TABLE OF CONTENTS

SEMINAR ON CONFORMAL AND HYPERBOLIC GEOMETRY

by

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1. Motions in non-Euclidean Geometry.

We will consider metric spaces X such that the distance between any two points $x , y \in X$ is

$$\text{dist}(x,y) = \text{Inf. length of curves connecting them.}$$

A (local) isometry between any two such spaces is a map that preserves lengths of curves. A space X as above is complete if given any path

$$p : [0,1) \longrightarrow X$$

of bounded length, the limit $\lim_{t\to 1} (p(t))$ exists, i.e. X has "no holes". (Note that such a metric can be multiplied by a bounded Borel function).

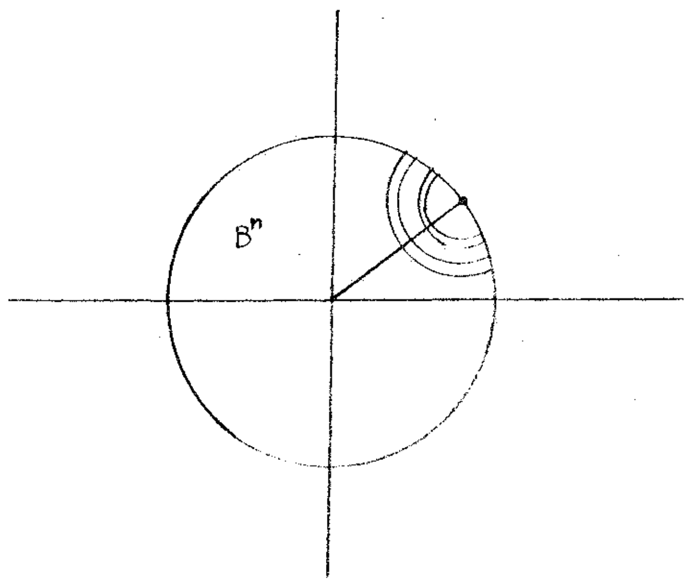
Theorem. For each $n \geq 2$, there exists a non-Euclidean infinite geometry. That is, a non-Euclidean (and non-compact) complete Riemannian n-manifold which is homogeneous with respect to points, directions, and 2-planes .

(The case $n = 1$ can also be considered. However, a "hyperbolic line" is isometric to the standard Euclidean line).

Proof. The proof is based on work of Steiner (1825) on inversions in $\mathbb{R}^n$.

Let $B^n \subset \mathbb{R}^n$ be the (closed) unit ball centered at the origin, let M_n, be the group generated by inversions in $\mathbb{R}^n$, and let $G = G_n$ be the subgroup of M_n consisting of inversions that preserve B^n. We call M_n, and G, the Möbius group.

Steiner showed that inversions preserve circles, 2-spheres,...,(n-1)-spheres. They also preserve angles. One proves easily that the elements of G are just rotations about points, composed with transformations that move points along rays.

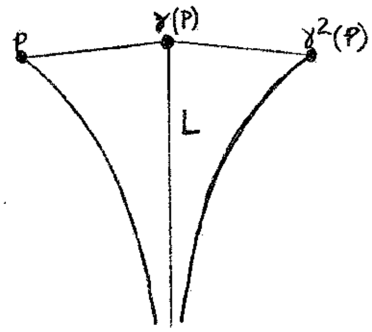


Since G clearly acts transitively on each ray, it follows that G acts transitively on B^n.

Now, let $x \in B$ = interior of B^n, and fix a Riemannian metric on T_xB. If $\gamma \in G$, then the derivative $d\gamma$ maps T_xB onto $T_{\gamma(x)}B$. If we give $T_{\gamma(x)}B$ the induced metric and if we do this for all $\gamma \in G$, we obtain a complete Riemannian manifold B on which G acts by isometries.

We have then constructed a model for hyperbolic geometry.

Now, consider our space B "from the inside", and let $\gamma \in G$. Assume the points $p, \gamma(p), \gamma^2(p) \in B$ are not in straight line, and let L be a line that bisects the angle they form,



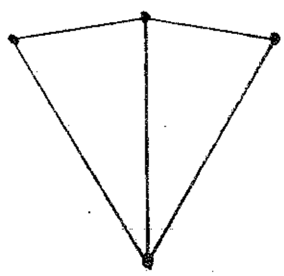
Look at the lines $\gamma^{-1}(L)$, L, $\gamma(L)$. There are 3 possibilities :

I) They intersect inside the sphere. (Generic).

II) They intersect at ∞, i.e. in ∂B. (Unstable).

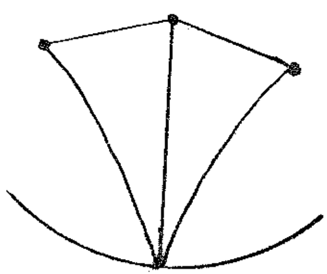
III) They do not intersect. (Generic).

Case I.



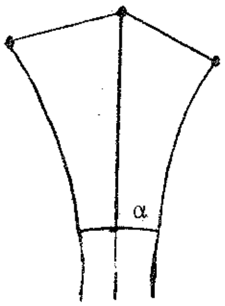
Rotation about a point. γ is elliptic.

Case II.



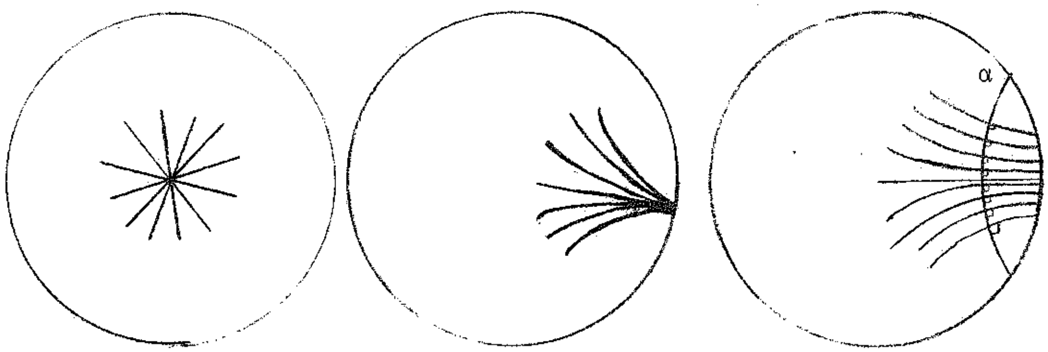
Fixes a point on the sphere at ∞. γ is parabolic.

Case III.



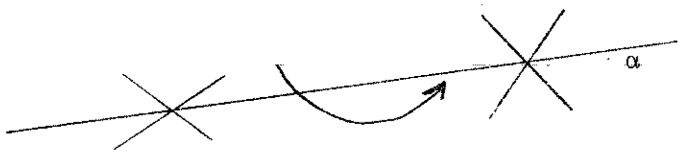
Leaves invariant the line α obtained by taking the segments of shortest distance between successive lines. (The common perpendicular). We have a rotation and a translation along α . γ is hyperbolic.

So, in dimension 2 , the isometries are rotations about a point and translations along a geodesic.



Elliptic Parabolic Hyperbolic

In dimension 3, the general case is rotation about a line and translation along the line

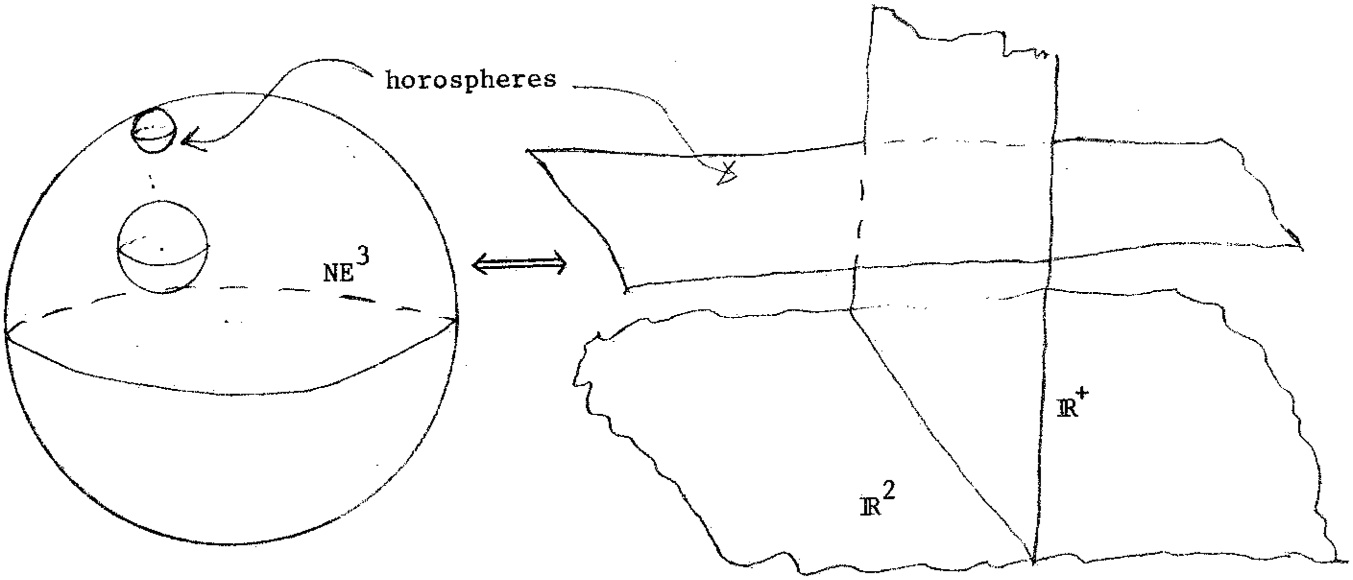


Inside the 3-dimensional non-Euclidean model NE^3 , we have all 3 of 2-dimensional geometries :

I) Spherical geometry.

II) Euclidean geometry : the horospheres.

III) Hyperbolic geometry : the geodesic planes.



The isometries of NE^3 are in 1 - 1 correspondence with conformal transformations of S^2 .

2. Triangle groups.

Given integers $p,q,r \geq 2$, let $\Gamma_{p,q,r}$ be the group with generators x,y,z, and relations

$$x^2 = y^2 = z^2 = 1$$

$$(xy)^p = (yz)^q = (zx)^r = 1$$

Then $\Gamma_{p,q,r}$ can be represented as a group of isometries of one of the 2-dimensional geometries :

I) Spherical geometry, if $1/p + 1/p + 1/r > 1$.

II) Euclidean geometry, if $1/p + 1/q + 1/r = 1$.

III) Non-Euclidean geometry, if $1/p + 1/q + 1/r < 1$.

For this, let P denote either the Riemann sphere S^2 , the Euclidean plane E^2 , or the model NE^2 for non-Euclidean geometry. Then, given integers p,q,r as above, there exists a triangle $T = T_{p,q,r}$ in P , bounded by geodesics, with interior angles π/p , π/q , π/r . (T lies in S^2 , E^2 or NE^2 according as $1/p + 1/q + 1/r - 1$ is positive, zero or negative).

Theorem (Poincaré(*)). The group $\Gamma_{p,q,r}$ is isomorphic to the group of isometries of P generated by reflections on the 3 edges of T . The triangle T serves as fundamental domain for the action of $\Gamma_{p,q,r}$ on P .

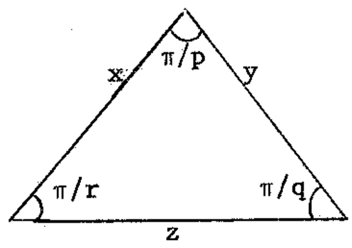
In other words, the various images of T under the action of $\Gamma_{p,q,r}$ give a tiling of P .

