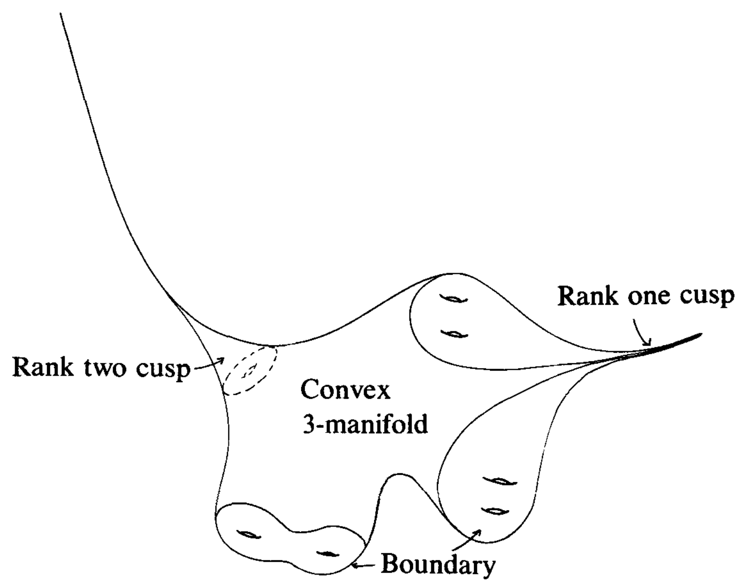

Fig. 3

References of these two papers include those to earlier work by Akaza and A. F. Beardon who also related $\delta(\Gamma)$ to the Hausdorff dimension of $\Lambda(\Gamma)$, for certain Fuchsian groups.

In [S2] we continued the development of these papers to study general properties of such measures, δ and D. We were able there to treat general Fuchsian groups (§ 6) and higher dimensional geometrically finite groups without cusps (§ 3). Discussions of that paper with Thurston (with crucial assists) allowed simplifications and completions of the discussion given here for the general geometrically finite case with cusps. We wrote the paper for dimension 3 but it clearly translates into dimension n.

Finally, in Thurston's hyperbolic geometry notes [T] one finds in revealing form the geometric information about geometrically finite groups required here. Namely the quotient by Γ of the convex hull of the limit set consists of a compact piece with boundary and exponentially skinny ends attached, one for each cusp (Fig. 3).

§ 1. Existence of the (geometric) measure μ

For any discrete group of hyperbolic isometries define *the absolute Poincaré series* by

$$g_s(x, y) = \sum_{\gamma \in \Gamma} \exp(-s \cdot \text{hyperbolic distance}(x, \gamma y)),$$

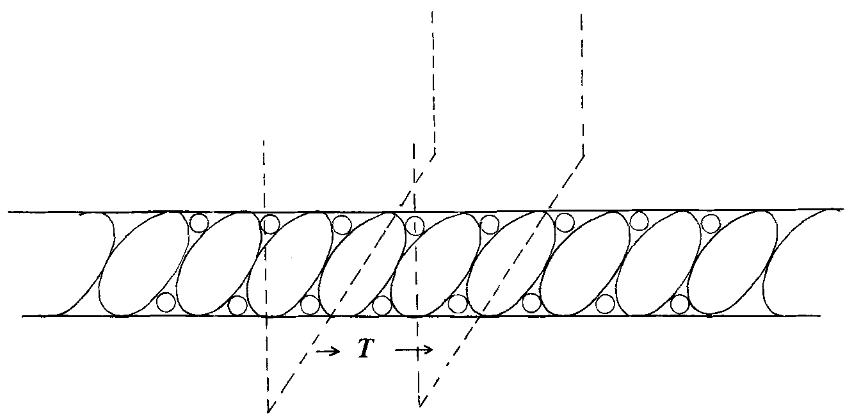

Fig. 4

and the *critical exponent* $\delta(\Gamma)$ as the infimum of the set of s where $g_s(x,y)<\infty$. Here x and y are points in hyperbolic space, $\mathbf{H}^3$.

Following a construction of Patterson [P] one places atomic masses along the orbit $\{\gamma y\}$ of y with weights

$$\frac{1}{g_s(y,y)}\exp(-s\cdot\text{distance}(x,\gamma y)).$$

The limit as $s\to\delta(\Gamma)$ defines a measure μ on the limit set $\Lambda(\Gamma)$ satisfying

$$\gamma^*\mu=|\gamma'|^\delta\mu,\quad \gamma\in\Gamma,\quad \delta=\delta(\Gamma).$$

(See [S2] § 1. Actually if $g_\delta(y,y)<\infty$ one has to increase each term in the sum for $g_s(x,y)$ by a factor $h(\text{distance}(x,\gamma y))$ of arbitrarily small exponential growth to make the series diverge at $s=\delta$. For $\varepsilon>0$, $k>0$, $h(x)$ satisfies $|h(s+t)/h(s)-1|<\varepsilon$ for $0\leq t\leq k$ and s sufficiently large.)

