CONFORMAL DYNAMICAL SYSTEMS*

by

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1. Iteration

In many parts of geometry and analysis the situation arises in which one considers a set of differentiable transformations of an underlying manifold obtained by iterated composition of a given set of initial or generating transformations. One is then interested in the possible positions and shapes of the images of a neighborhood of a general point by the total set of transformations. Questions as to whether the images of an initial neighborhood return infinitely often to intersect the initial neighborhood or whether the iterated images wander off to accumulate elsewhere -- perhaps even at ∞ if the underlying space is non-compact -- are the basic questions of topological dynamics.

The distribution of the images of a generic point relative to a given measure on the space is the subject of measurable dynamics or ergodic theory (a name especially used for a singly generated set of transformations -- because of the Von Neumann-Birkhoff ergodic theorem). The features relative to topology and measure theory can be quite different when the iterated transformations squeeze of distort the neighborhoods in an unbounded manner.

For example, let C denote a Cantor set of positive measure on the circle S^1 bounding the unit disk D. Let Γ denote the group of transformations of S^1 in $PSL(2,R)$ generated by the set of non-Euclidean reflections of the disk D which interchanges each

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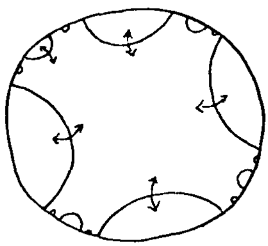
complementary interval I of the Cantor set with (S^1-I). Then every Γ orbit of a point on S^1 is dense in S^1. But the orbit of the Cantor set partitions a part of the circle having positive linear measure zero into disjoint copies of the Cantor set.



Γ is the group generated by reflections in the sides of non-Euclidean convex hull of the Cantor set.

On the other hand it is easy to show that a group of transformations of a manifold which do not distort distance at all and which has dense orbits acts ergodically -- given two sets A and B of positive measure some image of A intersects B in a set of positive measure.

In the above example the transformations have an unbounded amount of geometric distortion. Thus we are faced with the problem of understanding the distortion produced by large iterated compositions of transformations from a given set. If the set of transformations under study form a continuous group it is only the distortion transversal to the orbit which needs to be considered. To make the problem tractable the generating set will usually be finite or at least compact. For example the phenomenon of the example cannot happen for a finitely generated subgroup of projective transformations, PSL(2,R), acting on the circle. (See Theorem 1.)

Of course, for any composition of differentiable transformations the tangent map is the product of the tangent maps of the successive factors. Thus for the linear part of the distortion we will have a large, random product of matrices taken from a bounded set. Understanding such random products is a field in itself. The main tools are the boundary theory of Lie groups of Furstenburg as used in work of Margulis, the multiplicative ergodic theorem of Osilidec, and the Smale, Anosov, Pesin approach.

Actually, there is already a rich supply of dynamical examples when all the derivatives encountered are scalar multiples of orthogonal transformations (≡ conformal transformations). These include all differentiable examples in a space of one real dimension, all complex analytic examples on a one complex dimensional manifold, and any group of Moebius transformations of the n-sphere. For these conformal dynamical systems the understanding of the linear distortion simplifies to the problem of determining the scalar multiples or conformal factors. This is of course an abelian computation as for dimension one, and we shall see common features of 1-dimensional systems and conformal systems in higher dimensions.

After one has come to terms with the linear part of the distortion of an iterated composition (the tangent map) there remains the difficult problem of the non-linearity, the unbounded deviation of the iterated composition on a neighborhood from any linear approximation.

For conformal transformations the non-linearity problem is similar to the non-linearity problem in dimension-one. In dimension one a natural measure of non-linearity of a transformation f is $L(f) = (\ln f')' = f''/f'$ where prime denotes differentiation. The composition law $L(f\circ g) = (L(f)\circ g)\cdot g' + L(g)$ leads to a general principle of linearity for certain situations in dimension one. (See 2)). This principle is also valid for the volume distortion of an iterated composition of expanding maps.

Again in dimension one a natural measure of non-projectivity of a transformation f is the Schwarzian derivative $S(f) = L(f)' - 1/2(L(f))^2 = f'''/f' - 3/2(f''/f')^2$. The composition law $S(f\circ g) = (S(f)\circ g)(g')^2 + S(g)$ leads to certain results (David Singer, Misuriewicz, Guckenheimer).

2. Distortion lemmas, C^2 Denjoy theory.

We study the non-linear geometric distortion in an iterated composition $g_n = f_n\circ f_{n-1}\circ\dots\circ f_1$. Let f' denote a numerical measure of distortion which multiplies under composition. For example, volume distortion or linear distortion for conformal maps. Suppose the non-linearity measurement, grad(log f'), is bounded by N for each of the generating transformations f_i and that $\sum_{i=0}^{n-1}$ distance $(x_i, y_i) \leq L$ where x_0 and y_0 are two initial points and $(x_i, y_i) = (f_i(x_{i-1}), f_i(y_{i-1})) = (g_i(x_0), g_i(y_0))$.

Lemma 1. *The ratio of* g_n' *at the two points* x_0, y_0 *is bounded on both sides by* $\exp \pm(L \cdot N)$.

Proof. (classical)

$$|\log g_n'(x_0)/g_n'(y_0)| = |\log \prod_{i=1}^{n} f_i'(x_{i-1}) - \log \prod_{i=1}^{n} f_i'(y_{i-1})|$$

$$\leqq \sum_{i=1}^{n} |\log f_i'(x_{i-1}) - \log f_i'(y_{i-1})|$$

$$\leqq \sum_{i=1}^{n} N \cdot \text{distance } (x_{i-1}, y_{i-1}) \leqq N \cdot L.$$

Q.E.D.

Remark. In particular for an infinite sequence $f_1, f_2, \ldots$ so that $\sum_{i=0}^{\infty}$ distance $(x_i, y_i) = L < \infty$ we have for all n,

$|\log g_n'(x_0)/g_n'(y_0)|$ is bounded independently of n (by $L \cdot N$).

This simple computation has a second corollary for one-dimensional transformations. Let C be a collection of compositions (or words) in the generating transformations which satisfies $g_n = f_n \circ f_{n-1} \circ \ldots \circ f_1$ in C implies $g_{n-1} = f_{n-1} \circ \ldots \circ f_1$ in C. Let g_n' denote the linear distortion.

Lemma 2. $\{x \mid \sum_{g_n \in C} g_n'(x) < \infty\}$ *is an open set*.

Proof. (Schwartz). If $\lambda > 1$, one shows by induction on the word length n in C that there is $\delta > 0$ so that $g_n'(y_0) \leqq \lambda\, g_n'(x_0)$ for distance $(x_0, y_0) < \delta$. Namely,

$$
\begin{aligned}
|\log g_n'(y_0)/g_n'(x_0)| &\leq \sum_{i=1}^{n} N \text{ distance } (x_{i-1}, y_{i-1}) \\
&= \sum_{i=1}^{n} N \text{ distance } (g_{i-1}(x_0), g_{i-1}(y_0)) \\
&\leq \sum_{i=1}^{n} N\, g_{i-1}'(x_i^*) \text{ distance } (x_0, y_0) \\
&\leq N \lambda\delta \sum_{i=1}^{n-1} g_{i-1}'(x_0) \text{ by induction.}
\end{aligned}
$$

If $\sum_{g_i \in C} g_i'(x_0) < \infty$, then we can choose δ small enough to complete the induction. But then distance $(x_0, y_0) < \delta$ implies $\sum_{g_n \in C} g_n'(y_0) \leq \lambda \sum_{g_n \in C} g_n'(x_0) < \infty$.