Proof. Let $\hat{\Gamma}$ be the group of isometries of P generated by reflections on the 3-sides of T . It is easily seen that $\hat{\Gamma}$ is generated by 3 elements which satisfy the same relations satisfied by the generators $\Gamma = \Gamma_{p,q,r}$. So, there is a canonical homomorphism from Γ onto $\hat{\Gamma}$, and we want to see that it is in fact an isomorphism. We first form an abstract object X as follows : Let

$$\tilde{X} = \Gamma \times T ,$$

(*) In "Théorie des Fuchsien groupes". Tome II. See also [Milnor], "On the 3-dimensional Brieskorn manifolds $M(p,q,r)$" , in Annals of Maths. Studies No. 84, p. 175-225. Edited by L.P. Neuwirth.

taking one copy of T for each element in Γ ; we associate each generator of Γ to an edge of T ,



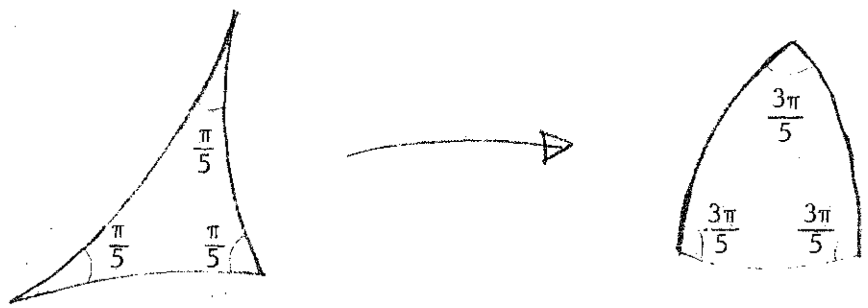
and we call this edge its domain. Now, define $X = \tilde{X}/\sim$, where $\sim$ is the relation : $(g,t) \sim (gx,t) \Leftrightarrow t \in$ domain x , similarly for y,z . Then X is a combinatorial surface; now we make it geometric : embed T in P , and use the homomorphism $\Gamma \to \hat{\Gamma}$ to define a map $\tilde{X} \to P$ by $(\gamma,w) \to \gamma(w)$. This induces a map $\pi : X \to P$ that, we claim, is a homeomorphism. For this, one shows first that this is a local homeomorphism (this can be verified by inspection on the three vertices of T , which are the only points where we could have difficulties); next, we observe that, since P is complete, every path in P can be lifted to a path in X , hence π is a covering projection. The rest of the theorem now follows because P is simply connected.

It follows from Poincaré's thoerem that in the spherical case, the group $\Gamma_{p,q,r}$ has order

$$\frac{\text{area }(S^2)}{\text{area } T} = \frac{4}{p^{-1}+q^{-1}+r^{-1}-1} .$$

In the Euclidean and the non-Euclidean cases, the group $\Gamma_{p,q,r}$ has infinite order. Note that the only triples (p,q,r) for which $p^{-1}+q^{-1}+r^{-1}$ is greater than 1 are $(2,2,r)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$. The corresponding groups $\Gamma_{p,q,r}$ are the dihedral group of order $4r$, the tetrahedral group, the octahedral group and the icosohedral group. (Beware that these names are often reserved for the corresponding subgroups of index 2 consisting of orientation preserving isometries). In the Euclidean case $p^{-1}+q^{-1}+r^{-1}=1$, we have only the triples (2,3,6) , (2,4,4) and (3,3,3). For all remaining possible triples, infinitely many of them, we are in the hyperbolic case.

Remark. In many cases, one can actually represent a triangle group $\Gamma_{p,q,r}$, with $p^{-1}+q^{-1}+r^{-1}<1$, in the spherical isometries. What happens is that we can often find a suitable integer $a \in \mathbb{Z}$ such that if we enlarge (one or more of) the angles of the triangle $T_{p,q,r}$ by this factor, we obtain a new "spherical" triangle (i.e. with sum of its angles > π) whose associated group of reflections is isomorphic to the original group $\Gamma_{p,q,r}$.

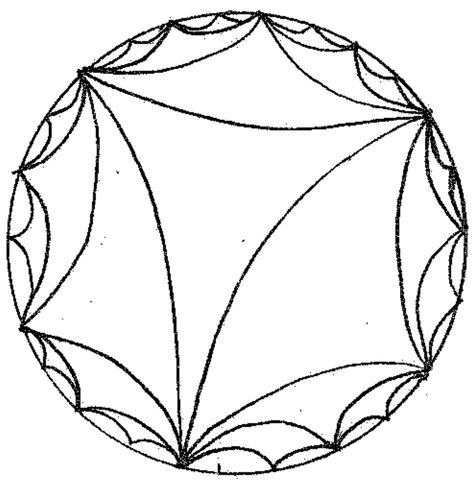


For example, $\Gamma_{5,5,5}$ is isomorphic to the group of isometries of S^2 generated by reflections on the edges of a triangle with angles $3\pi/5, 3\pi/5, 3\pi/5$. Note however, that in this case we do not have a tiling of S^2 , the various images of the triangle overlap, and they wind around the sphere an infinite number of times.

Now, if instead of looking at the full triangle groups $\Gamma_{p,q,r}$, we look at the subgroups $\Gamma^+_{p,q,r}$ consisting of orientation preserving maps, then $\Gamma^+_{p,q,r}$ is a subgroup of $PSL_2(\mathbb{R})$ (if $p^{-1}+q^{-1}+r^{-1}<1$), which is the group of all orientation preserving isometries of $\mathbb{H}^2$. Under the standard inclusion $PSL_2(\mathbb{R}) \longrightarrow PSL_2(\mathbb{C})$, they give orientation preserving isometries of hyperbolic 3-space $\mathbb{H}^3$. The isomorphism mentioned above can be proved using a field automorphism of $\mathbb{C}$ to change the angles, which then induces an automorphism in $PSL_2(\mathbb{C})$ taking the triangle group $\Gamma^+_{5,5,5}$ into the corresponding group of spherical isometries.

<u>Exercise.</u> Construct such a proof.

We have then seen that there exist finitely generated discrete groups of isometries of $\mathbb{H}^2$ with compact fundamental domain. If we do the same constructions with an ideal triangle, we get essentially the modular group $\Gamma_{\infty,\infty,\infty} \cong SL_2(\mathbb{Z})$. It has fundamental domain with finite area. (A subgroup of index 2 in $\Gamma_{\infty,\infty,\infty}$ is a subgroup of index 6 in $SL_2(\mathbb{Z})$)



<u>Exercise.</u> Show that given $\alpha_1,\ldots,\alpha_n$ with

$$\sum_{i=1}^{n} \alpha_i < (n-2)\pi$$

there exists a convex n-gon in $\mathbb{H}^2$, with angles $\alpha_1,\ldots,\alpha_n$.

Now, what about tilings of $\mathbb{H}^3$?, and of $\mathbb{H}^n$?.

<u>Theorem</u>. Consider a convex solid in $\mathbb{H}^3$ with dihedral angles π/n_i . Let Γ be the group with 1 generator x_i for each face and relations

$$x_1^2 = x_2^2 = \dots = 1$$

$$(x_1x_2)^{n_1} = (x_2x_3)^{n_2} = \dots = 1$$

then Γ gives a tiling of $\mathbb{H}^3$.

<u>Proof</u>. The same construction as for the triangle groups

$$\Gamma \times (\text{solid})/\sim$$

gives the tiling.

So now, what kinds of convex figures can be realized in $\mathbb{H}^3$?

<u>Theorem</u>. (Andreev. Math. USSR, Sbornik, 1970). Any decomposition of S^2 with angles $\leq \pi/2$ (with some fine print) can be realized uniquely by a convex solid in $\mathbb{H}^3$.

This shows that there are many tessellations of $\mathbb{H}^3$ by convex solids and reflection groups. (For example, by a regular dodecahedron with angles $\pi/2$. Same considerations with an ideal tetrahedra yield finite volume fundamental domain.

In higher dimensions, we have the arithmetic groups, to be treated later.

<u>Exercise</u>. Construct all groups possible using regular solids for $\mathbb{H}^n$, $n = 2,3,4$.

3. Tilings of $\mathbb{H}^n$ by arithmetic construction.

We show that there exist discrete subgroups of isometries $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ such that $\mathbb{H}^n/\Gamma$ is compact, thus giving a tiling of $\mathbb{H}^n$.

Lattices in $\mathbb{R}^n$ are discrete subgroups isomorphic to $\mathbb{Z}^n$. They are characterized up to a compact set of possibilities by two invariants :

S = minimal separation, the distance from the origin of the lattice point nearest the origin.

V = volume, the volume of a fundamental region of the lattice = det M where M is a matrix composed of basis vectors for the lattice.

Let L be the set of lattices in $\mathbb{R}^n$. We can topologize L as follows : two lattices are close $\Leftrightarrow$ can find bases such that the basis vectors are close. (Equivalently, one observes that $L \simeq GL_n(\mathbb{R})/GL_n(\mathbb{Z})$ and as such has an induced topology).

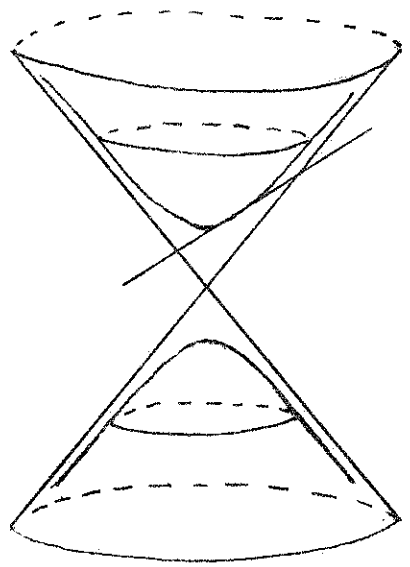
Proposition (Hermite-Mähler). A set of lattices L_o has compact closure in L if the minimal separation S is bounded from below and the volume V is bounded from above.

Lorentz model for hyperbolic space : Consider $\mathbb{R}^{n+1}$ with the quadratic form $\varphi = x_1^2 + x_2^2 + \ldots + x_n^2 - t^2$, and let G be the subgroup of invertible linear transformations of $\mathbb{R}^{n+1}$ leaving φ invariant

$$G = \{M \in GL_{n+1}(\mathbb{R}) \,\big|\, \varphi(M\vec{v}) = \varphi(\vec{v}) \ , \ \vec{v} \in \mathbb{R}^{n+1}\}$$

The locus of points satisfying $\varphi = -1$ is a two-sheeted hyperboloid, on which G acts. φ restricted to the tangent spaces to the hyperboloid is positive definite (see figure), this allows us to define a metric on the hyperboloid that is invariant under the group G.

Denoting one sheet of the hyperboloid by $\mathbb{H}^n$, we have G acting as a group of isometries of $\mathbb{H}^n$.



G acts transitively since it contains the full rotation group around the point $(0,0,..,0,1)$, and any point on $\mathbb{H}^n$ can be sent to $(0,0,...,0,1)$ by an element of the form $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$.

Now, we consider the quadratic form $\varphi = x_1^2 + \ldots + x_n^2 - \sqrt{2}t^2$ on $\mathbb{R}^{n+1}$. As before one constructs a model for $\mathbb{H}^n$ by taking one sheet of the hyperboloid defined by $\varphi = -1$. Let G be the subgroup of $GL_{n+1}(\mathbb{R})$ leaving φ invariant. Now define Γ_n to be the subgroup of G whose matrix entries lie in $\mathcal{O}(Q(\sqrt{2}))$, the integers of $Q(\sqrt{2})$. Thus

$$\Gamma_n = \{M = (a_{ij}) \mid a_{ij} = u_{ij} + m_{ij}\sqrt{2} \text{ and } \varphi(M\vec{v}) = \varphi(\vec{v})\}$$

<u>Theorem.</u> Γ_n is a discrete group of isometries and H^n/Γ_n is compact.

We first give a proof in the following analogous case :

Consider $\mathbb{R}^4$ with the quadratic form $\varphi = x_1^2 + x_2^2 + x_3^2 - 7t^2$ and let $G \subset GL_4(\mathbb{R})$ be the group preserving φ and Γ be the integer matrices of G. Then $\mathbb{H}^3/\Gamma$ is compact.

Proof. Since integral matrices are discrete in real matrices $\Gamma \subset G$ is discrete. Since G acts transitively on $\mathbb{H}^3$ with a compact stabilizer group, it suffices to show that G/Γ is compact; this implies H^3/Γ is compact. Note first that G/Γ can be identified with the G-orbit, $L_o \subset L$, of the basic integer lattice $\mathbb{Z}^4 \subset \mathbb{R}^4$.