We will see later § 4 that $\delta(\Gamma)$ is the Hausdorff dimension of $\Lambda(\Gamma)$ in the geometrically finite case, and the Poincaré series diverges at $s=\delta$. In particular the h-factor is not necessary (after the fact).

§ 2. μ has no atoms

We work in the upper half space model and we put a cusp at ∞. If the cusp has rank one, we may assume $\Lambda(\Gamma)$ is contained between two parallel lines $y=0$, $y=1$ and Γ contains the translation $T(x+iy)=(x+iy)+1$ (See [T], p. 8.21) (Fig. 4).

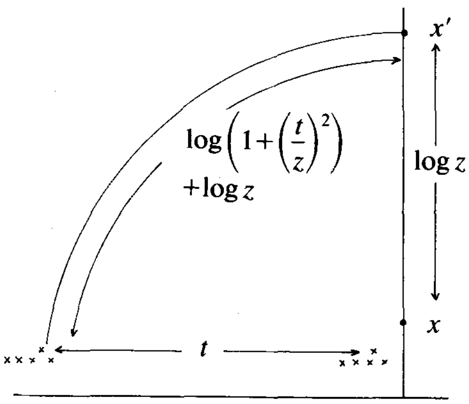

Fig. 5

We estimate $g_s(x',y)/g_s(x,y)$ in terms of z, where x is a fixed point and x' is at height z above x', as s approaches $\delta(\Gamma)$. Each sum over Γ is broken into a sum Σ' corresponding to the part of the orbit of y clustering down to a fundamental domain of T and then a sum over the integers Z corresponding to $\{T^n\}_{n\in Z}$.

The Σ' sums for x' and x are in the approximate ratio $z^{-\delta}$ for s near δ.

The entire sum for x is obtained by some (fixed) convergent series(3) multiplying the Σ' term. The same relation holds for x'. The convergent series for x' is comparable to the integral

$$\int_R \left(\frac{1}{1+(t/z)^2}\right)^{\delta} dt = z\int \left(\frac{1}{1+t^2}\right)^{\delta} dt.$$

Since the integral is convergent, $\delta>1/2$. More generally a cusp of rank k implies $\delta(\Gamma)>k/2$ (Fig. 5) (cf. [B2]).

The combined effect gives

$$g_s(x',y)/g_s(x,y)\sim z^{1-\delta}$$

(ignoring the effect of the factors $h(\text{distance}(x,\gamma y))$ which are smaller than any power of z for z large).

If we had put a rank 2 cusp at infinity the argument would involve a sum over a lattice (or a double integral) and the factor $z^{2-\delta}$ emerges.

If μ has an atom at the cusp the ratio would increase as fast as z^{δ} as $x'\to\infty$

(3) The convergence is implied by the existence of μ of finite mass and exponent δ.

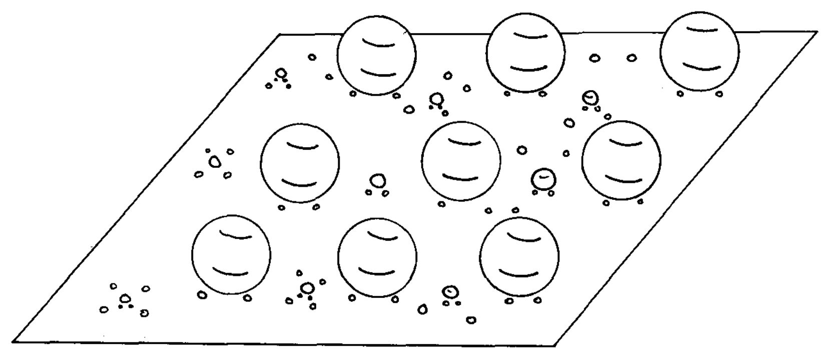

Fig. 6

vertically. (The h factors contribute nothing to the exponent.) Since $\delta>1/2$ in the rank one cusp case (and $\delta>1$ in the rank 2 cusp case) we have $\delta>1-\delta$ (and $\delta>2-\delta$ in the rank 2 cusp case). This contradiction proves:

PROPOSITION 1. *The measure μ constructed above has no atoms along orbits of parabolic cusps.*

Acknowledgement. I learned this rather nifty argument from Patterson who employed it in the Fuchsian case [P].

