J. DIFFERENTIAL GEOMETRY
18 (1983) 723–732

THE DIRICHLET PROBLEM AT INFINITY FOR A NEGATIVELY CURVED MANIFOLD

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We consider a complete Riemannian n-manifold M which is connected and simply connected, and has sectional curvatures k bounded between two finite negative constants, i.e., $-b^2 \leqslant k \leqslant -a^2$. Such a manifold shares many features with hyperbolic space, the carrier of non-Euclidean geometry. M is diffeomorphic to an open n-ball and has a natural compactification $\overline{M}$ which adds a topological $(n-1)$-sphere at infinity. The compactification $\overline{M}$ is *homeomorphic* to a closed n-ball. Since the ideal points of $\overline{M}$ are constructed by an infinite dynamical process (asymptotic classes of geodesics), it is rare that one may think of $\overline{M}$ as a differentiable object.

Probability theory provides a measure class on $\partial\overline{M}$, which is the "harmonic measure class". If $m \in M$, and $A \subset \partial\overline{M}$ is a Borel set, then define $\nu_m(A)$ to be the probability of a random path (see Appendix) starting at m tending to a limit in A. One knows (Prat [10]) that almost all paths tend to limits on $\partial\overline{M}$, and we shall give a quantitative discussion of the phenomenon below. One also knows that $m \overset{h_A}{\to} \nu_m(A)$ defines a bounded harmonic function (possibly constant) with values in $[0,1]$. By the maximum principle h_A is either $\equiv 0$, $\equiv 1$, or takes values in $(0,1)$. In any case, $\nu_m(A) > 0$ for one m if and only if $\nu_m(A) > 0$ for all m. Thus all the hitting measures ν_m on $\partial\overline{M}$ are absolutely equivalent and define one measure class, the "harmonic measure class".

Showing that the harmonic measure class is nontrivial would solve the following problem which the author learned from Yau and Malliavin.

Problem. Does a complete, simply connected, negatively curved manifold (with curvatures bounded between two finite negative constants) have *any* nonconstant bounded harmonic functions?

Conformal theory in dimension two implies the answer is affirmative there. For the universal cover of a compact negatively curved manifold of any dimension, the answer is also affirmative. More generally, a nonamenable covering of any Riemannian manifold has nonconstant bounded harmonic functions; Lyons-Sullivan (1982).

Received December 7, 1982, and, in revised form, June 28, 1983.

The aim of this paper is to solve the problem affirmatively in general. In fact, the Dirichlet problem relative to infinity can be solved. We use probability on Riemannian manifolds; independently and simultaneously M. T. Anderson solved the Dirichlet problem at infinity for negatively curved manifolds.[1]

Theorem 1. *Each continuous function on the sphere at infinity $\partial\overline{M}$ has a continuous harmonic extension to $M \cup \partial\overline{M}$.*

Of course, the harmonic extension is unique using continuity at infinity and the maximum principle on the interior. Thus M has many nonconstant harmonic functions which are bounded.

Theorem 1 is deduced from the following statement using "the Poisson formula".

Theorem 2. *The harmonic measure class on $\partial\overline{M}$ is positive on each nonvoid open set. In fact if m_i in M converges to m_∞ in $\partial\overline{M}$, then the Poisson hitting measures ν_{m_i} tend weakly to the Dirac mass at m_∞.*

The proof is based on a quantitative study of how the negative curvature pushes random paths out to infinity. The estimates, because of the variable curvature, do not show the harmonic measure class is equivalent to the geodesic measure class, and the author doubts that this is always so in high dimensions, e.g., $n \geqslant 4$.

Now we state what properties of the metric are used to prove these theorems. There are two, one corresponding to the upper bound and one to the lower bound on curvature.[1]

(A) First, we use that the geodesics rays from any point give a polar coordinate system for M and the geodesics diverge at least at a definite exponential rate. The negative upper bound $-a^2$ on curvature implies this. Then a is a lower bound for the exponential rate of divergence.

(B) Second, let $p(t, x, y; B)$ be the fundamental solution of the heat equation vanishing at the boundary of the unit ball centered at x_0 in M. We need the existence of a time $\tau > 0$ and a positive member $\mu(\tau)$ both independent of x_0, so that the total mass of $p(t, x, y; B)$ is $\geqslant \mu(\tau)$. This kind of estimate is true in the presence of a lower bound on the Ricci curvature (Cheeger, Yau (1980), Cheval-Feldman (1983)).