We will show that L_o is closed in L and that the minimal separation is bounded below while the volume is bounded above, so that by Mahler's proposition, L_o is a closed set with compact closure, hence compact. Thus G/Γ and hence $\mathbb{H}^3/\Gamma$ are compact.

S is bounded below on L_o : Reducing $x_1^2 + x_2^2 + x_3^2 - 7t^2 = 0$ modulo 8, it is clear that φ restricted to $\mathbb{Z}^4 - \{\vec{0}\}$ takes non-zero integer values. Since φ is continuous and $\varphi(\vec{0}) = 0$, the neighborhood U about the origin on which $\varphi \leq 1/2$ contains no point of $\mathbb{Z}^4 - (\vec{0})$. But L_o is the orbit of the lattice $\mathbb{Z}^4$ by G, which leaves φ invariant, so there is no point of any lattice $L \in L_o$ in U, and S is bounded below on L_o.

V is bounded above on L_o : The basic lattice $\mathbb{Z}^4$ has volume 1 and since elements of G leave φ invariant, they have determinant 1. Hence the volume is bounded by 1 on L_o.

L_o is closed in L : Let L_* be a lattice in L and suppose that there exists a sequence $\{L_i\}$ of lattices in $\mathcal{L}_o$ converging to L_*. Since φ is continuous and $\varphi|L_i$ takes integer values, $\varphi|L_*$ must take integer values.

Let $\{v_i\}$ be a basis for L_*. Then there exists $L_{i_o} \in \{L_i\}$ with basis $\{u_i\}$ such that u_i is close to v_i and $\varphi(u_i) = \varphi(v_i)$ as well as $\varphi(u_i \pm u_j) = \varphi(v_i \pm v_j)$, which implies that $\varphi(u_i,u_j) = \varphi(v_i,v_j)$. Hence the element of $GL_4(\mathbb{R})$ taking L_{i_o} to L_* leaves φ invariant and $L_* \in \mathcal{L}_o$.

<u>Remarks on the general case</u> : Since any integer can be expressed as the sum of four squares, the quadratic form $\varphi = x_1^2 + \dots + x_n^2 - 7t^2$ takes zero values on $\mathbb{Z}^n - \{\vec{0}\}$ for $n \geq 4$. Therefore, one uses the form $\varphi = x_1^2 + \dots + x_n^2 - \sqrt{2}t^2$ over the integers of $Q(\sqrt{2})$ and the above arguments go through with a standard modification.

Since $\mathcal{O}(Q(\sqrt{2}))$ is not discrete in $\mathbb{R}$, we need to work harder to show that $\Gamma \subset G$ is a discrete group of isometries. First notice that the integers $\mathcal{O}(Q(\sqrt{2}))$ from a discrete subring of the product ring $\mathbb{R}\times\mathbb{R}$ via the embedding $(m+n\sqrt{2}) \longrightarrow (m+n\sqrt{2}, m-n\sqrt{2})$. This induces an embedding of Γ in

$$G\ell_{n+1}(R \times R) \equiv G\ell_{n+1}(R) \times G\ell_{n+1}(R)$$

as a discrete group. Since Γ leaves $\varphi = x_1^2 + \dots + x_n^2 - \sqrt{2}\,t^2$ invariant, the image of Γ in the second copy of $GL_{n+1}(\mathbb{R})$ belongs to the orthogonal group of $\overline{\varphi} = x_1^2 + \dots + x_n^2 + \sqrt{2}\,t^2$, which is a compact group. Thus Γ must be discrete in the first factor $GL_{n+1}(\mathbb{R})$.

Remark. A more general theorem was proved by A. Borel in Vol(1) Topology. This makes use of the concepts of arithmetic and algebraic groups (ref. Borel : Introduction aux Groupes Arithmetiques).

What does this proof yield in $\mathbb{H}^1$? It implies the existence of solutions m , n to Pell's equation $n^2-am^2 = \pm 1$ which implies the existence of units in the integers $\mathcal{O}(Q\sqrt{a})$.

Let $\varphi = x^2-at^2$ where a is a square-free integer. Then $\varphi|\mathbb{Z}^2-\vec{0} \neq 0$, and hence $\mathbb{H}^1/\Gamma$ is compact where Γ is the set of integer matrices leaving φ invariant. In particular, Γ is infinite. But Γ is isomorphic to the units of $\mathcal{O}(Q(\sqrt{a}))$ (i.e. the integers $m+n\sqrt{a}$ of norm ± 1 , $m^2-an^2 = \pm 1$) via the homomorphism $m+n\sqrt{a} \to \begin{pmatrix} m & an \\ n & m \end{pmatrix}$. Often the units are difficult to find in practice. For example, in $\mathcal{O}(Q\sqrt{94})$ the smallest unit after 1 is given by $2{,}143{,}295 + 221064\sqrt{94}$!

Remark. Apparently Fermat sent these examples to the English mathematicians (especially Pell), presumably contributing to the strong tradition in number theory then.

In infinite dimensions, one can construct groups of hyperbolic motions by inversions in spheres, just as in the finite dimensional case. There exists an Hilbertian hyperbolic geometry taking the unit ball of Hilbert space and the group generated by inversions. Then taking $\{\Gamma_n \subset \Gamma_{n+1} \subset \dots\}$ any sequence of cocompact groups constructed by the above theorem, the group $\Gamma_\infty = \bigcup_{n=1}^{\infty} \Gamma_n$ is a discrete subgroup of isometries of $\mathbb{H}^\infty$.

Problems. 1) What is a maximal discrete group containing Γ_∞ ?

What is its fundamental domain ?

2) Is there a discrete group with bounded fundamental domain ? (This corresponds to cocompact in finite dimensions).

<u>Exercise</u> 1. Show that the rank of the group of units in the integers of an algebraic number field K is given by $\frac{C}{2}+R-1$ where C (resp. R) is the number of complex (resp. real) embeddings of K in $\mathbb{C}$. (This is a form of the Dirichlet unit theorem)

<u>Exercise 2</u>. Consider a finite group Γ acting irreducibly on $\mathbb{R}^n$: $\rho : \Gamma \to \mathrm{Aut}(\mathbb{R}^n)$. An integral form of ρ is determined by a lattice invariant by Γ. Show that the number of equivalence classes of integral forms is finite.

<u>Exercise 3</u>. (Corollary to Ex. 2) : Show that the set of ideal classes in an algebraic number field is finite.

These exercises are all applicationsof the compactness of certain sets of lattices.

How does one pass from the discrete cocompact groups of isometries constructed above to manifolds ? Since Γ_n is discrete, it acts properly discontinuously on $\mathbb{H}^n$ and $\mathbb{H}^n/\Gamma_n$ is a Hausdorff space. H^n/Γ_n is a geometrically non-singular manifold if Γ_n has no torsion, or equivalently if Γ_n acts without fixed points.

<u>Proposition</u>. The group Γ_n has subgroupsof finite index without torsion.

<u>Proof</u>. We take the case where Γ is the group of integer matrices preserving the form $x_1^2+x_2^2+x_3^2-7t^2$. Let $\Gamma' = \Gamma \cap \{\text{integer matrices} \equiv 1 \pmod 3\}$. Now, we

show that any finite subgroup F of $GL_n(\mathbb{Z})$ injects into $GL_n(\mathbb{Z}/3\mathbb{Z})$. This implies that the kernel of reduction mod 3 has no finite subgroup, hence no torsion, and thus Γ' is torsion-free.

First, reduce to the case where F acts irreducibly on $\mathbb{R}^4$. Then introduce an F-invariant metric normalized so that the shortest non-zero lattice vector of F has length one. Now, the orbit of a closet lattice point on unit sphere gives a spanning set for $\mathbb{R}^n$. But these vectors are at most distance 2 apart and hence can't be identified mod 3.

4. Construction of fundamental domains.

There are two natural ways of constructing a fundamental domain for a given discrete group of isometries of $\mathbb{H}^n$, one is that of Dirichlet, the other is that of Ford. Before describing these methods, let us give some of Poincaré's motivations for studying discrete groups (the reference is [Ford], automorphic functions, Chelsea, 1929). Consider a 2^{nd}-order differential equation

$$\eta'' + A(w)\ \eta' + B(w)\eta = 0 \tag{1}$$

where A,B are holomorphic functions in a region S of the complex w-plane. Given 2 linearly independent solutions $\rho_1(w)$, $\rho_2(w)$ of this equation at a point w_o, any other solution is of the form,

$$\rho(w) = a\rho_1(w) + b\rho_2(w) \tag{2}$$

where a,b are constants. Now consider a loop L in S based at w_o; if we continue ρ_1, ρ_2 analytically along L, we obtain a new pair ρ_1', ρ_2' of "monodromic" solutions at w_o. By (2), these are of the form

$$\rho_1'(w) = a\rho_1(w) + b\rho_2(w)$$

$$\rho_2'(w) = c\rho_1(w) + d\rho_2(w)$$

with $ad - bc \neq 0$. (Since ρ_1' and ρ_2' are also linearly independent). Therefore, the ratio ρ_1'/ρ_2' defines a linear transformation

$$w \longrightarrow \frac{a\rho_1(w) + b}{c\rho_a(w) + d}$$

The set of all linear transformation that arise in this way, i.e. by considering all loops based at w_o, forms a group Γ that Poincaré called the group of the differential equation. If we start with a different pair of solutions, we obtain a conjugate of Γ by a linear transformation.

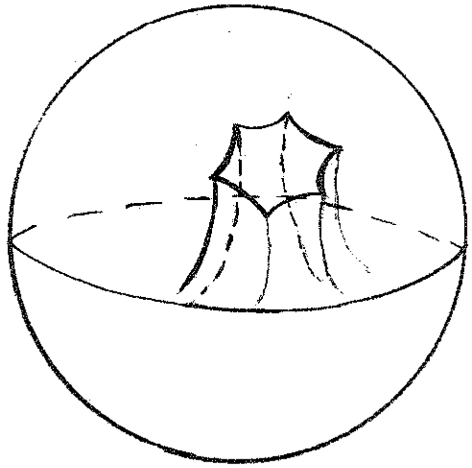
If the region S is simply connected, then any solution of (1) at w_o is analytic in all of S, so the group Γ is trivial. Otherwise, Γ may be finite, infinite or even a continuous group.

Now consider the inverse function F of the ratio of two solutions, $F(\rho_1(w)/\rho_2(w)) = w$. This is, generally speaking, a multivalued function, and Poincaré's question was, when is F single valued ? For this he noted that if ρ_1', ρ_2' are monodromic solutions of (1) obtained by analytic continuation of ρ_1, ρ_2 along a certain loop, then

$$F(\rho_1'/\rho_2') = F(\rho_1/\rho_2)$$

or in other words, F is invariant under the action of Γ (i.e. F is a Γ-automorphic function). If we assume that F is single valued, then it is a (non constant) holomorphic function on some region $S' \subset S$, invariant under Γ. Look at the Γ-orbit of a point in S', then this orbit cannot have an accumulation point in S', otherwise F would be constant. Therefore, the group Γ acts properly discontinuously on S'.

These groups are discrete in the sense we have been talking about, except that they have a fundamental domain that hits the sphere at ∞ in a fundamental domain for the action of the group on this sphere. They were called Kleinian groups by Poincaré. He called them Fuchsian if they also have real coefficients, or preserve a circle in the complex plane.



We now return to the construction of a fundamental domain for a group Γ of isometries of $\mathbb{H}^n$. As we mentioned before, there are two natural ways.

i) Dirichlet way. It makes sense if you have a discontinuous group of isometries on a metric space

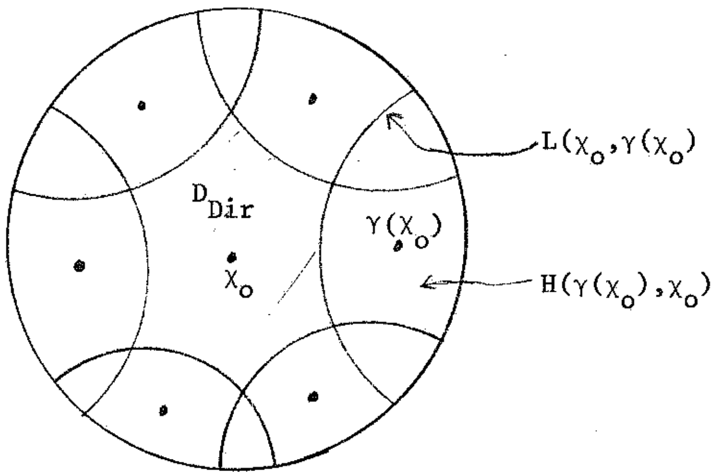
ii) Ford way. This is based on the Euclidean metric, and you see how much the metric is distorted. It has the advantage that it also makes sense for points at ∞.