§3. The support of μ and divergence of the Poincaré series at δ

For our geometrically finite group there are finitely many horospheres whose orbits under Γ are disjoint and which rest on all the parabolic points (Fig. 6).

This follows from the Margulis decomposition of any hyperbolic manifold into thick and thin parts ([T] p. 5.55). If we form the hyperbolic convex hull of the limit set $\Lambda(\Gamma)$, remove these horoballs, and divide by Γ the result is compact ([T] p. 8.20).

It is clear then that any geodesic starting in the convex hull and heading towards a non-parabolic limit point leaves the horoballs infinitely often. Thus it reenters the compact region in the quotient infinitely often.

This shows the complement of the parabolic orbits in $\Lambda(\Gamma)$ consists of radial limit points ([S2] § 5) (a result of Beardon and Maskit who call these points of approximation).

By § 1 μ gives full measure to the radial limit set. The elementary Corollary 20 of [S2] § 5 then shows:

PROPOSITION 2. *The Poincaré series $g_s(x, y)$ diverges at $s=\delta$.*

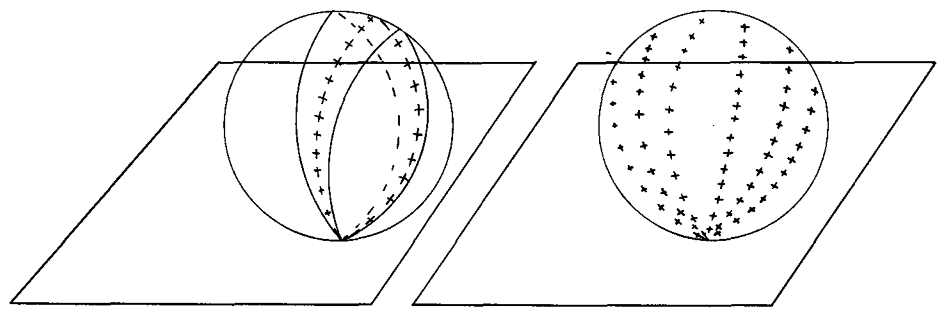

Fig. 7

Remark. This divergence was achieved by Patterson [P] in the Fuchsian case using a spectral analysis of the Laplacian. He speculated on the likelihood it could be obtained by more elementary methods.

§ 4. The uniqueness of μ and the ergodicity of μ, $\mu\times\mu$, and dm_μ

Now consider the associated invariant measure for the geodesic flow dm_μ mentioned in the introduction. By § 3 this measure is recurrent and so it is ergodic (see [S2] § 4 and § 5 for these points).

The ergodicity of dm_μ is equivalent to the ergodicity of Γ acting on $\mu\times\mu$. The ergodicity of $\mu\times\mu$, implies the ergodicity of Γ acting on μ.

These ergodicity statements are valid for any finite measure ν which satisfies $\gamma^*\nu=|\gamma'|^\delta\nu$ and which has no atoms along parabolic cusps. But there can be no atoms for any ν because of the divergence established in § 2.

To see this, consider the orbit of a point x on the horosphere of § 3 (Fig. 7).

Because the convex hull of $\Lambda(\Gamma)$ less the interiors of horoballs mod Γ is compact this orbit is a bounded hyperbolic distance from any point in the convex hull outside the horoballs.

In particular in the sum for the Poincaré series for $g_\delta(x,y)$ where x belongs to the convex hull the $\mathbf{Z}$ (or $\mathbf{Z}+\mathbf{Z}$) terms in the orbit of x lying on one horosphere add up to a quantity commensurable with the diameter of the horosphere raised to the δ power.

PROPOSITION 3. *Σ(diameters of horosphere)$^\delta=\infty$ where the sum is taken over the orbit of any cusp. (Diameter in metric on unit ball model.)*

This proves the ν above (which is finite) has no atoms along the cusp orbits. Thus

dm_ν, $\nu\times\nu$, and ν are ergodic. This implies $\nu=\mu$ because $m=(\mu+\nu)/2$ is also ergodic (for the same reason) and the ratio of μ or ν to m is invariant.