From these general remarks it is easy to derive much of the C^2 Denjoy theory. For example,

(i) (Denjoy) A C^2 diffeomorphism of the circle without a periodic point has only dense orbits (and thus by Poincaré is topologically conjugate to a rotation).

(ii) (Rosenberg and Sullivan) A complex analytic homeomorphism of a neighborhood of an invariant rectifiable closed curve in $\mathbb{C}$ with no periodic points on the curve has only dense orbits on the curve.

(iii) (Sacksteder) In a C^2-codimension one foliation of a compact manifold by simply connected non-compact leaves all leaves must be topologically dense.

These are all results of topological dynamics. There are also direct corollaries in these three cases relative to the natural one-dimensional measure and measurable dynamics.

Theorem 1. In each of the examples above there is no set of positive measure which intersects each orbit (or leaf) in at most one point. Namely, these dynamical systems are recurrent or conservative -- for every set of positive measure A infinitely many iterated transformations bring part of A back to A.

Problem. Is case (iii) ergodic? (i) and (ii) are known to be (Katok, Herman,...)

Proofs and generalizations.

(i) (Denjoy) If a C^2 diffeomorphism f of the circle has an invariant Cantor set, the complementary intervals have finite total length, and are wandering. By Lemma 1 the derivatives of n-fold compositions at various points in one interval are commensurable. Thus the total sum of derivatives along one orbit is comparable to the total length of intervals and so is finite. This is valid even at the endpoints of one interval. By Lemma 2 the sum of derivatives along one orbit is finite on a neighborhood of one of these endpoints x_0. In fact from the proof for an interval I about x_0 $|(f^n)'y|$ is bounded by a constant times $|(f^n)'x_0|$, and so it tends uniformly to zero. By recurrence at x_0 some $f^n(x_0)$ lies close to x_0. Thus we can assume $f^n I \subset I$ and we have a periodic point. This contradicts the assumption, proving f is topologically transitive.

(ii) (Rosenberg-Sullivan) The same argument as in (i) works here. One point of some subtlety perhaps is that the one-dimensional calculations are valid for measuring lengths of images of rectifiable arcs by f^n because f is a conformal map.

Problem. (a) Is such a statement as (ii) true for a complex analytic homeomorphism defined near any invariant topological curve?

(b) Is there a real analytic homeomorphism of the circle giving a Denjoy type example (no periodic points but not topologically transitive)?

(iii) The proof is similar to that of (i) using the holonomy pseudogroup on one transversal which generates a collection C of transformations with (by compactness) a finite number of generators. If there is a non-trivial minimal set of leaves it must intersect the transversal in an invariant Cantor set. The complementary intervals must wander because the Cantor set is a minimal closed invariant set. Now we are in position to do the argument in (i) line by line using the C-orbit here in place of the f^n-orbit there. One constructs a contracting periodic point and thus holonomy in one leaf. This contradicts simply connectivity.

Remark. Lest the reader think this type of argument (invented by Schwarz and Saksteder for the above case) goes forever we mention the following example. If D denotes the endomorphism $\theta \to 2\theta$ on the circle for each irrational α there is (Douady, Sullivan, Thurston) a minimal D-invariant Cantor set $K(\alpha)$ where D identifies exactly 2 endpoints and the order structure of $x, Dx, D^2x, \ldots$ for a general point of $K(\alpha)$ is that of the orbit of a point under the rotation by α.

However, one can easily prove

(iii)′ There is no proper infinite closed invariant set for an expanding endomorphism D of S^1 on which D is a homeomorphism.

Proof. (Douady) An infinite compact metric space cannot have a self homeomorphism which uniformly decreases the distance between sufficiently near pairs of points.

Theorem 1 Proof. A dynamical system is conservative if and only if for almost all points x the sum of volume distortions evaluated at x of all transformations defined at x is a divergent series. (Exercise).

In real dimension one we have by Lemma 2 this convergence set is open. The set is invariant by definition. But we have already proven above these three dynamical systems (i), (ii), (iii) are minimal (all orbits are dense). Thus if this set is non-void it is everything. Then again we construct periodic points which contradict minimality.

3. Analytic functions and mappings.

In the 1880's Poincaré introduced the subject of discrete subgroups Γ of complex linear fractional transformations $w \to aw + b/cw + d$ acting conformally on the plane or Riemann sphere. These were the monodromy groups of 2nd order differential equations $a(z)w'' + b(z)w' + c(z)w = 0$ where the inverse function of the multivalued (ratio) solution defined in $C - \{\text{zeroes of } a(z)\}$ was single valued in some domain. Thus Poincaré investigated complex analytic functions $F(w)$ invariant or automorphic under the action of Γ.

The natural domains of definition of the automorphic functions $F(w)$ were the connected components of the complement of the Poincaré limit set to which every orbit of Γ in C clustered.

In the first part of this century Fatou and independently Julia studied the topological dynamics associated to the iterated compositions of a complex analytic self mapping of the sphere, $z \to R(z)$. Fatou was interested in analytic functions satisfying functional equations associated to rational substitutions of the form $z \to R(z)$. For example, $F(R(z)) = \lambda \cdot F(z)$, studied for $|\lambda| \neq 1$ by Koenig, Shroder, and by Poincaré in the nineteenth century, and for $|\lambda| = 1$ by C.L. Siegel in the 1940's.

Again the natural domains of definition for these functions satisfying functional equations were the connected components of the complement of a limit set associated to the dynamics of $z \to R(z)$. This limit set -- now called the <u>Julia set</u> -- is defined to be the complement of those points which have a neighborhood where the conformal factors (in the spherical metric) of the iterates R, $R \cdot R$, ... are uniformly bounded (the stable points).

We will mention some common features of these two situations related to the conformal properties of the mappings involved.

First the topological dynamics. For the Poincaré limit set $\Lambda(\Gamma)$ of <u>any</u> discrete group Γ one knows the Γ orbit of any point is dense in $\Lambda(\Gamma)$. In <u>complete analogy for the Julia set</u> $J(R)$ <u>of the rational map</u> $z \to R(z)$ <u>one knows</u> (Fatou-Julia) the backward orbit of any point in $J(R)$ is dense in $J(R)$.

On the complement $\Omega(\Gamma)$ of the Poincaré limit set $\Lambda(\Gamma)$ the action of Γ is properly discontinuous (for each compact $K \subset \Omega(\Gamma)$ there are only finitely many $\gamma \in \Gamma$ so that $\gamma K \cap K \neq 0$). One can then form a quotient Riemann surface $\Omega(\Gamma)/\Gamma$. A famous theorem of Ahlfors (1965) is that this Riemann surface for a group Γ of d-generators has finite type (it is obtained from a compact Riemann surface by removing finitely many points.) In particular, the components of the complement of the limit set $\Lambda(\Gamma)$ fall into finitely many Γ

orbits ($\leq 2d-2$), and it is impossible for one of these orbits to be a wandering disk.

The topological picture of the action of $z \to R(z)$ on the complement of Julia limit set $J(R)$ was less well know until recently. For example, could one component be a disk D so that all the images D, $R(D)$, $R \cdot R(D)$, ... are disjoint? If such a wandering disk exists, the total area on the sphere is finite. Thus $\sum_n |R^{n\prime}(z)|^2 < \infty$ for almost all z in D.