Dirichlet way.

Fix a point $x_o \in \mathbb{H}^n$ which is not fixed by any element of Γ except the identity. Look at its orbit, then the Dirichlet's fundamental domain D_{Dir} of Γ consists of all points closer to x_o than to any other point in its orbit, that is,

$$D_{Dir} = \{\chi \in \mathbb{H}^n | \rho\ x, x_o) \leq \rho(x, \gamma(x_o)) \quad \text{for every} \quad \gamma \in \Gamma - \{1\}\}$$

where $\rho(,)$ is the hyperbolic distance.



If given 2 points $x_1, x_2 \in \mathbb{H}^n$, we let $L(x_1,x_2)$ be the hyperplane

$$L(x_1,x_2) = \{x \in \mathbb{H}^n \mid \rho(x,x_1) = \rho(x,x_2)\}$$

then $L(x_1,x_2)$ divides $\mathbb{H}^n$ in two convex half-spaces $H(x_1,x_2)$ and $H(x_2,x_1)$, where

$$H(x_1,x_2) = \{x \in \mathbb{H}^n \mid \rho(x,x_1) < \rho(x_1,x_2)\}$$

and $H(x_2,x_1)$ is defined similarly.

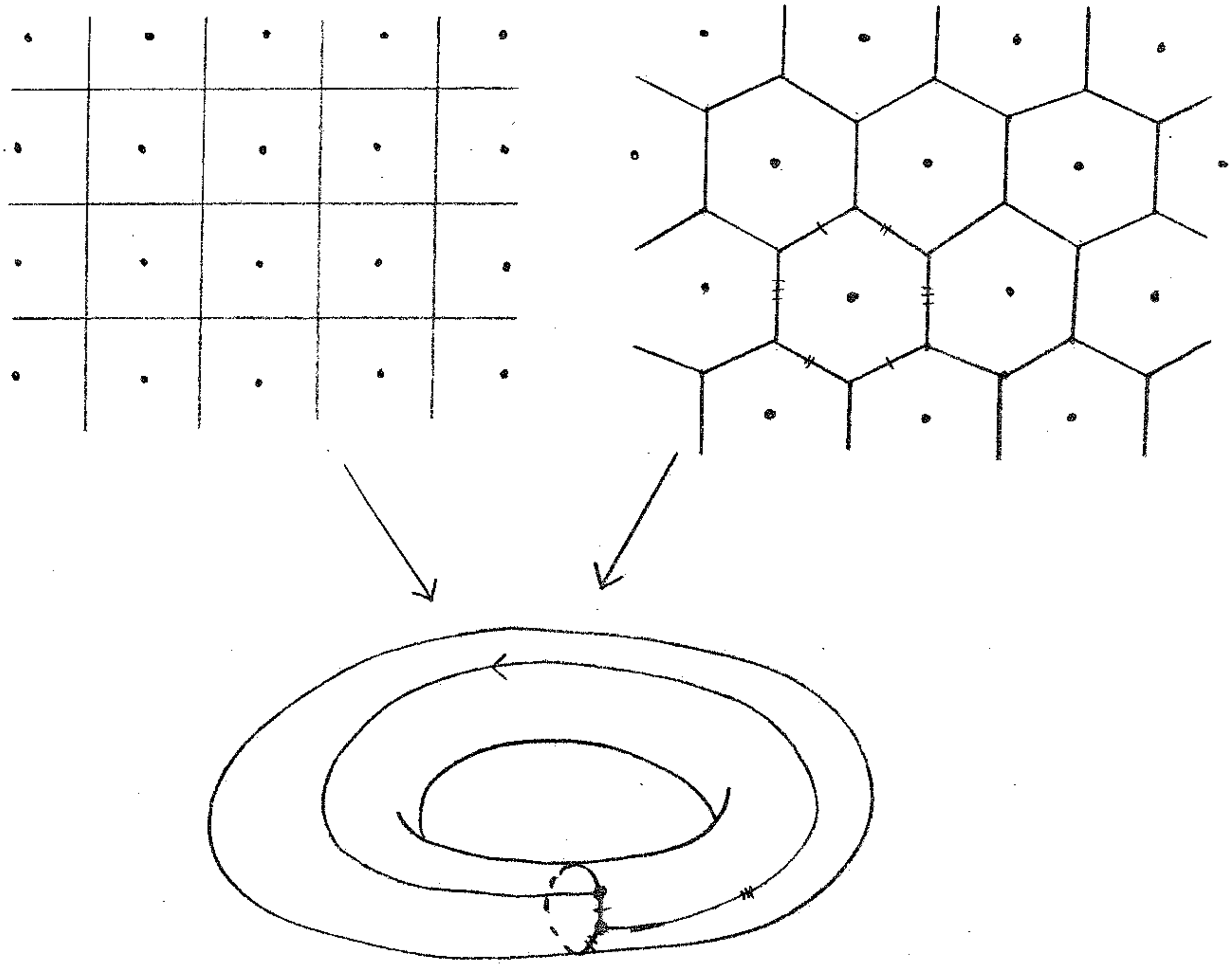
Returning to our fundamental domain for Γ, one clearly sees that the interior of D_{Dir} is,

$$\mathring{D}_{Dir} = \bigcap_{\gamma \in \Gamma - 1} H(x_o,\gamma(x_o))$$

which shows that D_{Dir} is convex .

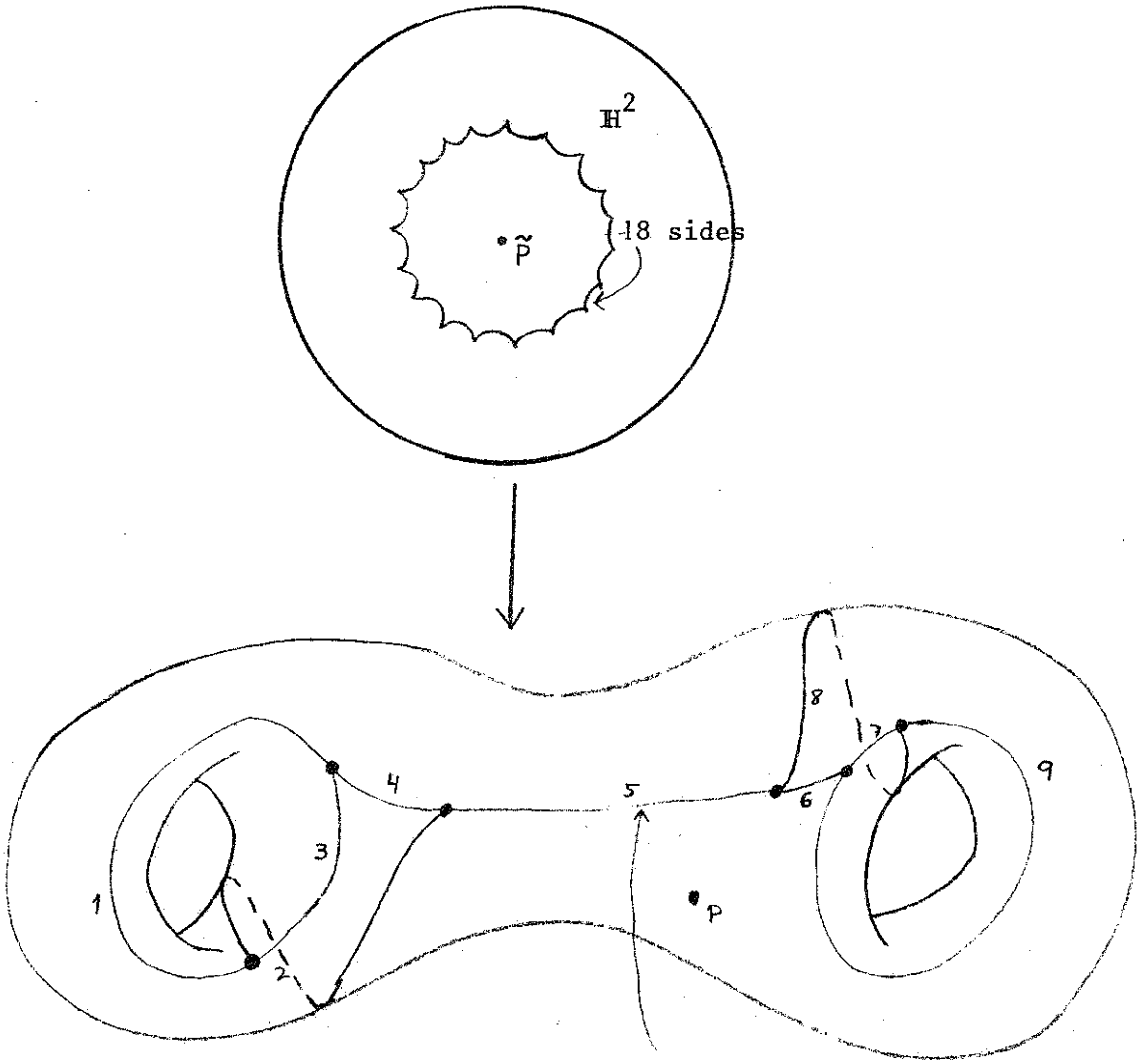
<u>Remark</u>. If we consider the ordinary lattice L_o in $\mathbb{R}^2$, the construction above yields the dual lattice. However, this is a special case. The generic

one, moving L_o slightly, yields an hexagon as fundamental domain.



<u>Exercise.</u> Consider a group $\Gamma_g \subset SL_2(\mathbb{R})$ that yields a surface of genus g. Show that in the generic case, D_{Dir} has $12g-6$ sides. (Hint. Use the Euler characteristic)

Example. Γ_2



Cut locus as viewed from p .

Exercise. How many different ways there are of glueing the 18-gon to get different 2-holed-tori. (Different means up to orientation preserving diffeomorphism of the surface).

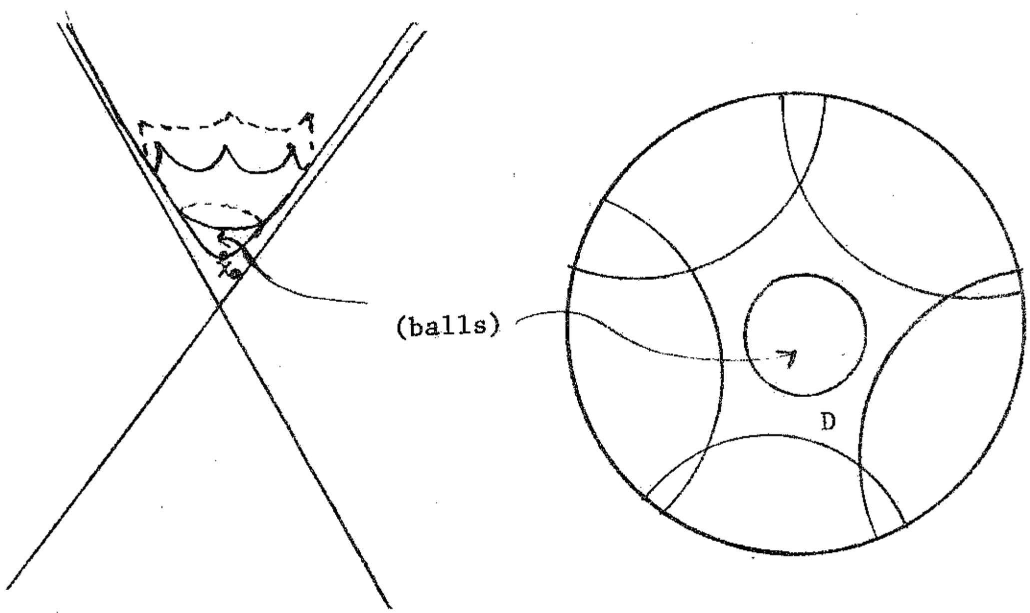
There are about 9. This will appear in a joint paper by Jörgensen and Natainen.