So we have proved,

THEOREM 1′. *There is one and only one probability measure μ on $\Lambda(\Gamma)$ satisfying for $\gamma\in\Gamma$, $\gamma^*\mu=|\gamma'|^\delta\mu$. Moreover $\mu\times\mu$ is ergodic for the action of Γ on $\Lambda(\Gamma)\times\Lambda(\Gamma)$.*

§ 5. The eigenfunction φ_μ

If the hyperbolic isometry σ^{-1} translates our fixed point x to a variable point p, the linear distortion $|\sigma'|$ on the sphere (with metric coming from rays emanating from x) only depends on p. Define a function φ_μ at p by

$$\varphi_\mu(p)=\int_{\text{sphere}}|\sigma'|^\delta\,d\mu.$$

If $\lambda=\delta(\delta-2)$, φ_μ is an eigenfunction of the Laplacian with eigenvalue λ. (See [P] and [S2] § 7.)

By definition φ_μ is Γ invariant. Thus by compactness and the continuity of φ_μ the size of φ_μ is controlled on the convex hull of $\Lambda(\Gamma)$ mod Γ outside the cuspidal ends.

If we reconsider the calculation of § 1, with the cusp in question put at infinity and ignoring the factors h(distance) which are not needed now after § 2, we see an estimate for $\varphi_\mu(p)$ in the cusp

$$\varphi_\mu(p)\asymp\begin{cases} z^{1-\delta} & \text{rank one cusp}\\ z^{2-\delta} & \text{rank two cusp}\end{cases},$$

where z is the height of p above x. (And $\asymp$ means the ratio is bounded by two constants.)

These estimates may be viewed in two ways. First, by definition φ_μ is the μ convex combination of basic λ-eigenfunctions (the horospherical functions) normalized to be 1 at x. The basic eigenfunctions behave like the functions $c\exp(-\delta\cdot(\text{hyperbolic distance}(p,y)))$ where y is near infinity (c is $\exp(\delta\cdot(\text{hyperbolic distance}(x,y)))$). Thus a calculation analogous to that of § 1 consisting of breaking μ up over the action of T leads to the above estimate.

Second, and more literally our unique μ is constructed from orbit Dirac mass approximations. Using these approximations the calculation of § 1 becomes the estimation of φ_μ.

Using the estimate on φ_μ we can prove

PROPOSITION 4. *φ_μ is square summable on a unit neighborhood of the convex hull of $\Lambda(\Gamma) \bmod \Gamma$.*

There is no problem on the compact part. The cuspidal ends have cross-sectional areas decreasing like $\exp(-(r\cdot\text{hyperbolic distance}))$ where r is the rank of the cusp. (See figure 3 and [T].) The growth of φ_μ is by the estimate of this section no more than $\exp((r-\delta)\cdot\text{hyperbolic distance})$. We use Harnack for positive eigenfunctions to extend the estimate to a unit neighborhood. We recall $\delta>r/2$ from § 2.

Since $2(1-\delta)<1$ if $\delta>1/2$ and $2(2-\delta)<2$ if $\delta>1$, φ_μ^2 is summable in each type of cuspidal end.

Remark. We don't use it here, but if $1<\delta$, φ_μ is also square summable outside the convex hull. In other words,

PROPOSITION. *If Γ is a geometrically finite group, and if $1<\delta$, φ_μ defines a positive square-summable eigenfunction on $\mathbf{H}^3/\Gamma$ for the eigenvalue $\delta(\delta-2)$.*

Proof. If $\delta>1$, φ_μ is actually decreasing in a rank one cuspidal end of the convex hull $\bmod \Gamma$. Then φ_μ belongs to L^2 on the ∂ of the unit neighborhood (the only noncompact part being the rank one cuspidal end).

We may assume the boundary of the neighborhood lifted to $\mathbf{H}^3$ is convex, smooth, and has bounded curvature. Consider the projection of points outside the neighborhood to the boundary. The points on a surface at distance R are compressed together by a factor e^{-R}. So the area element is larger by a factor of e^{2R}. The value of φ_μ has decreased by a factor of $e^{-\delta R}$. Thus the integral over the outside is a double integral

$$\int_{R=1}^{\infty}\int \varphi^2\, d\sigma_R\, dR = \int_1^{\infty} e^{(-2\delta+2)R}\int_\sigma \varphi_\mu^2\, d\sigma\, dR$$

where σ is the boundary of the unit neighborhood and σ_R is the surface at distance R. The integral is finite since $\delta>1$, it being understood the integral is only taken over a fundamental domain for the action of Γ.

§ 6. The finiteness of dm_μ and proof that $D=\delta(\Gamma)$

The invariant measure for the geodesic flow dm_μ is by definition ([S2]) just the arc length along all geodesics summed via the measure $\mu\times\mu/|x-y|^{2\delta}$ on their endpoints.