One is tempted to use the ideas of 2) to arrive at a contradiction in the manner of Denjoy's theorem for C^2 diffeomorphisms of S^1 (which analogously asserts there is no wandering interval on S^1.)

Such an elementary proof is not available at the present. One proves the following theorem by using the measurable Riemann mapping theorem (Ahlfors-Bers 1960) to construct, if the conclusion is false, an infinite dimensional space of complex analytic self-mappings of $\mathbb{C} \cup \infty$ with a given degree, and thus a contradiction.

Theorem 2. (Sullivan [1982]). Under the forward iteration of a rational map of degree d, $z \to R(z)$ the connected domains of the complement of the Julia limit set $J(R)$ map into finitely many cyclic orbits of domains.

These cyclic stable regions can be classified into five types. The first two types, attractive basins and parabolic basins have fundamental domains for the equivalence relation: $x \sim y$ if and only if $f^n x = f^m y$ some $n,m \geq 0$. The third type, superattractive basins do not, but they are foliated by the closures of the classes of the equivalence relation, $x \approx y$ if and only if $f^n x = f^n y$, $n \geq 0$. The last two types are rotation domains, Siegel disks or Herman rings, which are foliated by the closures of forward orbits. (See figure 1.)

(i) An attractive basin D arises from an attractive periodic cycle γ with nonzero derivative of modulus less than one, $\gamma = z \cup fz \cup \ldots \cup f^{n-1}z$, $f^n z = z$, $0 < |(f^n)'(z)| < 1$, and D consists of the components of $W_s(\gamma) = \bigcup_{x \in \gamma} \{y \mid \lim_{n\to\infty} \text{distance } (f^n y, f^n x) = 0\}$

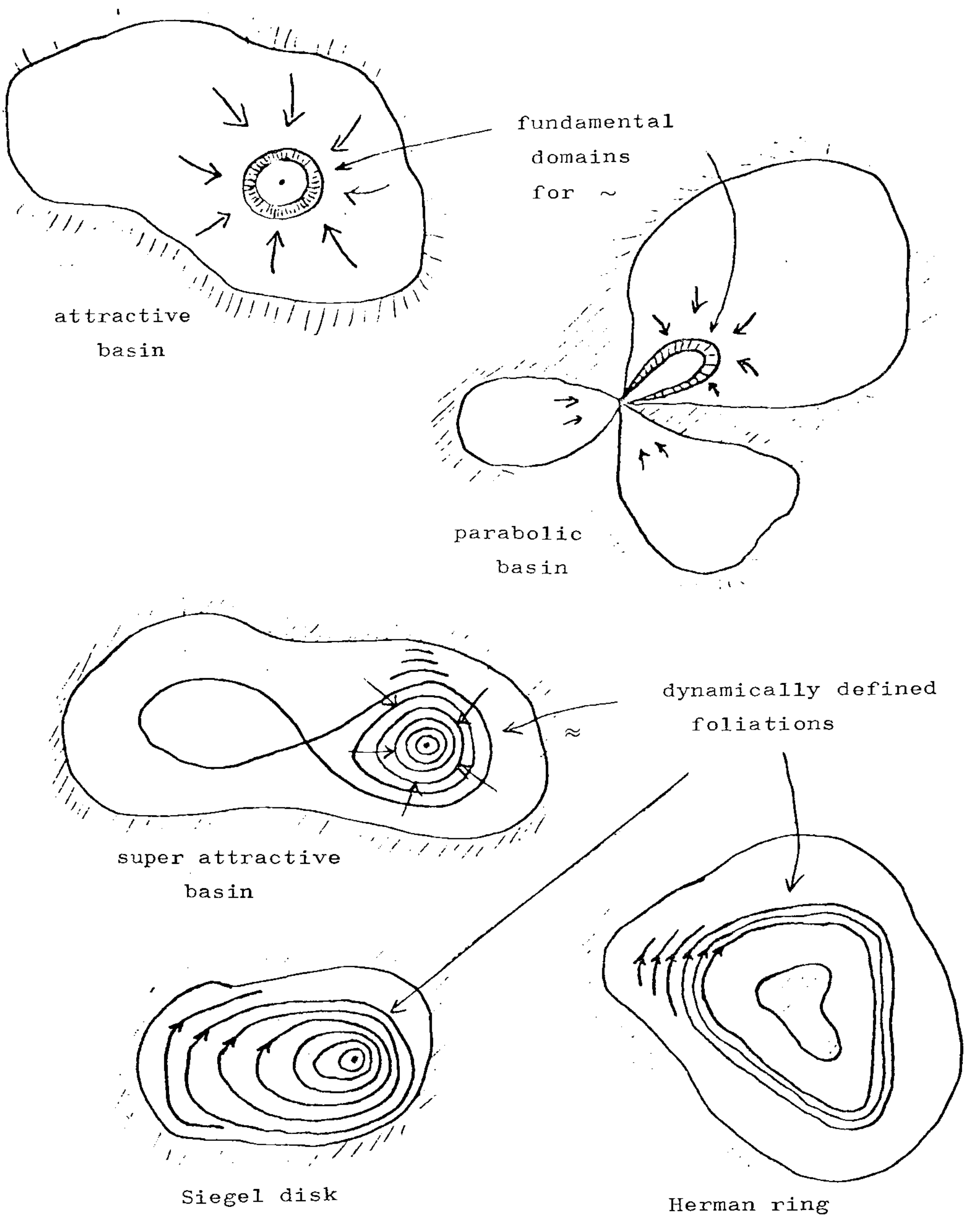

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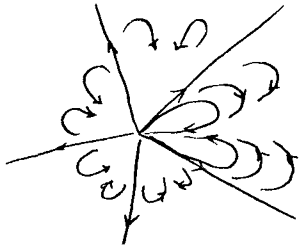
figure 1

containing points of γ. Fatou [1918] showed that such a D must contain a critical point of f. Thus there are no more than 2d-2 attractive basins for an endomorphism of degree d.

If we remove from D the inverse orbit of γ, $\{\bigcup_{n\geq 0} f^{-n}\gamma\}$, the set of $\sim$ equivalence classes ($x \sim y$ if and only if $f^n x = f^m y$) defines a torus with branch points corresponding to the critical points of f. This follows easily from the local model of f near γ, where near a fixed point of a power of f we have $z \to \lambda z$, $|\lambda| < 1$.

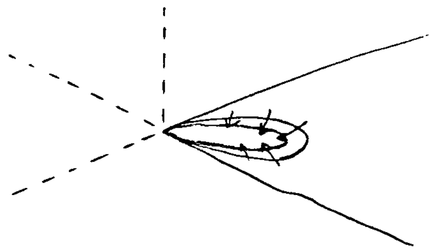
(ii) A <u>parabolic basin</u> D arises from a non-hyperbolic periodic cycle γ with derivative a root of unity, $\gamma = z \cup f(z) \cup \ldots \cup f^{n-1}z$, $z = f^n(z)$, $((f^n)'(z))^m = 1$, γ is contained in the frontier of D, and each compact in D converges to γ under forward iteration of f. (Fatou [1918].) The local picture of the dynamics consists of parabolic sectors arranged around the fixed point of a power of f which in local coordinates is $z \to z + z^{\ell} + \ldots$. Fatou [1918], Camacho [1979].



local model

$z \to z + z^{\ell} + \ldots$

The local model produces a fundamental domain for the global dynamics on D because all orbits in D tend to γ. Looking at the local picture then shows the quotient of D by the $x \sim y$ equivalence classes is a union of twice punctured sphere with branched points coming the critical points of f lying in D (there must be at least one critical point in D, Fatou [1918]).



fundamental domain

(iii) A superattractive basin D is defined just like an attractive basin but now the derivative of the power of f having a fixed point is zero. Now points arbitrarily near the attracting cycle are identified by f and there is no true fundamental domain for the $\sim$ equivalence classes. The more precise relation $x \approx y$ if and only if $f^n x = f^n y$, $n \geq 0$,defines a foliation of $D' = D$ - inverse orbit of γ by the closures of the $\approx$ equivalence classes. The leaves are compact 1-manifolds which are not necessarily connected and which have n-prong singularities at the inverse orbit of other critical points in D. The local linearization near a superattractive fixed point shows the leaves near γ are nearly concentric closed curves around the points of γ. The rest of the foliation of D' is obtained by applying f^{-1} to this concentric foliation near γ.