1. Notation and sketch of proof

Here we describe the structure of the proof of Theorem 2. The required details are given in §2 and the Appendix. The basic idea is easy to state and to understand. Consider a fixed point m in M and the distance $d(x)$ as measured

[1] In [12] Kifer describes his solution of the problem under a slightly stronger bounded curvature condition.

from m. The exponential divergence of geodesics from m means the level spheres have a definite convexity towards m. Thus a randomly moving particle has a definite tendency to move away from m.

We imagine this rate to be linear in time. But then the apparent angle swept out by the particle is an exponentially decreasing function and so integrable. Thus the particle converges in angle at infinity. Moreover, if the particle starts at x_0 far from m, this convergence tends to take place near the ray $(\overrightarrow{xm_0})$. Thus the harmonic measure class on ∂M is nontrivial, and the idea of the proof is complete.

A technical problem arises because the motion of a random particle is not locally rectifiable. To remedy this and to make the calculation simpler, we replace each random path $w(t)$ by a random sequence $(x_0, x_1, \cdots)$. The sequence is within distance 1 of every point of the path and has for our purposes an equivalent probability structure. (Also this construction makes sense on any complete Riemannian manifold; see Lyons-Sullivan (1982) for refinements.)

Inductively, given a finite sequence $x_0, x_1, \cdots, x_n$ the next point x_{n+1} is distributed by a probability measure $\mu(x_n, \tau)$ in the unit ball B_{x_n} centered at x_n, and so on. The measures $\mu(x_n, \tau)$ are defined in one way using random paths, and a stopping time which directly implies the probability structure on sequences will have our required properties.

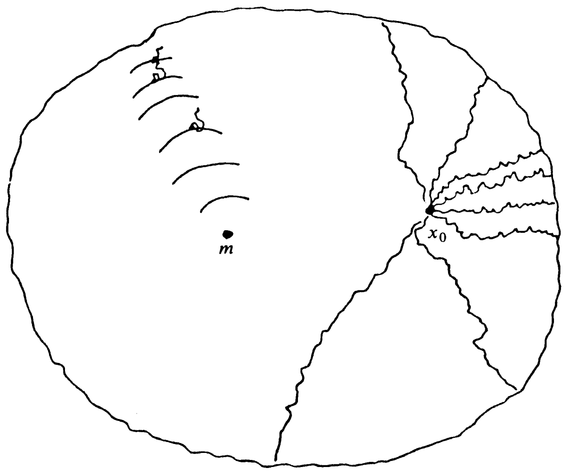
The measure $\mu(x_n, \tau)$ is also defined by the heat kernel $p(\tau, x_n, y; B_{x_n})$ in B_{x_n} plus a boundary measure $\nu(\tau, x_n)$ on ∂B_{x_n}—the time τ harmonic measure of ∂B_{x_n} relative to x_n. This allows one to show for an appropriate $\varepsilon > 0$ the expected value of $e^{-\varepsilon d(x)}$ over B_{x_n} relative to $\mu(x_n, \tau)$ is majorized by $ce^{-\varepsilon d(x_n)}$ for some $c < 1$. A trivial calculation then shows the expected value of

$$e^{-\varepsilon d(x_0)} + e^{-\varepsilon d(x_1)} + \cdots + e^{-\varepsilon d(x_n)} + \cdots$$

over all sequences starting at x_0 is $\leqslant Ce^{-\varepsilon d(x_0)}$ for $C = 1 + c + c^2 + \cdots$. Thus the sum of the infinite series is finite for almost all sequences, and even small for a definite fraction of sequences if $d(x_0)$ is large.

But because of the exponential divergence of geodesics determined by the negative curvature $-b^2 \leqslant k \leqslant -a^2$, this sum, for ε smaller than a, majorizes the total angle swept out by a sequence as viewed from m.

By construction the random path and its associated sequence have the same limit at ∞. The above estimates show definite proportions of sequences hit infinity in small disks about the end point of any given ray. These estimates prove Theorem 2, which in turn formally implies Theorem 1.



2. The proof using the notation of §1

(1) The vector field $\operatorname{grad}(d(x))$ is the unit outward normal field to spheres concentric about m. From the Jacobi equation and property (A) of the metric the field diverges apart at an exponential rate at least as quickly as in the space of constant curvature $-a^2$. This latter rate is a. Thus the divergence or volume distortion of this field is $\geqslant (n-1)a$, where $n =$ dimension M. In other words $\Delta d(x) = \operatorname{div} \operatorname{grad} d(x) \geqslant (n-1)a$.