Similarly since φ_μ is the μ sum for ξ in the sphere at infinity of the basic horospherical eigenfunctions φ_ξ, φ_μ^2 at a point is a $\mu\times\mu$ sum of products $\varphi_\xi\cdot\varphi_\eta$. This product is invariant under translations along the geodesic connecting ξ and η and its value on the geodesic is commensurable to $1/|\xi-\eta|^{2\delta}$ since each φ_ξ is 1 at the center of the ball.

If we smear out the arc length mass along each geodesic uniformly on its unit tubular neighborhood and add these according to $\mu\times\mu/|\xi-\eta|^{2\delta}$ we obtain a smooth measure whose density function is dominated by a constant $\cdot\varphi_\mu^2$. Thus the total mass of dm_μ in the quotient by Γ is dominated by the $\int\varphi_\mu^2$ on a unit neighborhood of the support of the set of geodesics both of whose endpoints lie in $\Lambda(\Gamma)$. In particular we have for any group Γ,

PROPOSITION 5. *The total mass of dm_μ is dominated by $\int_N\varphi^2 dy$ where dy is the Riemann measure and N is a unit neighborhood of (the convex hull of $\Lambda(\Gamma))/\Gamma$.*

Remark. When $\delta>1$, we can smear out the geodesics using a function like the product $\varphi_\xi\cdot\varphi_\eta$ restricted to an orthogonal plane (and normalized to have total integral 1). Then the above argument gives for any group

PROPOSITION. *If $\delta>1$, then dm_μ has finite total mass iff φ_μ is square integrable. In fact the total mass of dm_μ equals the L^2 norm of φ_μ.*

Remark. This smearing and comparison argument was supplied by Bill Thurston, on request.

Using § 5 we have the

COROLLARY. *For a geometrically finite group the invariant measure dm_μ for the geodesic flow has finite total mass.*

Then using [S2], Theorem 25 we have,

COROLLARY. *For a geometrically finite group the Hausdorff dimension of $\Lambda(\Gamma)$ is the critical exponent $\delta(\Gamma)$.*

Actually, quoting Theorem 25 [S2] gives the Hausdorff dimension of the radial limit set equals $\delta(\Gamma)$. Since (§ 2) we only add a countable number of points to get $\Lambda(\Gamma)$ the Hausdorff dimension doesn't change.

For the reader who desires a more self-contained argument we note that in the discussion below of the density function of μ the argument for Theorem 25 is recapitulated.

§ 7. The density function of μ

A metrical study of a measure μ involves the density function $\mu(\xi, r)=\mu$(ball of radius r and center ξ). The pairs (ξ, r) can be labeled by polar coordinates from the center of the ball. We think of (ξ, r) as corresponding to the endpoint $v(\xi, r)$ of a geodesic v from the center x pointing towards ξ of hyperbolic length $t=\log 1/r$.

THEOREM. *The density function $\mu(\xi, r)$ satisfies*

$$\mu(\xi, r) \asymp r^{\delta}\varphi_{\mu}(v(\xi, r))$$

where $\asymp$ means the ratio is bounded above and below, and ξ belongs to $\Lambda(\Gamma)$.

Proof. The inequality $\mu(\xi, r) \leq$constant $r^{\delta}\varphi_{\mu}(v(\xi, r))$ is true for all groups. This follows from the formula § 5, $\varphi_{\mu}(p)=\int|\sigma'|^{\delta}\,d\mu$, and the fact that $|\sigma'|\asymp 1/r$ on a ball of radius r about ξ.

If the other inequality were not true we would have a sequence (ξ_i, r_i) so that the ratios $\mu(\xi_i, r_i)/r_i^{\delta}\ \varphi_{\mu}(v(\xi_i, r_i))\to 0$.

Change the base point of the group to $v(\xi_i, r_i)$ and form a geometric limit group, ([T] p. 9.1). The measures $|\sigma_i'|^{\delta}\cdot\mu$ where σ_i^{-1} takes the center to $v(\xi_i, r_i)$ have total mass $\varphi_{\mu}(v(\xi_i, r_i))$ by definition. Thus dividing by the total mass we can form a limit measure ν which will satisfy $\gamma^*\nu=|\gamma'|^{\delta}\nu$ for each γ in the limit group.

Let us suppose $\xi_i\to\xi$ and ξ^* is antipodal to ξ. Since $|\sigma_i|\asymp 1/r_i$ on a disk of size $k\cdot 1/r$ centered at ξ on the sphere for any k we conclude that ν is concentrated at the antipodal point ξ^*. It follows that the limit group is a subgroup of the parabolic group associated to ξ^*.