Lee Mosher studied the case $g = 3$. There are 1726 different ways. In the case of $g = 4$, one guesses that there must be about 20 million !

Example. Let Γ_n be the group on matrices with entries of the form $(n_{i_j} + m_{i_j}\sqrt{2}), n_{i_j}, m_{i_j} \in \mathbb{Z}$, that perserve the quadratic form

$$\rho(x) = -\sqrt{2}\, t^2 + x_1^2 + x_2^2 + \dots + x_n^2 \ .$$

Choose the point $x_o = (1,0,\dots,0)$ as base point. Its isotropy group which is contained in $SO(n)$, consists of the signed permutations of the $x_i'S$. It has order $2^n n!$. Now, disregard these signed permutations and construct a Dirichlet domain for Γ_n with respect to x_o.



Theorem. 1) Γ_n is uniformly discrete. That is, there exists a ball contained in $D_{Dir}(\Gamma_n)$ of radius independent of n.

2) diam $D_{Dir}(\Gamma_n) \geq c \log n$, with c constant.

Exercise. What does the volume $\mathrm{Vol}(\Gamma_n)$ looks like as $n \to \infty$?

Sketch of proof. The idea to prove 1) is to show that we have a minimal distance from x_o to any other point in its orbit, independently of n. For this we note that the hyperbolic distance from a point P in the hyperboloid to the base point x_o is a function of its "height", i.e. of the t-component. In fact, if $\gamma \in \Gamma_n$ is a matrix operating on x_o, then the distance from x_o to $\gamma(x_o)$ is $\sim\log|\text{t-component}|$ of t-column of γ. Now,

$$\varphi(\text{t-column}) = -\sqrt{2} \quad \text{and} \quad \varphi(\text{x-column}) = 1 .$$

therefore, the t-component of an x-column is either 0 or $\geq 1+\sqrt{2}$ and the t-component of a t-column is

$$\begin{cases} 1 & \text{if all others are } 0 \\ \geq 1+\sqrt{2} & \text{Otherwise} \end{cases} .$$

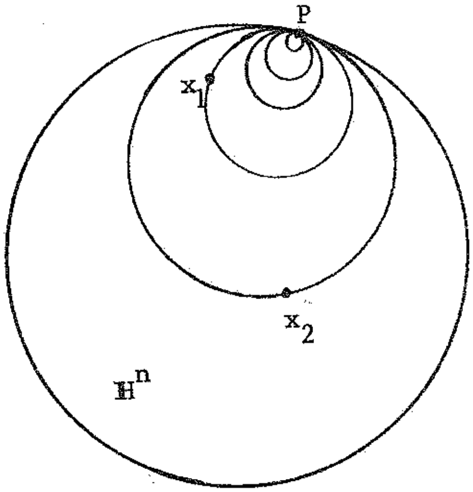
Actually, if many x_i's are non-zero, then t has to be very large. This gives minimal distance independent of dimension.

The idea for 2) is to find enough hyperplanes to chop off D_{Dir}, and then use the fact that the main diagonal $(1,\ldots,1)$ forms an angle θ of almost $\pi/2$ with respect to the principal axis (i.e. $\cos\theta = \frac{1}{\sqrt{n}}$) to show that, at least for one generating side, the t-component is $\geq \sqrt{n}$, so the length of the main diagonal in D_{Dir} tends to ∞ as $n \to \infty$

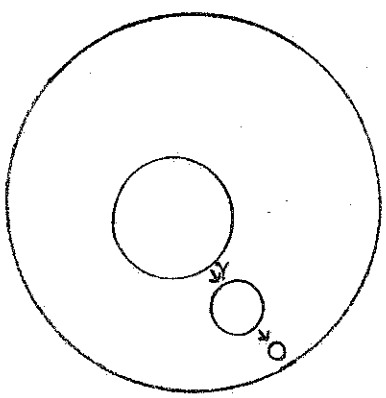
Ford's way.

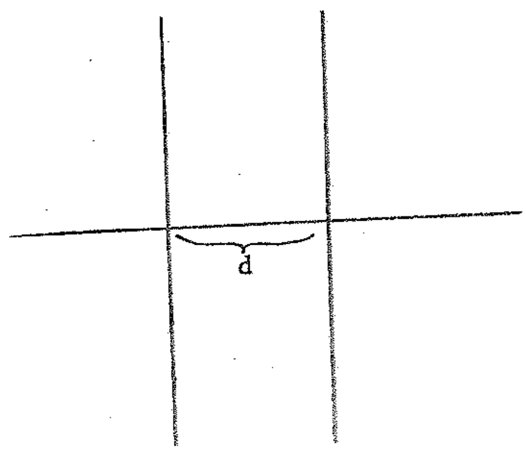
The fundamental domain for a group Γ obtained by Ford's method depends on the model for $\mathbb{H}^n$ that we use. If we use the unit ball model, we obtain the same domain as for Dirichlet's method. If we use the upper half-space model, we may get a different one. We use the ball model here.

Note that if we fix a point P at ∞, then we can look at all the horospheres based at P, and they give an ordering to all points in $\mathbb{H}^n$: given any two points in $\mathbb{H}^n$ you can decide which one is closer to P.

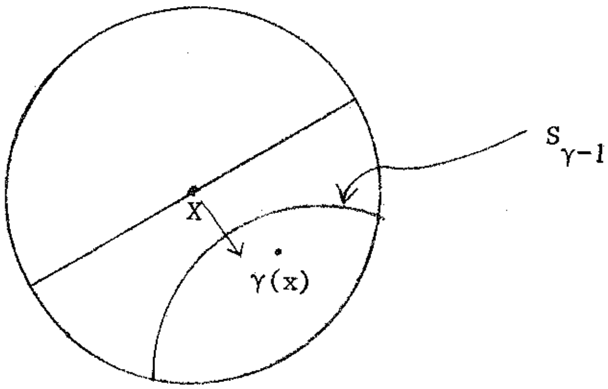


Ford's method is based on the Euclidean metric on $B^n \subset \mathbb{R}^n$. The idea is that if you pick a sphere inside B^n and you move it towards the boundary by an element $\gamma \in \Gamma$, the sphere gets smaller and smaller as you approach the boundary, so you can decide where in $\mathbb{H}^n$ you are by looking at $|\gamma'(x)|$.

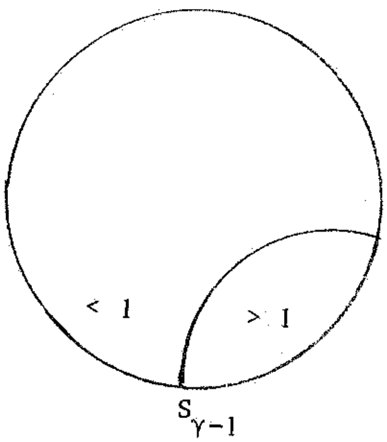


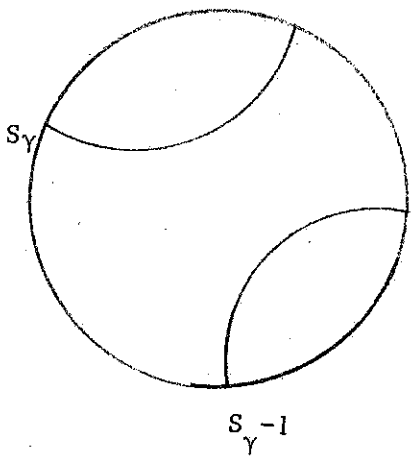


Any transformation in B^n is a rotation followed by a non-euclidean transformation. Suppose you have a translation by d, then this translation is a reflection on two hyperplanes whose distance is $d/2$. So if γ is a translation then it is given by a reflection in the hyperplane at origin, followed by a reflection on the bisecting hyperplane.



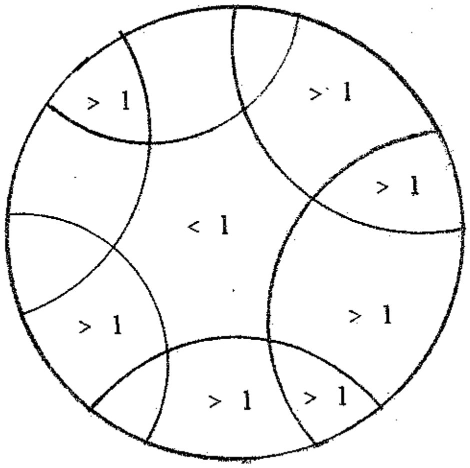
This bisecting hyperplane is Ford's isometric sphere $S_{\gamma^{-1}}$ of γ^{-1}, i.e. it is the set of points in B^n where $|\gamma^{-1}(x)| = 1$. In the inside all points have $|\gamma^{-1}(x)| > 1$, in the outside they have $|\gamma^{-1}(x)| < 1$. The transformation γ takes S_γ into $S_{\gamma^{-1}}$.





Now, to define Ford's fundamental domain of Γ, pick $x \in \mathbb{H}^n$ which is not fixed by any element of Γ. Then

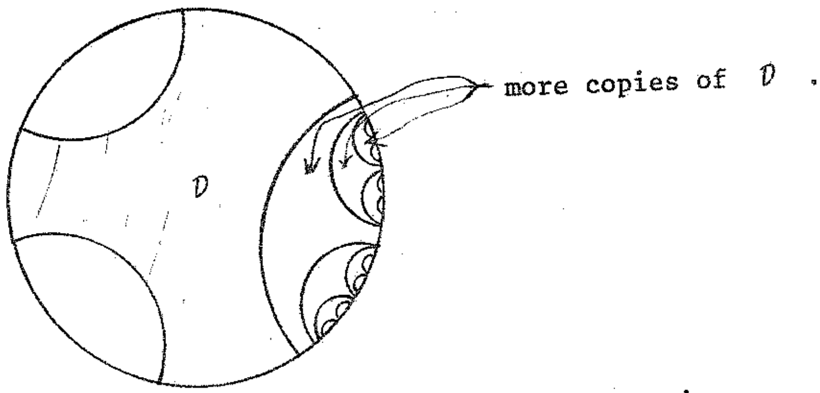
$$\mathcal{D} = \{x \mid |\gamma'(x)| \leq 1 \text{ for all } \gamma \in \Gamma\}$$



This gives the same fundamental domain as Dirichlet, but it also makes sense at ∞ (because the derivatives are defined).

To see that $\mathcal{D}$ is in fact a fundamental domain for Γ, first we note that if we pick two points x_1, x_2 inside $\mathcal{D}$, then they are not related by any element of Γ, because if there were a $\gamma \in \Gamma$ with $\gamma(x_1) = x_2$, then either $|\gamma(x_1)|$ or $|\gamma^{-1}(x_2)|$ would be > 1. Now, pick a point $x \in \mathbb{H}^n$. If all derivatives are < 1, you are in $\mathcal{D}$. Suppose some is > 1, then look at the maximum of all derivatives, say $|\gamma_o(x)|$. Then the chain rule implies that at the point $\gamma_o(x)$ all derivatives are < 1, so $\gamma_o(x) \in \mathcal{D}$.

<u>Example</u>. Consider the Schottky group Γ defined by reflections on 3 hyperplanes



This gives a "triangle" as fundamental domain, but it also gives a a fundamental domain in the circle at ∞ . The group Γ acts discontinuously on an open set of S^{∞} , the complement of this open set is the limit set of Γ .

Remark. The limit set of the Schottky group is a geometrically repeating Cantor set. [All Cantor sets are topologically the same. However, geometrically, there are wild Cantor sets (e.g. the Denjoy diffeomorphism) and geometrically repeated Cantor sets] .

5. Hyperbolic geometry as part of conformal geometry.

The theme of this chapter is to consider hyperbolic geometry as a part of conformal geometry. Conformal geometry also includes :

- Complex analytic transformations in $\mathbb{C}$ are conformal.
- Differentiable maps in $\dim_{\mathbb{R}} = 1$ are conformal; this contains the theory of codimension 1 foliations.
- 0 dimensional geometry (group $\longmapsto$ tree , etc).