Since $|\operatorname{grad}(d(x))| = 1$ and $\Delta e^f = (|\operatorname{grad} f|^2 + \Delta f)e^f$, if $0 < \varepsilon < (n-1)a$, then $\Delta e^{-\varepsilon d} \leqslant c(\varepsilon)e^{-\varepsilon d}$ for some $c(\varepsilon) < 0$. (Putting $f = -\varepsilon d$, $(|\operatorname{grad} f|^2 + \Delta f) \leqslant \varepsilon^2 + \varepsilon(-(n-1)a) = \varepsilon(\varepsilon - (n-1)a)$.)

(2) For each x in M let $\mu(x, t)$ be the probability measure in the unit ball B about x, which is defined by the formula

$$\mu(t, x) = p(t, x, y; B)\, dy + \nu(t, x)$$

of Appendix (a). Here $p(t, x, y; B)$ is the heat kernel vanishing on ∂B, and $\nu(t, x)$ is the boundary measure so that

$$h(x) = \int p(t, x, y) h(y)\, dy + \int \varphi\, d\nu$$

for all bounded harmonic functions in B with continuous boundary values φ.

In Appendix (d) one sees that all these measures are positive and that the mass of $p(t, x, y; B)\, dy$ is monotone decreasing. Now let $t = \tau$ be the time given by property (B) of the metric.

Proposition. *For $0 < \varepsilon < (n-1)a$ there exists $c < 1$ so that*

$$\int e^{-\varepsilon d(x)}\, d\mu(x_0, \tau) \leqslant c e^{-\varepsilon d(x_0)} \quad \textit{for any } x_0 \textit{ in } M.$$

Proof. Write $e^{-\varepsilon(d(x)-d(x_0))}$ as a harmonic function H on B_{x_0} with the same boundary values plus a function U vanishing at the boundary. By the definition of $\mu(x, \tau)$, $\int H\,d\mu(x_0, t) = H(x_0)$. Since U vanishes at the boundary, we are left to calculate $\int p(\tau, x_0, y; B)U(y)\,dy = V(x_0, \tau)$.

Since $\Delta H = 0$, we have $\Delta U \leqslant c(\varepsilon) < 0$ by (1). Thus[2]

$$\Delta\left(\int p(\tau, x_0, y; B)\,U(y)\,dy\right)(x_0) = \int p(\tau, x_0, y; B)\Delta U(y)\,dy$$
$$\leqslant c(\varepsilon)\cdot \text{total mass}\, p(\tau, x_0, y; B)\,dy.$$

For $t \leqslant \tau$ the second factor being decreasing is $\geqslant$ total mass $p(\tau, x_0, y; B)\,dy$ which is $\geqslant \mu(\tau)$ by our basic assumption (B) on the metric. Using the heat equation we have

$$\frac{\partial}{\partial t}V(x_0, t) = (\Delta V)(x_0, t) \leqslant c(\varepsilon)\mu(\tau).$$

Thus

$$V(x_0, \tau) \leqslant V(x_0, 0) + c(\varepsilon)\cdot\mu(\tau)\cdot\tau = U(x_0) + c(\varepsilon)\cdot\mu(\tau)\cdot\tau,$$

where $c(\varepsilon) < 0$, $\mu > 0$, $\tau > 0$.

Putting all this together,

$$\begin{aligned}\int e^{-\varepsilon d(x)}\,d\mu(x_0, \tau) &= e^{-\varepsilon d(x_0)}\int e^{-\varepsilon(d(x)-d(x_0))}\,d\mu(x_0, \tau)\\ &= e^{-\varepsilon d(x_0)}\left(\int H(x)\,d\mu(x_0, \tau) + \int U(x)\,d\mu(x_0, \tau)\right)\\ &= e^{-\varepsilon d(x_0)}\left(H(x_0) + \int U(x)p(\tau, x, y; B)\,dy\right)\\ &\leqslant e^{-\varepsilon d(x_0)}(H(x_0) + U(x_0) + c(\varepsilon)\cdot\mu(\tau)\cdot\tau)\\ &= e^{-\varepsilon d(x_0)}(1 + c(\varepsilon)\mu(\tau)\tau).\end{aligned}$$

So $c = 1 + c(\varepsilon)\mu(\tau)\cdot\tau < 1$ works for the statement of the proposition.