Since the ξ_i belong to $\Lambda(\Gamma)$ so does ξ. For a geometrically finite group one can check that one of these limit groups is either not entirely parabolic or contains nontrivial elements fixing ξ.

That is either a subsequence $v(\xi_i, r_i)$ stays in the compact part of the (convex hull)/Γ or it goes out a cuspidal end. The limits in these two cases differ from a parabolic group at ξ^*. This proves the theorem.

§ 8. The packing measures

If U is an open set in a metric space Λ and $\psi(r)$ is a positive function define the packing measure ν_{p} (relative to ψ) by

$$\nu_{\mathrm{p}}(U)=\lim_{\varepsilon\to 0}\sup_{P(\varepsilon)}\sum_i \psi(r_i)$$

where $P(\varepsilon)$ is the collection of packings of U with radii $\leqslant\varepsilon$. (A packing of U is a collection $B_1, B_2, \ldots$ of metric balls in U of radii $r_1, r_2, \ldots$ so that the sum of two radii is less than the distance between centers.)

The outer measure determined by the values $\nu_p(U)$, $\nu_p(X)=\inf_{X\subset U}\nu_p(U)$, defines a countably additive Borel measure because $\nu_p(X\cup Y)=\nu_p(X)+\nu_p(Y)$ if X and Y are separated by a positive distance [R].

Now we produce finite positive measures equivalent to packing measures.

Suppose μ is a finite positive measure with compact support in Euclidean n-space say. Define the *density function* of μ, $\mu(x, r)$, by

$$\mu(x, r) = \mu(\text{ball of radius } r, \text{ center } x).$$

PROPOSITION. *There is a set Λ of full μ measure so that μ is equivalent to the $\psi(r)$ packing measure of Λ if there are constants c, C so that*

$$c \leqslant \frac{\mu(x, r)}{\psi(r)} \leqslant_{\text{i.o.}} C$$

for almost all x and $r\leqslant r_0$. ($\leqslant_{\text{i.o.}}$ means that the inequality holds for a sequence of $r_i\to 0$, depending on x.)

Proof. If $B_1, B_2, \ldots$ is a packing of $U\subset\{x\colon \text{inequalities hold}\}=\Lambda$ by balls of radii $r_1, r_2, \ldots\leqslant\varepsilon\leqslant r_0$ then

$$\sum \psi(r_i) \leqslant \frac{1}{c}\sum \mu(B_i) \leqslant \frac{1}{c}\mu(U).$$

Thus the $\psi(r)$ packing measure ν_p is a finite measure and for every Borel set $\nu_p(X)\leqslant(1/c)\mu(X)$.

For each x in Λ, there is a sequence of balls B_i with radii $r_i\to 0$ so that $\mu(x, r_i)\leqslant C\psi(r)$. By the covering lemma ([F], Theorem 2.8.14) there are arbitrarily fine coverings of any $X\subset\Lambda$ by balls centered on X and falling into $k(n)$ collections of disjoint balls. One of these disjoint collections must contain $1/k(n)\cdot\mu(X)$ of the mass of μ. Thus for this packing $\Sigma\,\psi(r_i)\geqslant(1/C\cdot k(n))\mu(X)$.

For any fixed neighborhood U of X sufficiently fine such coverings define packings contained in U. Thus $\nu_p(U)\geqslant(1/C\cdot k(n))\mu(X)$ for any $U\subset X$. So $\nu_p(X)\geqslant(1/C\cdot k(n))\mu(X)$.

Thus ν_p and μ are equivalent and the Radon-Nikodym derivative $d\mu/d\nu_p$ lies in the interval $[c, C\cdot k(n)]$.

Remark. If one replaces the hypothesis on μ in the proposition by

$$c \underset{\text{i.o.}}{\leqslant} \frac{\mu(x,r)}{\psi(r)} \leqslant C, \quad r \leqslant r_0, \quad \mu \text{ a.a. } x,$$

then μ is equivalent to the *covering* Hausdorff measure relative to the function $\psi(r)$. This fact is well-known [F] and the proof is similar to the one above.

COROLLARY (of proof). *If μ is a probability measure satisfying*

$$\liminf_{r\to 0} \frac{\mu(\xi,r)}{\psi(r)} \quad \textit{is finite and positive,}$$

there is an increasing sequence of subsets $\Lambda_n \subset \operatorname{supp}\mu$ so that μ is equivalent to a weak limit of the packing measures ν_n of Λ_n.