(iv) A Siegel disk is a stable regions which is cyclic and on which the appropriate power of f is analytically conjugate to a rotation of the standard unit disk. Siegel [1942] proved these occur near a non-hyperbolic periodic point if $1/\pi$-argument of the derivative is far from the rationals. Far from the rationals means $|\vartheta - p/q| > c/q^\nu$ for some $c > 0$, $\nu > 0$, and all p/q reduced fractions.

Fatou and Julia showed that if such regions existed their frontiers were contained in the union of the ω-limit sets of critical points.

Siegel disks around the origin occur already in the family $z \to \lambda z + z^2$, $|\lambda| = 1$. However, they do not occur when $1/\pi \arg \lambda$ is sufficiently Liouville because then there are periodic points tending to zero in this case (an easy calculation).

(v) A Herman ring is a stable region similar to a Siegel disk. Now we have a periodic cycle of annuli and a power of f is analytically equivalent to an irrational rotation of the standard annulus. Again the frontier is contained in the ω-limit sets of critical points. Such examples were found by Michel Herman in the family

$$z \to \frac{e^{i\theta}}{z} \cdot \left(\frac{z-a}{1-\bar{a}z}\right)^2$$

appropriate θ, a small. Herman uses Arnold's theorem [1960] about real analytic conjugations of real analytic diffeomorphisms of the circle to rigid rotations when the rotation number is like a Siegel number. Note that both Siegel disks and Herman rings are foliated by the closures of orbits and the leaves are closed curves.

More dynamical properties.

(i) One knows there are only finitely many cyclic stable regions described in (4) Sullivan [1982]. But it is a problem to find the sharp upper bound in terms of the degree. Is it $2d-2$?

(ii) Also for polynomials one knows each bounded stable region is simply connected (apply the maximum principle to $f, f^2, \ldots$). Thus polynomials do not have Herman rings.

(iii) An amusing corollary of the classification of stable regions in (4) is the following: if all critical points of f are eventually periodic but none are periodic then the Julia set of f is all of $\bar{\mathbb{C}}$. (Because each type of cyclic region besides the superattractive basin requires a critical point with an infinite forward orbit.) Examples of this type are $z \to \left(\frac{z-2}{z}\right)^2$ and the quotient of some higher degree endomorphism of a one-dimensional torus by the equivalence relation $x \sim -x$.

(iv) Fatou and Julia showed f on $J(f)$ is topologically transitive. (In fact, for any z in $J(f)$ the inverse orbit $\bigcup_{n\geq 0} f^{-n}(z)$ is dense in $J(f)$.) If no critical points tend to $J(f)$ or touch it Fatou showed some power of f is expanding on $J(f)$. He surmised the dynamical structure was continuous in the coefficients for such examples (now called Axiom A or expanding systems -- see below) and guessed that this property should be true except for special values of the parameters.

Even when $J(f)$ is contaminated by critical points one may think of $J(f)$ as the "hyperbolic" part* of the dynamics. The Siegel disks and Herman rings are in the "elliptic" part of the dynamics. The attractive basins and the parabolic basins are the properly discontinuous part of the dynamics. The superattractive basins are both wandering and of elliptic character.

Newton's method. We recall that it is still a difficult problem to find the zeroes of a complex polynomial $f(z)$. Newton's iterative method $z \to z - f/f'$ provides a natural example where the dynamics above is encountered.

For a general polynomial f of degree $d, N(z) = z - f/f'$ is a rational map of degree d. The zeroes of f determine fixed points of N where $|N'z| < 1$. Thus they determine attracting domains for the dynamics of N. In practice one also finds other periodic attracting domains for $N(z)$.

It is hoped one could account for these in general. Then if one could in addition show the Julia limit set has 2-dimension measure zero, a general understanding of Newton's iteration for almost all points would ensure, Figure 2.

We note that Fatou [1919] proved each contracting periodic domain for a rational map contains a branch point. In curious analogy David Singer proved the analogous result for a smooth endomorphism of the circle with a negative Schwarzian derivative (1975).

Problem: Find a common explanation of Singer's and Fatou's theorem.

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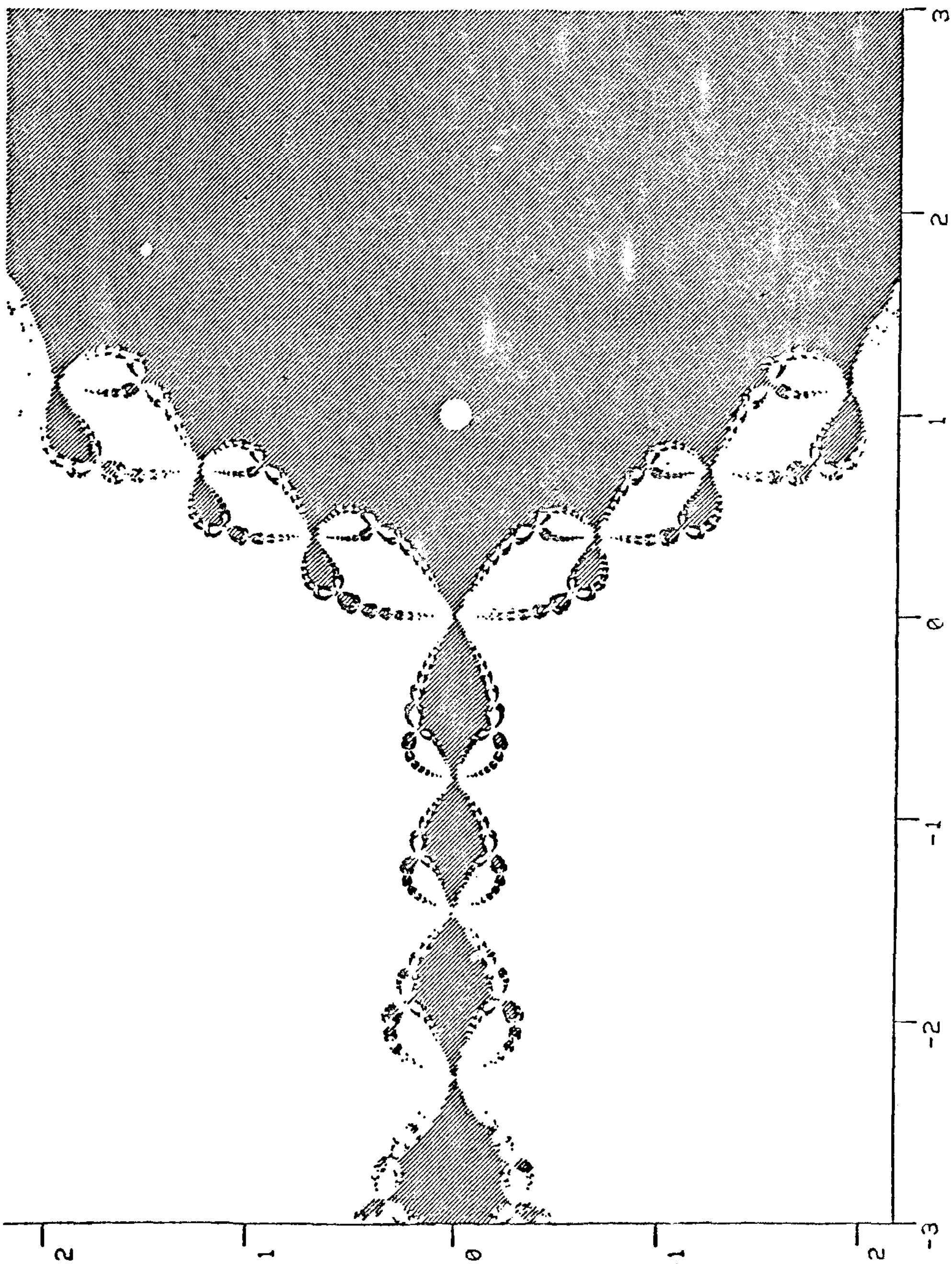
*The words "hyperbolic" and "elliptic" are meant to suggest chaotic and rigid structure respectively in the dynamics.