(3) Now for each x_0 in M consider the space $\{(x_0, x_1, \cdots, x_n)\}$ of sequences $x_0, x_1, \cdots, x_n$ in M where distance $(x_{m+1}, x_m) \leqslant 1$. Put a probability measure $P_n^{x_0}$ on these so that inductively $P_1^{x_0} = \mu(x_0, \tau)$ and inductively

$$P_n^{x_0} = \int \mu(x_{n-1}, \tau)P_{n-1}^{x_0},$$

where the formula makes sense using the fibration

$$\{(x_0, x_1, \cdots, x_n)\} \to \{(x_0, x_1, \cdots, x_{n-1})\}.$$

[2] The heat operator $e^{\Delta t}$ and Δ commute, and U is in the domain of Δ.

Namely, $P_{n-1}^{x_0}$ is defined on the base $\{(x_0, x_1, \cdots, x_{n-1})\}$ by induction; on each fibre $\{x\colon \operatorname{dist}(x, x_{n-1}) \leqslant 1\}$ we have the measure $\mu(x_{n-1}, \tau)$, and we add these up using $P_{n-1}^{x_0}$ to get $P_n^{x_0}$ on the total space $\{(x_0, x_1, \cdots, x_n)\}$.

(4) We calculate the expected or average value relative to $P_n^{x_0}$ of the function on sequences

$$e^{-\varepsilon d_0} + e^{-\varepsilon d_1} + \cdots + e^{-\varepsilon d_n},$$

where $d_n = \operatorname{distance}(x_n, m)$. By the proposition

$$\int e^{-\varepsilon d_n}\, d\mu(x_{n-1}, \tau) \leqslant c e^{-\varepsilon d_{n-1}} \quad \text{for } c < 1.$$

Thus

$$\int e^{-\varepsilon d_n} dP_n^{x_0} \leqslant c \int e^{-\varepsilon d_{n-1}} dP_{n-1}^{x_0} \leqslant c^n \int e^{-\varepsilon d_0} dP_0^{x_0} = c^n e^{-\varepsilon d_0},$$

and the average value of the sum is majorized by

$$e^{-\varepsilon \operatorname{dist}(x_0, m)}\left(1 + c^2 + \cdots + c^n\right).$$

(5) The measures $P_n^{x_0}$ determine a probability measure P^{x_0} on the compact space S^{x_0} of all sequences $(x_0, x_1, x_2, \cdots)$ with fixed x_0 and $\operatorname{dist}(x_m, x_{m+1}) \leqslant 1$ which is uniquely defined by the equations: nth projection $P^{x_0} = P_n^{x_0}$. (These equations define a decreasing family of compact convex sets in the space of probability measures on S^{x_0}. The intersection of these is nonempty. This proves existence. Uniqueness follows because a continuous function on S^{x_0} is approximately constant on fibres of this nth projection for n sufficiently large.)

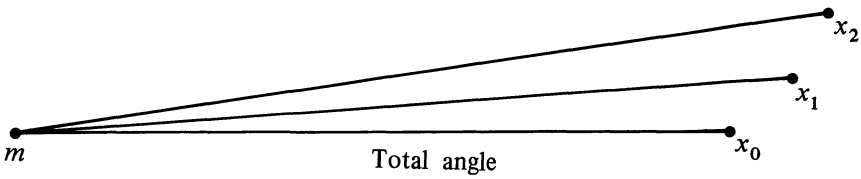
(6) By the monotone convergence theorem and (4) above the integral of $\lim_n(e^{-\varepsilon d_0} + \cdots + e^{-\varepsilon d_n})$ over S^{x_0} with respect to P^{x_0} is majorized by

$$e^{-\varepsilon \operatorname{dist}(x_0, m)}\left(1 + c + c^2 + c^3 + \cdots\right).$$

Thus again by monotone convergence, $e^{-\varepsilon d_0} + e^{-\varepsilon d_1} + \cdots$ is finite for P^{x_0} almost all sequences $(x_0, x_1, \cdots)$.

The size of the angle between x_m and x_{m+1} is majorized by $e^{-\varepsilon d_m}$ if $\varepsilon \leqslant a$ by our other basic assumption on the metric. Thus for P^{x_0} almost all sequences, the total angle from m swept out is finite. Moreover, measuring angular distance θ between an arbitrary ray from m and the one through x_0 we have shown

$$(*) \qquad \begin{aligned} &\text{probability}\left(\text{total angle of } (x_0, x_1, \cdots) > \theta\right) \\ &\qquad \leqslant \left(\frac{1}{1-c}\right)\frac{1}{\theta} e^{-\varepsilon \operatorname{dist}(x_0, m)}. \end{aligned}$$



Total angle

(7) Now consider $\mathcal{P}^{x_0}$, the space of continuous paths starting at x_0. Denote by μ_{x_0} the Wiener measure on the paths. The Wiener measure is characterized by the Markoff property and the property that it projects to the heat density $p(t, x, y)\,dy$ under the map which sends every path to its position at time t. (See the book of Stroock and Varadhan [11] and the Appendix for more details.)