Proof. Let Λ_n' be the set where the liminf belongs to $[1/n, n]$, and the lower inequality is true for $r \leqslant 1/n$. Then let $\Lambda_n \subset \Lambda_n'$ be such that $\mu(B \cap \Lambda_n)/\mu(B) > 1/2$ for balls B centered on Λ_n of radius $< r_n$. By the above proposition μ/Λ_n is equivalent to the packing measure ν_n of Λ_n. By the density theorem [F] 2.9.11 it follows easily that $\nu_n \to \mu$.

§ 9. Proof of Theorem 2

Let μ be the canonical geometric measure on the limit set of a geometrically finite Kleinian group. The metrically defined density function of μ, $\mu(\xi, r)$, is controlled by the theorem of § 7 which says

$$\mu(\xi,r) \asymp r^\delta \varphi_\mu(v(\xi,r)), \quad r \leqslant r_0.$$

Now suppose all cusps have rank $\geqslant \delta$ $(=D)$. Then by the estimates of § 5, φ_μ is bounded from below. By ergodicity of the geodesic flow § 6 almost all geodesics enter the compact part of the convex hull (mod Γ) infinitely often. Thus there are constants c, C so that

$$c \leqslant \frac{\mu(\xi,r)}{r^\delta} \underset{\text{i.o.}}{\leqslant} C.$$

Thus by § 8 μ is equivalent to the packing measure associated to r^δ. In particular the packing measure ν_{p} is finite and positive. Since a finite packing measure for r^δ clearly satisfies

$$\gamma^*\nu = |\gamma'|^\delta \nu, \quad \gamma \in \Gamma$$

we have $\mu = \nu_{\mathrm{p}}$ by uniqueness of such measures.

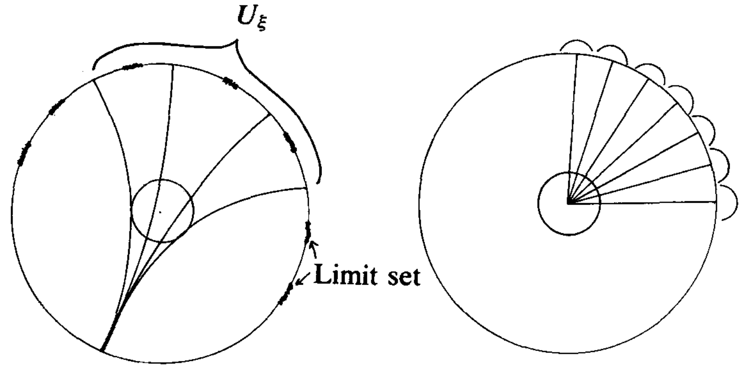

Fig. 8

Note that if some cusp has rank $>\delta$, the φ_μ is unbounded and the ratio $\mu(\xi, r)/r^\delta$ is unbounded from above for almost all ξ (again using ergodicity of the geodesic flow). Applying the covering lemma as in the proof of the proposition in § 8 shows the Hausdorff covering measure is zero. Thus $\nu_p \neq \nu_c$.

Similarly if all cusps have rank $\leqslant\delta$ we arrive at the dual inequalities

$$c \underset{\text{i.o.}}{\leqslant} \frac{\mu(\xi, r)}{r^\delta} \leqslant C$$

and μ equals the Hausdorff covering measure relative to r^δ. Also if some cusp has rank $<\delta$, then the ratio $\mu(\xi, r)/r^\delta$ is not bounded from below. Using the covering lemma shows $\nu_p = \infty$.

This proves most of the entries in the table following Theorem 2.

For the rest we recall Beardon's result (indicated in § 3) that $\delta > k/2$ (k any rank of a cusp), and note that if $D=2$ the Poincaré series diverges at $s=2$ so Γ cannot have any points of discontinuity and thus there are no rank one cusps.

§ 10. Counting orbits of the geodesic flow: entropy

Let us estimate the maximal number $N(\varepsilon, t, B)$ orbits of the geodesic flow which start in a ball B and are ε apart some time before time t (Fig. 8). (We only consider geodesics both of whose endpoints are in the limit set $\Lambda(\Gamma)$.)

For each ξ in $\Lambda(\Gamma)$ define a set U_ξ as indicated in the figure 8 (a).