In some sense this is a one dimensional discussion since conformal maps have one degree of freedom.

Consider the Poincaré ball model for hyperbolic space and let Γ be a discrete group of hyperbolic isometries.

Definition. The domain of discontinuity, Ω_Γ , for Γ is :

$$\Omega = \{x \in \text{ball} \mid \exists \text{ compact neighborhood } K \text{ of } x \text{ such that } \operatorname{card}(\gamma \in \Gamma \mid \gamma K \cap K \neq \phi) \text{ is finite}\}$$

The Poincaré limit set, Λ_Γ , is the complement in the hyperbolic ball of the domain of discontinuity, Ω_Γ .

Remarks.

1) Λ_Γ is a closed set.

2) Since Γ is discrete, the interior of the hyperbolic ball is contained in Ω_Γ and hence $\Lambda_\Gamma \subset \partial(\text{ball})$ = the sphere at infinity.

Remarkable Proposition. If C is any closed Γ-invariant set on ∂(ball) = sphere, then Γ acts discontinuously on sphere $\setminus C$ (C must contain at least 2 points).

Proof (due to Gromov). Form the hyperbolic convex hull, $H(C)$, of C in the ball. Define a map φ : sphere $\setminus C \rightarrow$ frontier$(H(C))$ by dropping each point perpendicularly to the frontier of the hull. Since C is invariant under Γ, φ is a Γ-invariant map and one has an equivariant image of sphere $\setminus C$ inside the ball where Γ acts discontinuously.

Corollary. In the sphere at infinity, there is a unique minimal, closed Γ-invariant set.

Proof. Two such minimal sets cannot intersect, otherwise they would not be minimal. The orbit of a point of a minimal set has to be dense, hence Γ is not discontinuous on a minimal set. Now by the proposition, the action of Γ is discontinuous on the complement of an Γ-invariant set, so it follows that there can be only one such minimal set.

In all, these considerations rule out the elementary groups : finite, parabolic with exactly one fixed point and axial with exactly two fixed points. They rule out the cases of no, one, or two limit points.

From the point of view of topological dynamics Λ and Ω behave very differently. On Ω, Γ acts discontinuously with no recurrence - points wander off to the sphere at infinity under Γ. On Λ every point has a dense orbit, hence Λ is sometimes called the non-wandering set.

Here, we have considered more special behavior where each of the two sets behaves better than in general. One also has the following examples :

Example. Consider the shift on an infinite sequence of $0,1$. Here the non-wandering set is not minimal.

Example. Consider the hyperbolic transformation $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ on $\mathbb{R}^2 \setminus 0$. This gives a non-discrete action on the wandering set.

Corollary. The Poincaré limit set Λ is either nowhere dense (i.e. interior $\Lambda = \phi$) or all of the sphere at infinity.

Indeed, if Λ had interior, the complement of $\mathring{\Lambda}$ would be a closed invariant subset C of the sphere and by the proposition Γ would act discontinuously on sphere $\setminus C = \mathring{\Lambda}$.

Corollary (left as exercise). For any relatively open set U of Λ there exist finitely many $\gamma_1, \ldots, \gamma_n \in \Gamma$ with $\Lambda = \gamma_1(U) \cup \ldots \cup \gamma_n(U)$.

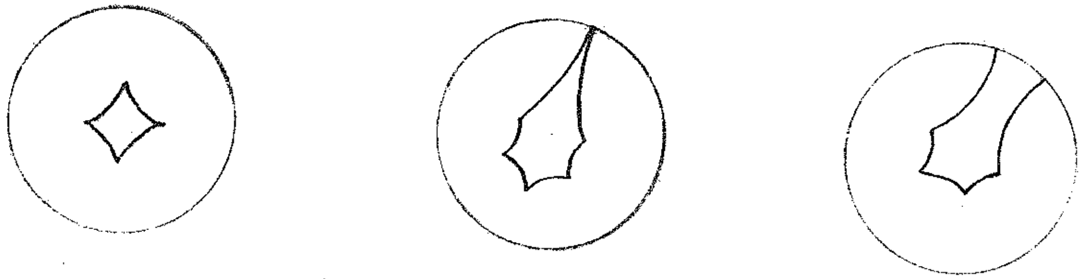
The geometrical properties of Λ will be interesting. According to the last corollary, it suffices to look at a small piece of Λ .

Example. If Γ is cocompact, then Λ = sphere .

We have another description of Λ . Take any point of hyperbolic space and look at its orbit under Γ . Then the limit set Λ will be the set of cluster points of this orbit. (This does not depend on the point chosen).

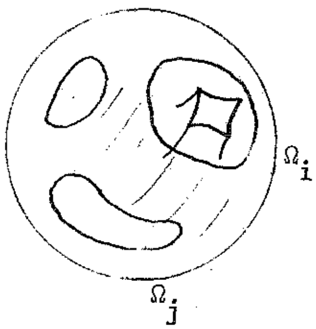
We have the following general picture for discrete groups Γ acting on the hyperbolic ball and having a fundamental domain with finitely many faces.

Γ cocompact | Γ cofinite volume



Λ = sphere | Λ = sphere | Λ ⊄ sphere

We have the following 3-dimensional picture :



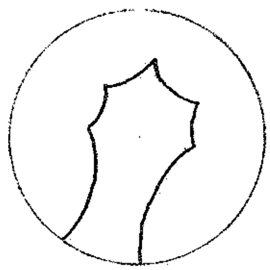
The fundamental domain intersects the sphere at infinity in a polygon, which gives a fundamental domain for the action of Γ on Ω . The fundamental domain is contained in a component Ω_i of Ω .

In dimension 2, discrete groups of hyperbolic isometries are called Fuchsian groups. We have the following picture :

Fuchsian group of the first kind (cocompact or cofinite volume groups)

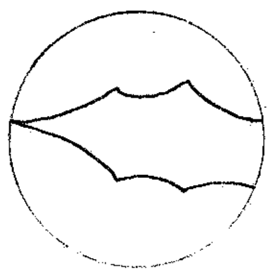
$\Lambda = S^1$ | $\Lambda = S^1$

Fuchsian groups of the second kind :



Λ = Cantor set

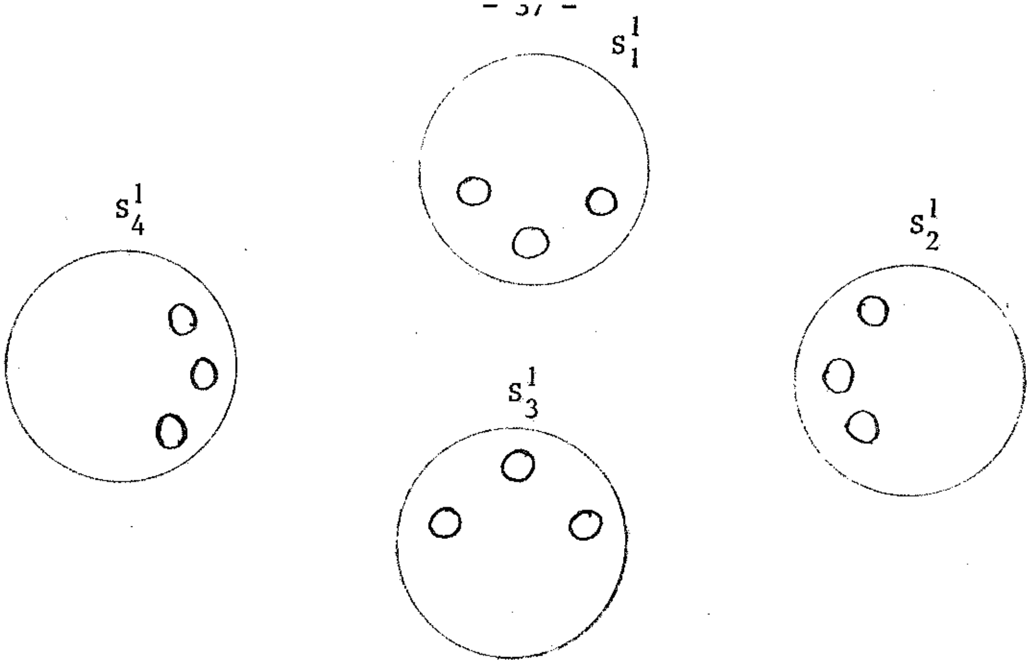
There is also the following case :



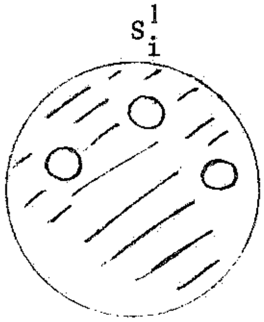
We will rule this out for the moment, since in this case the limit set Λ is not geometrically self-similar.

Exercise (for computor). Let Γ be the group generated by $z \to z+3$ and $z \to -\frac{1}{z}$. Determine the limit set of Γ . (Λ_Γ is of the complicated kind since $\Lambda_\Gamma \neq S^1$ and a fundamental domain for Γ has cusps).

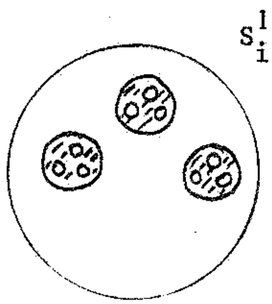
We now consider Γ generated by inversion in some collection (not necessarily finite) of disjoint circles. Consider the case of four circles (S_i^1 , $i = 1,\ldots,4$) . Reflecting in these circles gives :



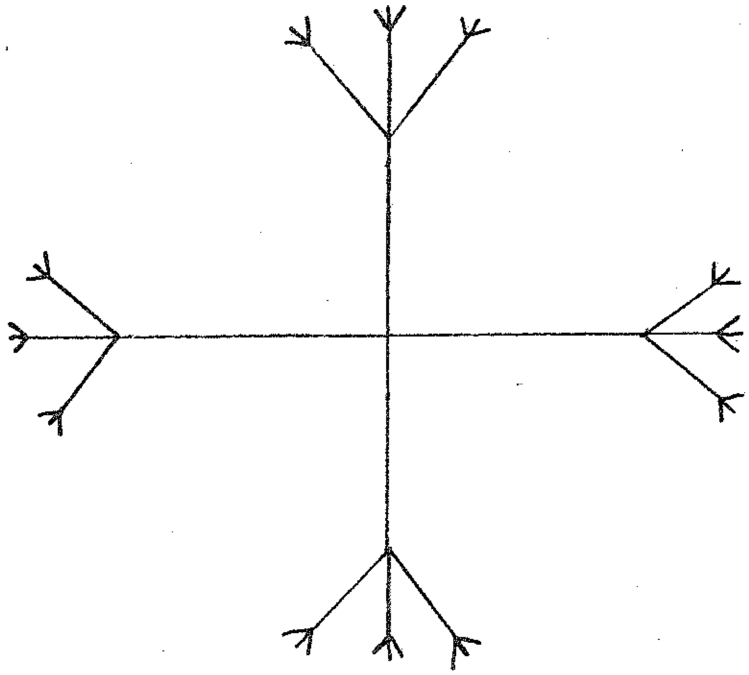
A fundamental domain for the action of Γ on S^2 is given by the intersection of the exteriors of each of the S_i^1 . Reflection about each of the 4 circles yields four addition fundamental domains of the form :



Reflecting again about each of the four circles S_i^1 yields twelve fundamental domains of the same form



This yields a "tree" of fundamental domains



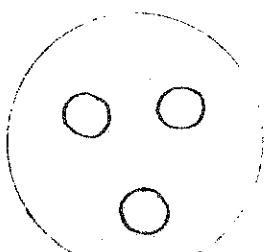
Now the fundamental domains cover everything except a Cantor set. Thus the limit set is this Cantor set.

These groups were first studied by Shottky around 1875 and are called the classical Shotthy groups. They gave the first examples of Cantor sets.

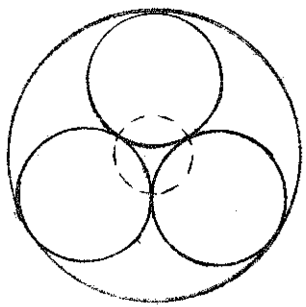
The group Γ is the "free" group on four generators of order 2. There are no relations other than $gen^2 = id$. (To see this let a normal subgroup of Γ act on the tree constructed above).