For a path $\omega(t)$ starting at $\omega(0)$ define $\tau_1(\omega) \in (0, \infty)$ to be the minimum of τ (as in (2)) and $\inf\{t\colon \text{distance}(\omega(t), (0)) = 1\}$. Define $\tau_n(\omega)$ inductively as the minimum of $\tau_{n-1}(\omega) + \tau$ and $\inf\{t\colon \text{distance}(\omega(\tau_{n-1}), \omega(t)) = 1\}$. Note that

$$\tau_n(\omega) = \tau_{n-1}(\omega) + \tau_1(\omega(\tau_{n-1}(\omega))).$$

Define a map {continuous paths starting at x_0} to {sequences starting at x_0} by

$$\omega(t) \xrightarrow{\pi} (\omega(0), \omega(\tau_1), \omega(\tau_2), \cdots),$$

where $\omega(0) = x_0$.

Proposition. (i) *The map π projects Wiener measure μ_{x_0} on the measure P^{x_0} constructed in* (5).

(ii) *For almost all paths* $\lim \tau_n(w) = \infty$.

Proof (1) To prove (i) we need only to show

$$\omega \underset{\pi_n}{\rightarrow} (\omega(0), \omega(\tau_1), \cdots, \omega(\tau_n))$$

projects Wiener measure onto the measure μ_{x_0} constructed in (3). This is true for $n = 0$; the heat kernel at time 0 gives the Dirac mass at x_0. If true for $n - 1$, it will be true for n if and only if the conditional distribution of $\omega(\tau_n)$ given $\omega(\tau_{n-1}) = x_{n-1}$ say is $\mu(x_{n-1}, \tau)$. But $\mu(x_{n-1}, \tau)$ is defined by random motion started at x_{n-1} and stopped at τ or at the first hit on boundary of the unit ball around x_{n-1} (Appendix). By the Markov property this measure agrees with the conditional measure on $\omega(\tau_n)$ for a path stopped at time τ_{n-1} at x_{n-1} and restarted up and run until τ_n. (In fewer words $\mu(x_0, \tau) = (\pi_1)_* W^{x_0}$ and

$$\tau_n(\omega) = \tau_{n-1}(\omega) + \tau_1(\omega(\tau_{n-1}(\omega)))$$

by definition.)

So by the Markov property the conditional of $\omega(\tau_n(\omega))$ given$\omega(\tau_{n-1}(\omega))$ is μ_x where $x = \omega(\tau_{n-1}(\omega))$.) This proves (1).

(2) The probability $\tau_1(\omega) \geqslant \tau$ is the total mass of $p(\tau, x, y; B)$ where $\omega(0) = x$. This is $\geqslant \mu(\tau)$. Consider those paths for which $\tau_{n+1} - \tau_n < \tau$ for k values of $n = n_1, n_2, \cdots, n_k$. The measure of this set is $\leqslant (1 - \mu(\tau))^k$ by the Markov property. Thus the measure of the set of paths with $\tau_{n+1} - \tau_n < \tau$ eventually is zero. In other words for almost all paths, τ_{n+1} is τ more than τ_n infinitely often. This proves $\lim \tau_n = \infty$ for W^{x_0} almost all paths. This completes the proof of the proposition.

(3) On each path ω we have selected a sequence $(x_0, x_1, \cdots)$ and shown the sequence converges in angle almost surely because the total angle swept out is finite almost surely. Note that $d_n \to \infty$ almost surely because $e^{-\varepsilon d_n} \to 0$ almost surely.

We conclude almost all paths converge in angle at infinity and by (7), part (i), and (6) we have a lower bound (*) on the proportion of paths which converge near a given ray (mx_0). This estimate gives the second statement of Theorem 2 which of course implies the first statement. Theorem 1 now follows formally using the "Poisson formula" to solve the Dirichlet problem. If φ on $\overline{M}$ is continuous, then

$$h_\varphi(m) = \int \varphi(y)\, d\nu_m(y)$$

is a harmonic function (Appendix) continuous on $M \cup \partial\overline{M}$ by Theorem 2.