Newton's method for $z^3-1=0$ $\quad z \to z - f/f' = 1/3\,(2z+1/z^2)$

Figure 2
(constructed by J.P. Eckman)

Fractal geometry of limit sets. We construct conformal measures of exponent δ in the Julia set of any rational map and use these to discuss the Hausdorff geometry in the expanding case. This discussion in the analogous case of Kleinian groups was carried out in Sullivan [1980] motivated by papers of Bowen [1980] and Patterson [1976].

Theorem 3. There is a positive, finite measure μ on the Julia set $J(R)$ of a rational map $z \to R(z)$ satisfying for some real number $\delta = \delta(R)$

$$\mu(R(A)) = \int_A |R'(z)|^\delta \, d\mu(z) \qquad (*)$$

for any Borel set $A \subset J(R)$ where R is injective. Moreover, $0 < \delta(R) \leq 2$ and if any δ satisfies (*) for some measure μ then $\delta \geq \delta(R)$.

Proof. If $J(R) = \bar{\mathbb{C}}$ take Lebesgue measure. If $J(R) \neq \bar{\mathbb{C}}$ by Fatou there is an open set U in the complement of $J(R)$ so that the inverse branches of R^{-n} are defined and the inverse images $R_i^{-1}U$, $R_i^{-1}(R_j^{-1}U)$,.. are all disjoint (with the exception possibly of one sequence of choices of inverse branches when U belongs to a disk or annulus on which R is equivalent to an irrational rotation) and converge towards $J(R)$.

For each x in U let $I(x)$ denote all the $R_i^{-n}(x)$ (except for the exceptional sequence if present). If $y \in I(x)$ satisfies $R^n(y) = x$ then define $d(y) = |(R^n)'(y)|^{-1}$ (in the spherical metric). Since all the inverse images of U are disjoint and the area of the sphere is finite $\sum_{y \in I(x)} d(y)^2 < \infty$ a.e. x in U by Lebesgue monotone convergence (solution of exercise above).

Let x_0 in U be any point where the series converges and define $\delta = \inf\{s \mid \sum_{y \in I(x_0)} d(y)^s < \infty\}$. Note $\delta > 0$ because there $\sim d^n$ points in the n^{th} level and the factors $d(y)$ are decreasing no faster than $\sim K^{-n}$ where $|R'(z)| \leq K$ on $J(R)$. Suppose for now $\sum_{y \in I(x_0)} d(y)^\delta = \infty$.

Define measures μ_s by putting atomic masses of weight $d(y)^s$ at y and normalizing to total mass 1. Let μ be any weak limit of the μ_s as $s \searrow \delta$. By our divergence assumption μ is supported at $J(R)$.

If U is a neighborhood of z_0 in $J(R)$ where R is injective, then R is bijective between $I(x_0) \cap U$ and $I(x_0) \cap R(U)$. (If z_0 is a critical point then U can be arranged so that R is k to one between $I(x_0) \cap U$ and $I(x_0) \cap R(U)$.) In the first case further assume U is chosen so that $|R'z|$ is a constant λ up to a factor near 1. (In the second case assume U is chosen so that $|R'z|$ is $< \varepsilon$.)

Thus in the first case we have $\mu_s(R(U))$ is $\lambda^\delta \mu_s(U)$ up to a factor near one. (In the second case $\mu_s(R(U)) < k \cdot \varepsilon^\delta$.) ($s$ is near δ.)

Letting $s \searrow \delta$ and then shrinking U we deduce for any atomic parts of μ we have

$$\mu(\{Rx\}) = |R'(x)|^\delta \mu(\{x\}). \qquad (**)$$

We may remove the critical points from consideration and consider (*) in the locally injective part. Then letting $s \searrow \delta$ and shrinking U we deduce (where R is locally injective) $dR^*\mu/d\mu = |R'|^\delta$ a.e. μ and this proves (*).

If $\sum_{y \in I(x_0)} d(y)^\delta$ does not diverge, introduce new weighting factors $h(d(y))d(y)^s$ for masses placed along $I(x_0)$ to define μ_s where

(i) $\sum_{y \in I(x_0)} h(d(y))d(y)^\delta = \infty$, (recall $\Sigma\, d(y)^\delta = \infty$ for $s < \delta$)

(ii) $h(x)$ is a positive function of x increasing to $+\infty$ as $x \searrow 0$ in such a way that for all $\varepsilon > 0$ and $0 < \lambda < \infty$, $|h(\lambda x)/h(x)| \in [1-\varepsilon, 1+\varepsilon]$ for $0 < x \leqq x_0(\varepsilon, \lambda)$. Thus $\alpha > 0$ implies $h(x) \leq x^{-\alpha}$ for x sufficiently small.

Now carry through the argument as above. By (ii) the new factor $h(d(y))$ introduces only a factor near 1 in the computations. (In the second case one has to divide a neighborhood of the critical point into countably many nice annuli and calculate using ii) and $\delta > 0$.) This completes the proof of the existence of conformal measures on any Julia set.

Since the conformal measures form a closed set in the weak topology on the probability measures we can go to the minimal dimension, by definition $\delta(R)$. If $\delta(R)$ were zero there could be no atoms by (**) above (since μ is a finite measure and full orbits are infinite). Then working where R is d to 1 if d = degree R we deduce $R_*\mu = d\mu$ contradicting total mass μ = total mass $R_*\mu$. This proves Theorem 3.

Say that R <u>is expanding on the Julia set if for each</u> x <u>in</u> $J(R)$ <u>there is an</u> n <u>so that</u> $|(R^n)'(x)| > 1$. Then it is easy to see that some fixed m $|(R^m)'(x)| > 1$ for all x in $J(R)$. We now work with R^m and denote it by R.