<u>Remark</u>. A non-trivial normal subgroup $N \subset \Gamma$ has the same limit set as Γ i.e. $\Lambda_N = \Lambda_\Gamma$. This is because Λ_N is invariant under Γ ;
(N normal $\Rightarrow N\Gamma\Lambda_N = \Gamma N\Lambda_N = \Gamma\Lambda_N \Rightarrow \Gamma\Lambda_N = \Lambda_N$) now apply Gromov's proposition.

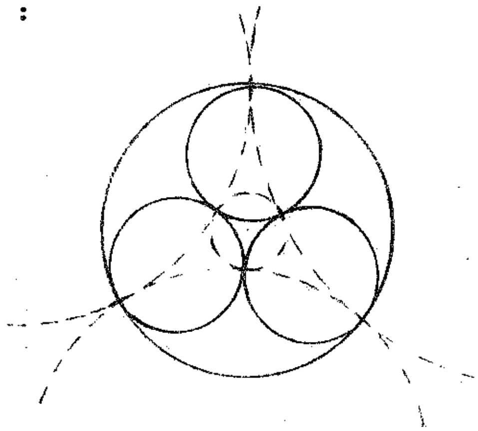
Thus the picture given by four disjoint circles is structurally stable, and is equivalent to the following picture :



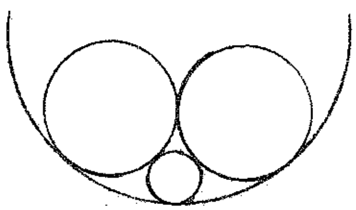
However, things are essentially different when the circles touch or intersect. For example, consider the group generated by reflections in the following four circles, three of which have a



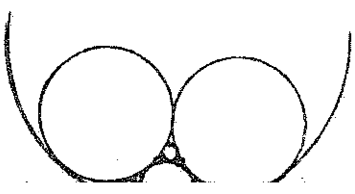
common orthogonal circle. The subgroup generated by the three inner circles is Fuchsian, having the common orthogonal circle as its limit set. Similarly, for the the other 3-tuples of circles, we have the limit sets being the common orhtogonal circles :



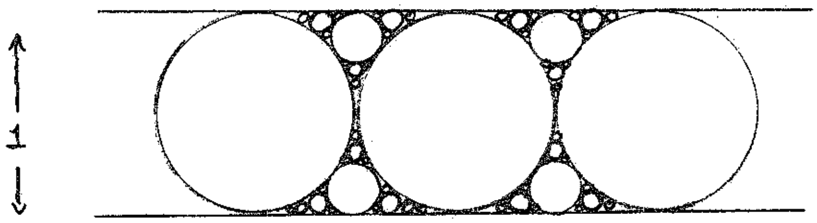
Thus the limit set coming from these four subgroups is



Now inverting one of the four circles of the limit set in the circle not contributing to it, one gets the Apollonian packing :



Sending the point x to ∞ (conformal transformation) gives the picture :



Exercise. Find the position and size of all circles. Show that the circles on the bottom line rest at rational points p/q and have diameter $1/q^2$. In this case the group Γ with this limit set is a subgroup of $PSL_2(Z[i])$.

Given a packing, define the packing constant δ such that

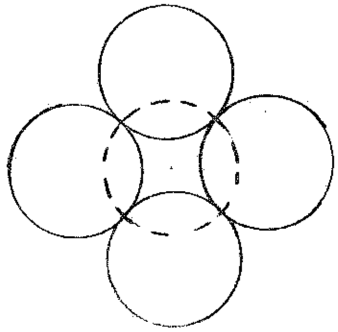
$$\varphi(s) = \sum_{\text{circles}} r_i^s = \begin{cases} \infty & \text{for } s < \delta \\ < \infty & \text{for } s < \delta \end{cases}$$

Unsolved problem. Is δ minimal for the Apollonian packing ?

Exercise. Given a packing with $\Sigma r_i < \infty$, show that the Lebesgue measure of the complement is positive.

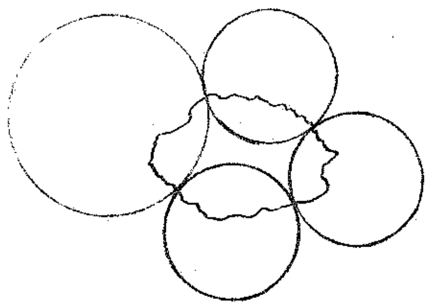
Proposition. Let Γ be a group of homeomorphisms of S^1 with no finite orbits. Then there exists a unique minimal closed invariant set. (The proof is analogous to the hyperbolic geometric case).

Now, back to our four-circle Shottky group. In the case where the four circles have a common orthogonal circle, it is invariant and the limit set of the reflection group is this circle.



(1)

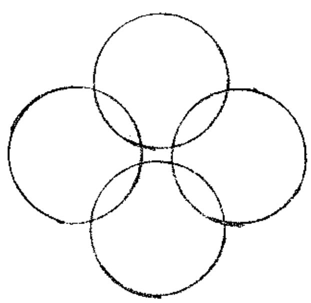
In general, there is no common orthogonal circle. Under iteration of reflections, the circles converge to the limit set, a very wiggly curve, pictured here :



(2)

There is a homeomorphism between pictures 1 and 2 and hence the two curves are homeomorphic. However, as we will see later, the second curve is non-rectifiable.

Another case is that of over-lapping circles :



If there is a common orthogonal circle this is the limit set, and one gets a complicated fundamental domain inside this circle.

Definition. The limit set Λ of a discrete group Γ has an expanding conformal cover if at each point $x \in \Lambda$, $\exists \gamma \in \Gamma$ such that $|\gamma' x| > 1$. This implies that $\exists$ a finite cover $U_1, \ldots, U_n$ of Λ with $\gamma_i : U_i \to \mathbb{C}$ such that $(U_i \cap \Lambda) \to \Lambda$ is an expanding conformal map. In the case of reflections in circles that overlap, it is clear that the limit set has an expanding conformal cover - simply take the U_i to be the interior of the discs bounded by the generating circles. We will see that the existence of an expanding cover of Λ forces Λ to "look alike everywhere" in a sense which we will make precise.

Definition. A limit set Λ is quasi-self similar if $\exists K$, $\exists r_o$ such that $\forall x \in \Lambda$ and $\forall r < r_o$

$$1/r[\Lambda \cap B(x,r)] \underset{\substack{\text{K-quasi}\\ \text{isometry}}}{\hookrightarrow} \Lambda$$

A K quasi-isometry is a bijection that distorts distances between $\frac{1}{K}$ and K.

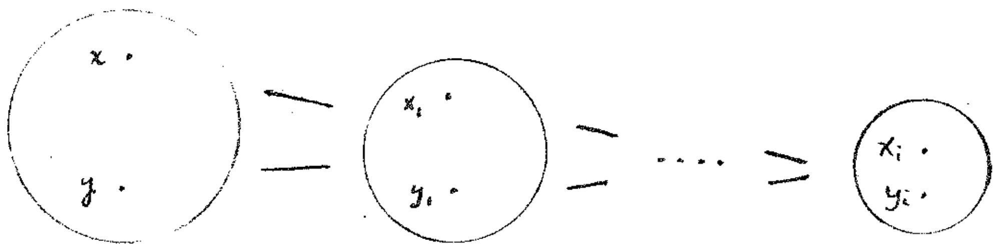
Thus roughly speaking, Λ is quasi-self similar if each small piece can be expanded to a standard size and then mapped into Λ by a K quasi-isometry.

Proposition. A set with an expanding conformal cover is quasi-self similar.

Thus in the case of the limit sets of Shottky groups, overlapping circles imply the existence of an expanding conformal cover of Λ which implies that Λ is quasi-self similar.

Distortion lemma (proof of Proposition in the compact case). If Λ is a closed compact set with a finite expanding cover, then Λ is K quasi-self similar.

Proof. Let r_o be the Lebesgue number of the expanding conformal cover and $x \in \Lambda$. If $r < r_o$, $B(x,r)$ lies inside a disc of the expanding conformal cover. Apply the corresponding conformal map to get a bigger $B(x_1,r_1)$. Iterate until the diameter of $B(x_n,r_n)$ is no longer smaller than r_o . The total distortion is given by the chain rule : the distortion of the product equals the product of the distortions of the γ_i . Now go backwards :



At each step there is compression, and the total compression goes as a geometric series :

$$\log \text{ratio} = \Sigma \left|\log \gamma'(x_i) - \log \gamma'(y_i)\right|$$

which by the mean value theorem gives

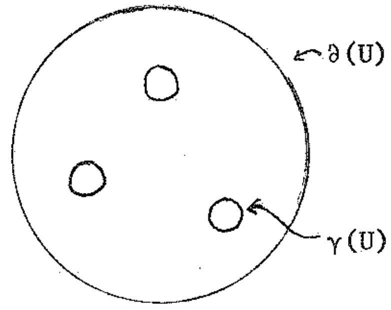
$$= \Sigma(\log \gamma')'(x_i^*) \, |x_i - y_i|$$

$$< K(\gamma', \gamma'')$$

This depends on γ' and γ'' and not on the choice of point x or the radius r .

Now, we mention another limit set, the Julia-Fatou limit set.

General remark. There is another formulation of the Poincaré limit set that will be useful in defining the Julia-Fatou limit set. Start with a point of discontinuity on the sphere. Choose a compact neighborhood U of x . Regarding this point as ∞ and forgetting the finitely many γ with $U \cap \gamma U \neq \phi$ we get :

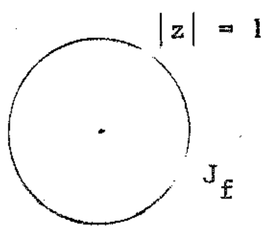


Thus we have a family of mappings $\{\gamma\} \subset \Gamma$ such that $\{\gamma\}|U$ forms a compact family of mappings $U \to S^2$. The Poincaré limit set is the complement of the set of <u>normal points</u> (i.e. points with compact neighborhoods U on which $\{\gamma\}|U$ forms a compact family of mappings.

Now, let $f : S^2 \to S^2$ be a $\mathbb{C}$-analytic self mapping of the Riemann sphere (e.g. a polynomial, which fixes ∞).

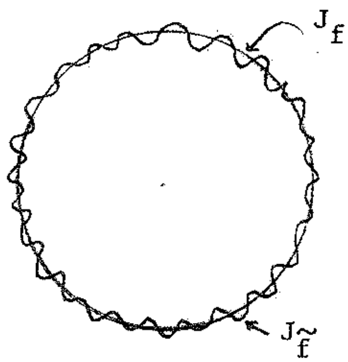
<u>Definition</u>. The Julia set, J_f , of an analytic self-mapping of S^2 is the complement of the set $\{x | \exists\, U \ni x$ such that the family $f , f \circ f, \ldots$ forms a compact set in $\Omega(U)$ in the topology of uniform convergence$\}$.

<u>Example</u>. Consider the polynomial $f(z) = z^2$. Points of norm > 1



tend to ∞ under iteration, while points of norm < 1 tend to 0 . The Julia set is the circle $|z| = 1$. Notice that the backward orbit of any point x clusters at the Julia set.

Suppose now that we perturb $f(z)$ slightly to $\tilde{f}(z) = z^2+\varepsilon$. Then the Julia set $J_{\tilde{f}}$ is still topologically a circle, but geometrically it differs from S^1 since it is non-rectifiable.



The Julia set $J_{\tilde{f}}$ is still quasi-self similar, but not rectifiable

<u>Corollary</u>. If $|f'z| > 1$ for all $z \in J_f$, then J_f is quasi-self similar.

(Since $|f'z| > 1$ there exists an expanding conformal cover and thus $J_{f'}$ is quasi-self similar).

<u>Example</u>. Consider $f(z) = e^z$ as a mapping of $\mathbb{C}$. It has been proved that $J_f = \mathbb{C}$. It is unknown whether or not this action is ergodic.