Appendix (*potential theory and probability theory on M*)

(a) Potential theory on M. For each compact connected region $B \subset M$ with smooth boundary let $p(t, x, y; B)$ denote the fundamental solution of the heat equation vanishing on the boundary of B. For each continuous function φ on ∂B let ϕ be the harmonic extension to B.

Define a measure $\nu(t, x; \partial B) = \nu$ on ∂B for x in B so that

$$\phi(x) = \int_B p(t, x, y; B)\phi(y)\, dy + \int_{\partial B} \varphi(\xi)\, d\nu.$$

The minimal heat kernel on M is defined to be

$$p(t, x, y; M) = \sup_\alpha p(t, x, y; B_\alpha),$$

where B_α varies over all compact connected smooth sub regions. The minimal heat kernel of interior B for B compact is $p(t, x, y, B)$.

(b) Probabilistic completeness of M. A Riemannian manifold is said to be *complete in the sense of probability* if the minimal heat kernel satisfies $\int p(t, x, y; M)\, dy = 1$ for all x in M.

One knows that a metrically complete Riemannian manifold is complete in the sense of probability if the Ricci curvature is bounded from below (Yau). From now on assume that M is complete in the sense of probability.

(c) Continuous paths in M. A path in M is a continuous map ω of $[0, \infty)$ into M. Let Ω be the set of all paths with the compact open topology and the corresponding Borel sets.

(d) The Wiener measure on paths. For each x in M there is the Wiener probability[3] measure μ_x on $\Omega_x \subset \Omega$, the paths starting at x ($\Omega_x = \{\omega \in \Omega \mid \omega(0) = x\}$). Just one of the measures μ_x determines all the others μ_y as well as all of the potential theory $\{p(t, x, y; B)\}$, the bounded harmonic functions on M, etc.

(1) Namely, if T_a denotes the time a shift map on paths $T_a\omega(t) = \omega(t + a)$, then

$$(T_a)_*\mu_x = \int (p(a, x, y)\, dy)\mu_y,$$

i.e., μ_x shifted by T_a is the convex combination of the other μ_y averaged by the heat density. In particular, if we map Ω_x to M by evaluating at time a we get the heat density $p(a, x, y)\, dy$.

(2) If $t > 0$, x belongs to B as in (a) and we map Ω_x to B by $\omega \to \omega(t)$ if $\omega(s) \in B$ for $0 \leqslant s \leqslant t$ (otherwise $\omega \to \omega(\tau)$ where $0 \leqslant \tau < t$ and τ is the maximum time so that $\omega(s) \in B$ for $0 \leqslant s \leqslant \tau$), then μ_x maps to $\mu(x, t) = p(t, x, y; B) + \nu(x, t)$ defined in (a).

(3) Finally, there is a bijective correspondence between bounded harmonic functions on M and the shift invariant elements of $L^\infty(\Omega)$. The measure class on Ω is that defined by dy on M and the collection μ_y. A representation formula is $h(x) = \int \varphi(\omega)\, d\mu_x$, where φ is a shift invariant bounded measurable function on Ω, and h is bounded harmonic on M. If h is given, then $\varphi(\omega)$ is defined by $\lim_{t \to \infty} h(\omega(t))$ which exists by the Martingale convergence theorem (Furstenberg).

Conclusion. Let M be as in the first part of the paper—complete, simply connected, with sectional curvatures between two negative constants; $-b^2 \leqslant k \leqslant -a^2$. Then M is probabilistically complete (Yau) and the above discussion is valid. We have shown using the random sequences that almost all paths

[3] Assuming the completeness in the sense of probability, $\int p(t, x, y)\, dy = 1$ for all $t \geqslant 0$, x in M.

converge in angle—namely to a point on the sphere at infinity $\partial\overline{M}$. If ϕ is any continuous function (or even any bounded Borel function), define φ on Ω almost everywhere by $\varphi(\omega) = \phi(\lim_{t\to\infty}\omega(t))$. Then φ is a shift invariant, and the representation formula becomes the Poisson formula. This is so because the hitting measure ν_x on $\partial\overline{M}$ is by definition the image of μ_x by the map $\omega \to \lim_{t\to\infty}\omega(t)$.

For (3) see Dynkin & Yushkevich (1968), Doob (1953), and Furstenberg (1971). For the analytical constructions of (1) and (2) of μ_x see Malliavin (1975), McKean (1969) and the elegant Martingale formulation of Stroock-Varadhan (1975).

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