Let $B(x,r)$ denote any ball of small radius r centered at a point x in $J(R)$. By the distortion lemma 1 there is an n so that if $B = B(x,r)$ and $B' = R^n(B)$, then B' has a definite size and $R^n\colon B \to B'$ is a "quasi-similarity" ($\equiv$ the ratio of derivatives at various point of B are comparable) because in the notation of Lemma 1 the $\{\text{distance } (x_i, y_i)\}$ form a geometric series. One deduces $\mu(B)$ is comparable with fixed bounds to r^δ. (This follows since all B' of a definite size have a definitely positive μ mass since μ is positive on open sets of $J(R)$ by topological transitivity.)

By a relatively simple general proposition (see Federer "Geometric Measure Theory" and §2 of Sullivan [1980]) such a measure μ is boundedly equivalent to the Hausdorff δ-measure. Thus the measure class of any conformal measure μ is determined by the geometric properties of the set $J(R)$.

We collect this information in

<u>Theorem 4</u>. <u>In the expanding case there is one and only one conformal measure</u> μ <u>on the Julia set</u> $J(R)$. <u>The exponent</u> $\delta = \delta(R)$ <u>of</u> μ <u>is the Hausdorff dimension and</u> μ <u>is a constant times the Hausdorff</u> δ<u>-measure</u> H_δ <u>on</u> $J(R)$. Moreover, $0 < \delta(R) < 2$.

<u>Corollary</u>. <u>The Hausdorff</u> δ<u>-measure is a finite and positive measure, the Hausdorff</u> δ<u>-measure of a ball of radius</u> r <u>in</u> $J(R)$ <u>is comparable to</u> r^δ, <u>and the Hausdorff</u> δ<u>-measure is ergodic relatively to</u> R <u>acting on</u> $J(R)$. (<u>In the expanding case</u>).

Proof of Theorem and Corollary. If μ_1 and μ_2 are conformal measures of exponent δ and total mass one so is $1/2\ (\mu_1 + \mu_2) = m$. The ratio of μ_1 to m is defined and is an R-invariant function. Since m being a conformal measure is ergodic (its measure class is determined by the geometry of $J(R)$) this Radon ratio is constant. Thus $m = \mu_1 = \mu_2$ and conformal measures of dimension are unique. By the same reasoning the exponent δ is unique being the Hausdorff dimension of $J(R)$. (Since Lebesgue measure $J(R) = 0$, see proposition below, $\delta < 2$.)

Now we know the Hausdorff measure H_δ by definition satisfies the defining equation (*) to be a conformal measure. In this expanding case we know it is also a finite measure. Thus by the above uniqueness H_δ is a constant times the unique normalized conformal measure. This completes the proof of the theorem.

The statements of the corollary follow directly from the above.

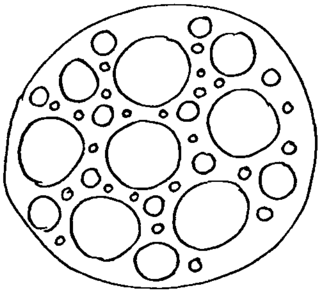
Remark. One can prove Theorem 3 and Theorem 4 for expanding rational maps using Markov partitions and Gibbs measures as Bowen did for quasi-fuchsian surface groups. We have chosen this way because Theorem 3 is more general and Theorem 4 is obvious once Theorem 3 is known. The Bowen proof, however, produces the μ by a method which evidently converges geometrically fast. Lucy Garnett [1983] used finite approximations to calculate the Hausdorff dimension $\delta(R)$ for the family $z \to z^2 + p$ for p real and small. A quadratic curve with minimum at $p = 0$ was found. Based on this calculation by Garnett I asked at this conference whether $\delta(R)$ varied smoothly or even real analytically as R varies in an analytic family of expanding examples. This was answered by Ruelle using Bowen's infinite procedure.

Theorem 5. (Ruelle, 1982) The Hausdorff dimension of $J(R)$ is a real analytic function of the coefficients of $z \to R(z)$ in any open connected set where each such map is expanding.

Problem. Is Theorem 5 true for analytic families of expanding Kleinian groups (for each x in the Poincaré limit set there is

$\gamma \in \Gamma$ so that $|\gamma' x| > 1)$?

We close with a remark about classifying these expanding groups. They determine compact 3-manifolds (with boundary) and Thurston has characterized which topological 3-manifolds arise in this way. The abstract group structure of the fundamental group Γ determines the topology of the limit set $\Lambda(\Gamma)$ and the topological action of Γ on $\Lambda(\Gamma)$. The simplest example is: Γ is a free group and $\Lambda(\Gamma)$ is the Cantor set of infinite words in Γ. In another class of examples (the acylindrical 3-manifolds) $\Lambda(\Gamma)$ is always homeomorphic to a Sierpinski curve obtained by removing from a 2-disk a dense collection of smaller disks.



Sierpinski Curve

This is the one dimensional analogue of the 0-dimensional Cantor set.

We ask now the question -- what invariants besides this topology determines the geometric realization of Γ as a discrete group?

Theorem 6. (Sullivan [1981]) The geometric realization of Γ as a discrete group in $PSL(2,\mathbb{C})$ with the expanding property is determined up to isomorphism by the Hausdorff dimension δ of $\Lambda(\Gamma)$ and the abstract measurable dynamics isomorphism class of Γ acting on $\Lambda(\Gamma)$ relative to Hausdorff δ-measure.

The proof uses the ergodicity of the action of Γ on $\Lambda(\Gamma) \times \Lambda(\Gamma)$ and a characterization of Moebius transformations as measurable transformations preserving the cross ratio of almost all 4 tuples of points. The analogue of this theorem for rational maps is not known -- maybe its proof will use the Schwarzian derivative.

<u>Remark</u>. Theorem 3 for all Kleinian groups and Theorem 4 for expanding groups go through just as above, as we mentioned before, using the "Poincaré series method" -- putting mass along the orbit. Now, however, Markov partitions are not always obviously available, and even when they are certain discontinuities arise which don't occur in the rational map case.

<u>Quadratic maps</u>. In the family $z \xrightarrow{R} \lambda z + z^2$ where $|\lambda| < 1$, one knows R is expanding on $J(R)$. (For these the critical point $-\lambda/2$ tends to the attractive fixed point 0.) Thus by Ruelle $D(\lambda)$ = Hausdorff dimension of $J(R)$ is a real analytic function for $|\lambda| < 1$.

<u>Theorem 7</u>. $D(\lambda)$ <u>is strictly greater than one and strictly less than two for</u> $0 < |\lambda| < 1$. (See Bowen [1980], for the analogous theorem on quasi fuchsian groups.) <u>Moreover</u>, $J(k)$ <u>is an Ahlfors quasi-circle</u>.

<u>Proof</u>. The Julia set moves continuously in $0 \leq |\lambda| < 1$ and so is always a Jordan curve. See Mañé, Sad, Sullivan [1982] where more is proved.

We need now show that $J(R)$ is not rectifiable and the Lebesgue measure is zero. The second case is ruled out by the following.

<u>Proposition</u>. <u>For an expanding</u> $z \to R(z)$ <u>the Lebesgue measure of</u> $J(R)$ <u>is always zero</u>. <u>Thus</u> $\delta(R) < 2$.

<u>Proof</u>. Take a density point x and radii $r_i \to 0$ so

$$m(J(R) \cap B(x,r_i))/m(B(x,r_i)) \to 1$$

where m is Lebesgue measure. Expand $B(x,r_i)$ up to a definite size B'_i using R^{n_i}. By the quasi-similarity lemma 1 we still have $m(J(R) \cap B'_i)/m(B'_i) \to 1$. A limit B' of the balls B'_i will satisfy $m(J(R) \cap B')/m(B') = 1$. Thus $B' \subset J(R)$. Then $J(R) = \bar{\mathbb{C}}$ which contradicts the expanding property.