Figure 8 (a) shows we need to consider rays emanating from B with endpoints in $U_\xi \cap \Lambda(\Gamma)$. Then a finite number of these U_ξ's will cover. Now if two geodesics are ε-

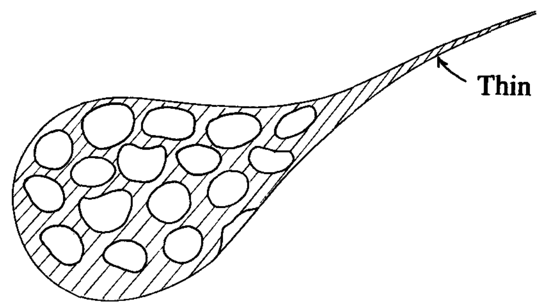

Fig. 9

apart in $\mathbf{H}^3/\Gamma$ some time before time t, their lifts to $\mathbf{H}^3$ will also be ε-apart some time before time t. Conversely, if two geodesics starting from nearly the same point in (a lift of) B are ε apart at some time $s<t$ they project to geodesics which at time s are either ε-apart in $\mathbf{H}^3/\Gamma$ or which lie in the ε-thin part of $\mathbf{H}^3/\Gamma$. (Because outside the ε-thin part the covering projection is an isometry on ε-balls.) (Fig. 9.)

Let us ignore this latter complication for the moment. If so, we end up counting the number of points in the limit set intersect U no two of which are closer than εe^{-t}.

We can use the canonical geometric measure to estimate this number. For example, if there are no cusps $\mu(\xi, r) \asymp r^\delta$ since φ_μ is bounded above and below §5. Then there are no more than $\asymp (2/\varepsilon)^\delta e^{\delta t}$ such points. Similarly, in a cover by ε balls there must be at least $\asymp (1/\varepsilon)^\delta e^{\delta t}$ such balls. If we take the log of this number divide by t, take the lim sup as $t \to \infty$ and then the limit as $\varepsilon \to 0$ we get Bowen's definition of the topological entropy of the geodesic flow (restricted to those orbits with both endpoints in the limit set). Thus we have (since the complication about ε-thin parts is not relevant if there are no cusps),

THEOREM. *For a geometrically finite group without cusps, the topological entropy*(¹) *of the geodesic flow (on the compact set of geodesics with endpoints in the limit set) is the Hausdorff dimension of the limit set* $\Lambda(\Gamma)$.

It is natural to ask what is the *measure theoretic entropy* relative to the finite invariant measure dm_μ constructed from the canonical geometric measure μ. We can see the answer is still the Hausdorff dimension (even in case there are cusps). In particular, dm_μ above is a measure which maximizes the entropy.

(¹) Calculated using the natural logarithm.

Let us divide the (convex hull)/Γ minus neighborhoods of cusps (the ε-thin part of $\mathbf{H}^3/\Gamma$) into finitely many cells of small diameter. By ergodicity relative to dm_μ, a geodesic only spends a small fraction f of its time in the thin part or near the walls of these cells (the shaded part of the figure).

Thus if we only distinguish orbits which are in different cells for more than a fraction $2f$ of the time up to time t we are again counting metrically separated orbits in $\mathbf{H}^3$ starting near one point of a lift B. The calculation of the exponential growth rate of the number goes as before with two further considerations.

(i) We are looking for $\varepsilon e^{-t'}$ separated points again but t' lies in $[(1-2f)t, t]$. Since f is arbitrarily small this complication won't matter for the growth rate.

(ii) Now the μ mass of a relevant ball of radius $e^{-t'}$ is $e^{-t'\delta}\varphi_\mu(v(\xi, t'))$. The factor φ_μ however has arbitrarily small exponential effect *for most geodesics* because dm_μ almost all geodesics are within $o(t)$ of the compact part ([S2] Corollary 19).

We find the exponential growth rate of the number of orbits counted using these partitions and looking most of the time works out to be δ. According to Feldman and [BK] (as explained to me by Dan Rudolph) these rates (which in our case are constantly δ) converge to the measure theoretic entropy. This proves the

THEOREM. *For a geometrically finite group the measure theoretic entropy*(²) *of the geodesic flow relative to the canonical measure dm_μ is the Hausdorff dimension of the limit set.*

Remark. (Pesin theory of exponents.) For the geodesic flow there is a familiar Hopf-Anosov picture of expanding and contracting foliations transverse to the flow. There is a uniform expansion (contraction) in two senses: for a time t flow at *each* point, *each* expanded (contracted) vector is expanded (contracted) by $e^t(e^{-t})$. The measure dm_μ conditioned to the leaves of the expanding (contracting) foliation is by construction equivalent to the geometric measure μ of dimension δ. It is expanded (contracted) uniformly by $e^{\delta t}(e^{-\delta t})$. Thus if one imagines Pesin theory of exponents and the Pesin-Margulis entropy formula being valid in a fractal dimension δ, the above result is consistent. Note that when $\mathbf{H}^3/\Gamma$ has finite volume, the answer 2 for the entropy does follow formally from the Pesin-Margulis formula.

(²) Calculated using the natural logarithm.

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