<u>Remark</u>. Because the curve is quasi-self similar it is a quasi-circle. (see Ahlfors book "Quasi-conformal Mappings.")

For the first possiblity using the Riemann mapping theorem on each component of the complement of $J(R)$, we obtain $D_2 \cup D_1 \xrightarrow{\varphi_1 \cup \varphi_2} \bar{\mathbb{C}} - J(R)$ analytic. Conjugating the dynamics back to $D_2 \cup D_1$ we obtain two $2 \to 1$ maps of the standard disk. The one for the component of $C - J(R)$ containg ∞ is $z \to z^2$. The one for the finite component is $z \to z \cdot \left(\frac{z-\lambda}{1-\lambda z}\right)$ (if the Riemann map sends the fixed point to the origin).

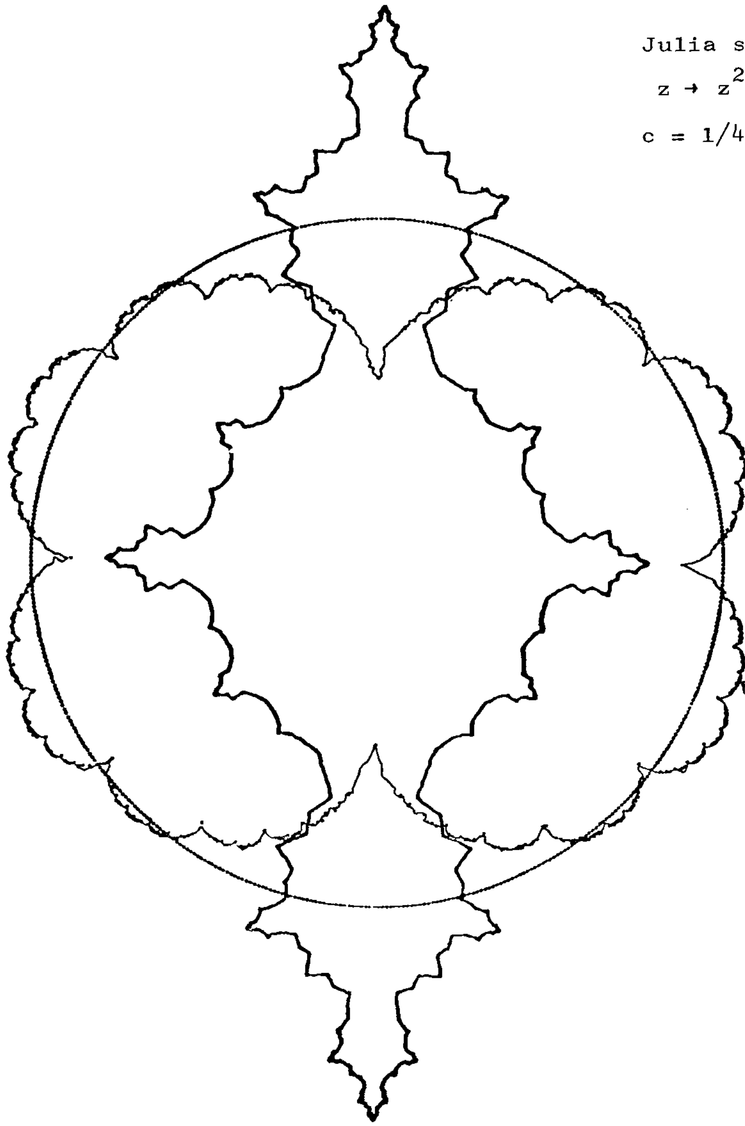
The Riemann maps φ_1 and φ_2 extend continuously to the boundary by Caratheodory's work. Moreover, if $J(R)$ is rectifiable these Caratheodory maps are non-singular with respect to arc length measure (by a harmonic measure argument). Thus we obtain a continuous and absolutely continuous conjugacy between $z \to z^2$ and $z \to z \cdot \left(\frac{z-\lambda}{1-\lambda z}\right)$ restricted to $|z| = 1$. But this is impossible. For both are essentially expanding and locally eventually onto and thus each one is ergodic with respect to the Lebesgue measure class on $|z| = 1$. Also each one preserves Lebesgue measure $d\theta$ (which is the harmonic measure of $|z| = 1$ relative to $z = 0$ which is fixed by each map.)

Thus the conjugacy sends $d\theta$ to $d\theta$ and must be a rigid rotation. This contradicts the fact that $z \to z \cdot \frac{z-\lambda}{1-\lambda z}$ has a varying derivative when $\lambda \neq 0$.

Remark. Since for $\lambda \neq 0$ we obtain for the simple map $z \to \lambda z + z^2$ (or $z \to z^2 + c$, $c \neq 0$) Julia sets which are non-rectifiable quasi-self similar fractal curves of Hausdorff dimension > 1 one is tempted to plot these on a computer. Here are some examples, Figure 3.

Bifurcations of conformal dynamical systems. It is very interesting (Figure 3, 4) to study the bifurcations of $J(R)$ for $z \xrightarrow{R} \lambda z + z^2$ as λ varies (see Douady-Hubbard, Mandelbrot,...). We have seen for $|\lambda| < 1$ $J(R)$ is a moving Jordan curve whose Hausdorff dimension is really varying. As λ hits the unit circle $J(R)$ even changes topologically.

If λ hits at a root of unity the curve pinches together at

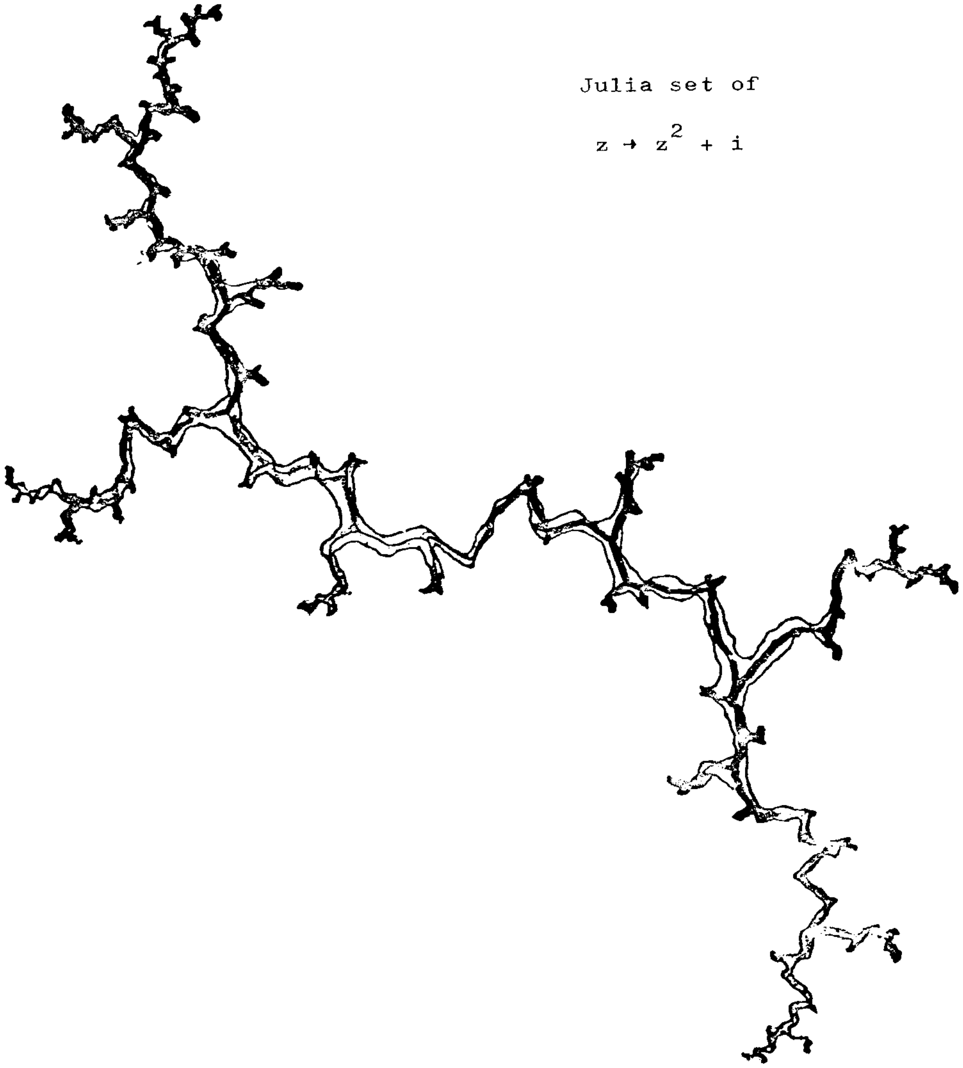


Julia set of

$z \to z^2 + c$

$c = 1/4, 0, .9$

figure 3



Julia set of

$z \to z^2 + i$

figure 4

the origin and all the preimages. The collapsed circle is still locally connected (Hubbard and Douady). If λ hits at an irrational angle which is far from the rationals $J(R)$ does not reach the origin which is contained in a Siegel rotation disk.

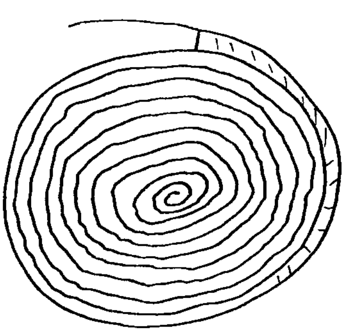
Problem. Is the boundary of the Siegel disk a Jordan curve, and is $J(R)$ locally connected? If so, the topological picture of $J(R)$ can probably be described using the invariant Cantor set for $\theta \to 2\theta$ mentioned above.

The case where λ hits at a Liouville angle leads to a non-locally connected $J(R)$. More precisely,

Theorem 8. (Douady-Sullivan) If $z \xrightarrow{R} \lambda z + z^2$ where $\lambda = e^{2\pi i\vartheta}$ with ϑ irrational but R is not linearizable near $z = 0$, then $J(R)$ is non-locally connected.

Proof. The Riemann map of the exterior of the unit disk to the exterior of $J(R)$ extends continuously to the boundary if $J(R)$ is locally connected (Caratheodory). Let C contained in S^1 be those angles (of exterior rays) which land at the fixed point $z = 0$. The Riemann map conjugates $z \to z^2$ to $z \to R(z)$.

If C were finite, some power of R^k would have an invariant line tending to zero. Since the derivative is an irrational rotation this line spirals in to the origin. It is then possible to find a region near zero which is invariant by R^{-k}. But R^{-k} is non-linearizable.



Invariant region (the snail enters its shell)

This is a contradiction. So C must be infinite.

Clearly C is closed. Also $\theta \to 2\theta$ (the action of $z \to z^2$ on rays) restricted to C must be a homeomorphism because R is a bijection on those rays which land at the irrational fixed point. But this contradicts the proposition in the first section (because an infinite compact metric space cannot have a homeomorphism which uniformly expands distances between sufficiently near points).

5. Characterization of complex analytic dynamical systems.

One says that a diffeomorphism of part of the plane is K-quasi-conformal if infinitesimal circles are mapped to infinitesimal ellipses of eccentricity $\leq K$. The definition can be extended to homeomorphisms or even branched coverings of any Riemannian surface.

Consider a collection C of transformations of a Riemannian surface S which is closed under composition and which are all K-quasi-conformal in the sense of the extended definition.

Theorem 9. *There is a complex analytic structure on S (compatible with its original quasi-conformal structure) so that all the transformations of the dynamical system C become complex analytic.*

Note. The condition of the theorem, K-quasi conformality of all the transformations in C is clearly necessary for the statement of the theorem (which asserts it is sufficient).

Corollary. *The dynamical systems determined by*

(i) *complex analytic self-mappings of* $\mathbb{C} \cup \infty$

(ii) *entire functions* $f: \mathbb{C} \to \mathbb{C}$

(iii) *a collection of Moebius transformations of* S^2, *are characterized by the condition of uniform quasi-conformality of all the iterated compositions.*

Proof. First one forms an invariant measurable conformal structure by a barycenter construction in the set of similarity structures on the tangent space at each point. Then one introduces complex analytic coordinates using the measurable Riemann mapping theorem.

Closing problem. In either of the conformal dynamical contexts, rational maps or finitely generated Kleinian groups are the expanding systems dense?

Bibliography

R. Bowen, "Dimension of quasi circles", Publ. IHES, Vol. 50 (1980).

A. Camacho, Dijon Conference Asteriks, 1979.

A. Douady, J. Hubbard, "Iteration of complex quadratic polynomials", CRAS, Paris 1982. Longer manuscript in preparation.

M. Fatou, (i) "Functions Satisfying Functional Equations," Bull. of French Math. Soc., 1919-1920.

(ii) "Iterating Entire Functions," Act. Math., Vol. 47.

L. Garnett, "Calculation of Hausdorff dimension," in preparation.

J. Guckenheimer, "Dynamics of rational maps," AMS Conference on Global Analysis, 1968.

G. Julia, "Iteration of rational maps," Journal of Math. Pure et Appliques, 1919.

B. Mandelbrot, "Form, Chance, Dimension," Free an, 1976 (New version, 1982).

R. Mañé, P. Sad, D. Sullivan, "On the dynamics of rational maps," submitted to Journal of the Ecole Normal Superieur.

M. Misuriewicz, "Mapping of the Interval," Publ. IHES, Vol. 48.

S.J. Patterson, "Limit Sets of Fuchian groups," Acta Math., 1976.

H. Poincaré, Collected Works, Vol. 1.

D. Ruelle, "Analytic Repellers," Journal of Ergodic Theory and Dynamical Systems, 1982.

R. Sacksteder, "Invariant measures for pseudogroups," Amer. Journal Math., 1965.

C.L. Siegel, "Iteration of complex analytic transformations," Annals of Math., 1942.

D. Sullivan, (i) "Discrete Conformal Groups and Measurable Dynamics," Proceedings of Poincaré Conference, Bloomington, Ind., 1980. BAMS 1982.

(ii) "Density at infinity..." Pub. IHES, 1980, Vol. 50.

(iii) "Quasi conformal homeomorphisms and dynamics," in preparation 1982; also C.R.A.S. (Paris), 1982.

M.V. Yakobson, "Question of topological classification of rational mappings of the Riemann sphere," Usp. Math. Nauk. 28, No 2, 247-248 (1973), also Math. Sb. (N.S.) 77, 105-124 (1968), and Math. Sb. (N.S.) 80, 365-387 (1967).

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