6. A dictionary : Limit set ⇔ Julia set.

We have two similar examples of conformal things :

i) A discrete group $\Gamma \subset PSL_2(\mathbb{C})$. It acts discontinuously on an open set of the sphere at ∞ . The complement of this open set in the sphere is the Poincaré limit set Λ_Γ of Γ .

ii) A $\mathbb{C}$-analytic map

$$f : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$$

with all its iterates $f \circ f$, $f \circ f \circ f, \ldots,$ and its Julia set J_f . If, for example, f is a polynomial, then there is a maximal open neighbourhood of ∞ which is mapped into itself, i.e. ∞ is an attractor, and J_f is the boundary of this neighbourhood.

We will develope a "dictionary" between these 2 examples.

1) $\Lambda_\Gamma \longleftrightarrow J_f$

2) Discreteness of $\Gamma \longleftrightarrow$ (to be discussed later).

3) Λ_Γ is the whole $\mathbb{C} \cup \{\infty\}$ or it is a Cantor set. J_f is $\mathbb{C} \cup \{\infty\}$ or it has no interior.

4) Λ_Γ is the minimal, closed, Γ-invariant subset of $\mathbb{C} \cup \{\infty\}$. J_f is a closed set, invariant under all iterations of f .

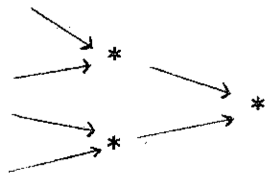
5) In the case of Λ_Γ , the orbit Γ_{x_o} of a point x_o is $\{x \in \Lambda_\Gamma : x = \gamma(x_o)$ for some $\gamma \in \Gamma\}$; for every $x \in \Lambda_\Gamma$, the orbit Γ_x is dense in Λ_Γ .

In the case of J_f ,we have three types of orbits :

i) The forward orbit. (Most often used in dynamics)

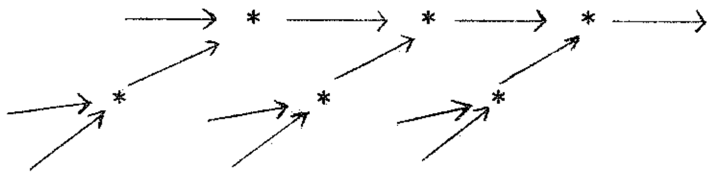
$$* \xrightarrow{f} * \xrightarrow{f} * \xrightarrow{f} \ \ldots .$$

ii) The backward orbit.



If f is linear, then f^{-1} is a Mobius transformation (so forget it).

iii) The full orbit. This is given by the equivalence relation $z \sim f^n(z)$.



This orbit is the biggest. It is relevant for studying ergodicity, and questions of invariance.

Proposition*). For all $x \in J_f$, the backward orbit of x is dense in J_f .

This is a consequence of the following lemma.

Lemma. Given an open set V in S^2 with $V \cap J_f \neq \emptyset$, there exists a N s.t. $f^n(V \cap J_f) = J_f$.

Proof. Consider an open neighbourhood V in S^2 of a point $x_o \in J_f$. Using

---

*) C.f. [Brolin] . Invariant sets under iteration of rational functions. (Lemma 2.2 and theorem 2.5). Arkiv für Math. 6, 1967.

Montel's argument in the proof of Picard's theorem, we have

$$\bigcup_n f^n(V) = \begin{cases} \mathbb{C} \cup \infty & \\ \mathbb{C} \cup \infty - \text{pt} & (\text{and } f \sim \text{polynomial}) \\ \mathbb{C} \cup \infty - 2 \text{ pts} & (\text{and } f \sim z^n) \end{cases}$$

for otherwise the f^n's would form a normal family at x_o ; none of these (at most) two points is in J_f . This implies $\bigcup_n f^n(V) \supset J_f$. Now use that the repulsive periodic points in J_f are dense (Brolin), to conclude that the union $\bigcup_n f^n$ is increasing, and therefore exists a N s.t. $f^N(V) = \bigcup_n f^n(V)$.

This proves the lemma. The proposition follows easily : Given a point $y \in J_f$ and a neighbourhood V of y , there exists a N s.t. $f^N(V) = J_f$, so V contains elements in the backward orbit of every point in J_f .

6) On each side of the dictionary, there is a "good" case, characterized by the expanding property. In Λ_Γ this means that for all $x \in \Lambda_\Gamma$ there exists $\gamma \in \Gamma$ s.t. $|\gamma'(x)| > 1$. In J_f this means that there exists N s.t. $|(f^N)'(x)| > 1$ for all $x \in J_f$. In both cases, "good" implies that there exists a finite expanding conformal cover.

<u>Corollary</u>. In the good case, Λ_Γ or J_f is quasi-self, similar.

This follows because you can take any small piece and start expanding it until it has full size. A simple calculation shows these iterated expansions are quasi-similarities. (See distortion lemma of §5).

The following is a new theorem.

<u>Theorem</u> (Sad). In the Julia case J_f , the set of good cases is dense.

(also open, by definition).

Main problem. Is the set of good cases dense in the set of all finitely generated Kleinian groups ?

So to speak, Thurston describes the groups in the boundary of good cases. If main problem were solved affirmatively, then the conjecture of Alfhors would be true and the finitely generated Kleinian groups would be well understood.

Remark. In the good case for Kleinian groups, the group Γ has fundamental domain $\mathcal{D}$ with a finite number of sides, and therefore Γ is finitely presented. This follows because the expanding property implies that $\mathcal{D}$ does not interest Λ_Γ so you can cut Λ_Γ out with a finite number of isometric spheres.

Now, we have an ergodicity result.

Definition. Topological transitivity for an expanding conformal cover means that there exists a dense full orbit.

Definition. A measure μ is conformal (of exponent δ) with respect to a collection of conformal maps $\{\gamma\}$ if

$$(*) \qquad \frac{d\gamma\mu}{d\mu} = |\gamma'|^{\delta}$$

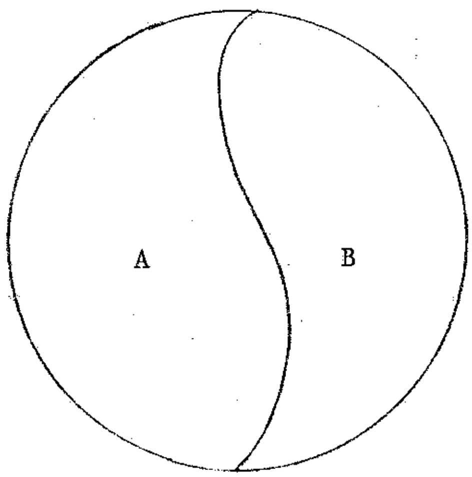
for some real $0 < \delta < \infty$. ($\frac{d\gamma\mu}{d\mu}$ is the Radon-Nikodym derivative). That is, if $\gamma^*\mu = |\gamma'|^{\delta}\ \mu$.

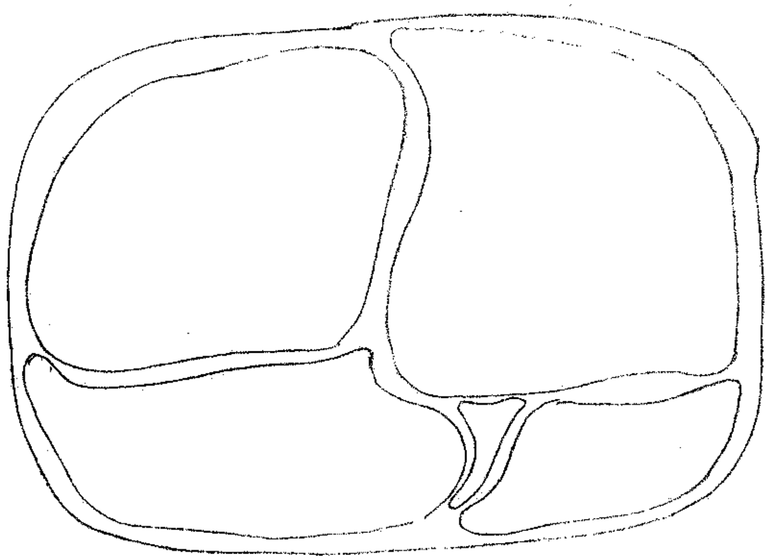
If we have bounded multiplicative error in (*) , we can modify μ to compensate, so this definition is stable under Lipschitz deformations of the geometry.

The following theorem applies to both examples.

Theorem. A conformal measure μ relative to a topological transitive expanding conformal cover is ergodic (i.e. if $X = A \cup B$ with A, B Borel sets, invariant under the system of transformations, then $\mu(A) \cdot \mu(B) = 0$).

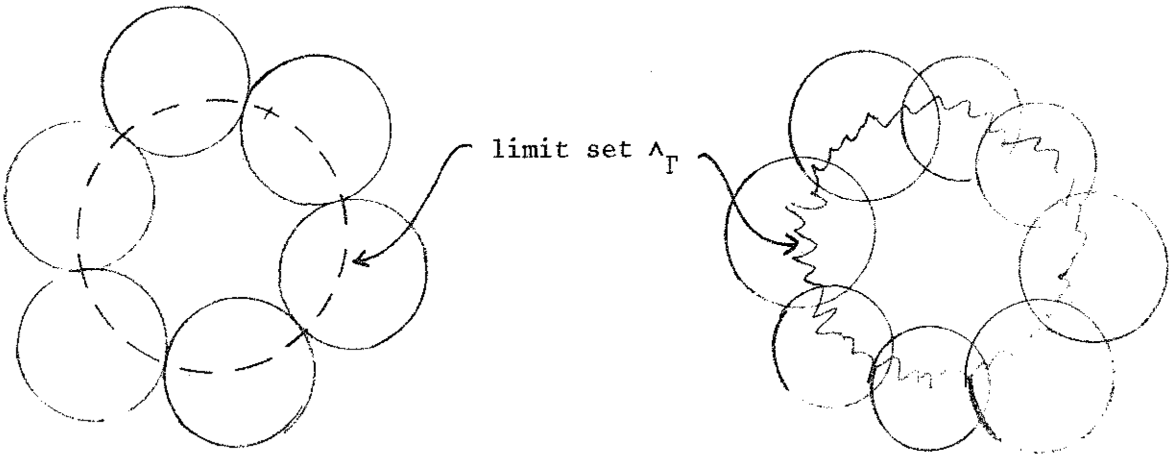
Proof. Let A, B be as above, and note that the amount of "black/white" in $A \cup B$ only changes a little if we have bounded distortion. Almost all points in B are density points, so apply the distortion lemma of §5 to one of these points to obtain in the limit a disc which is 100% black (full measure). In this way, we obtain open sets in B of full measure everywhere.





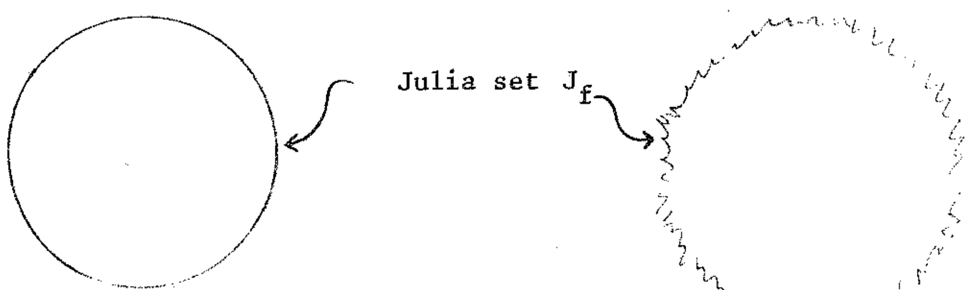
Doing the same with A, etc., we conclude that μ has finitely many ergodic components. The result now follows because there is a dense orbit.

Now, consider the examples



If there is a common orthogonal circle.

If there is no common orthogonal circle.



$z \longrightarrow z^2$

$z \longrightarrow z^2 + \varepsilon$

Theorem.*) If these curves are not circles (Platonic case), then they are not rectifiable.

The proof is similar for both examples. The first step was given by Bowen. We prove now the case of the Julia set. We will prove the result for Λ_Γ in §7.

Proof for J_f. Assume the curve is rectifiable, and let μ be the arc length. This μ satisfies

$$\frac{d\gamma\mu}{d\mu} = |\gamma'|^1$$

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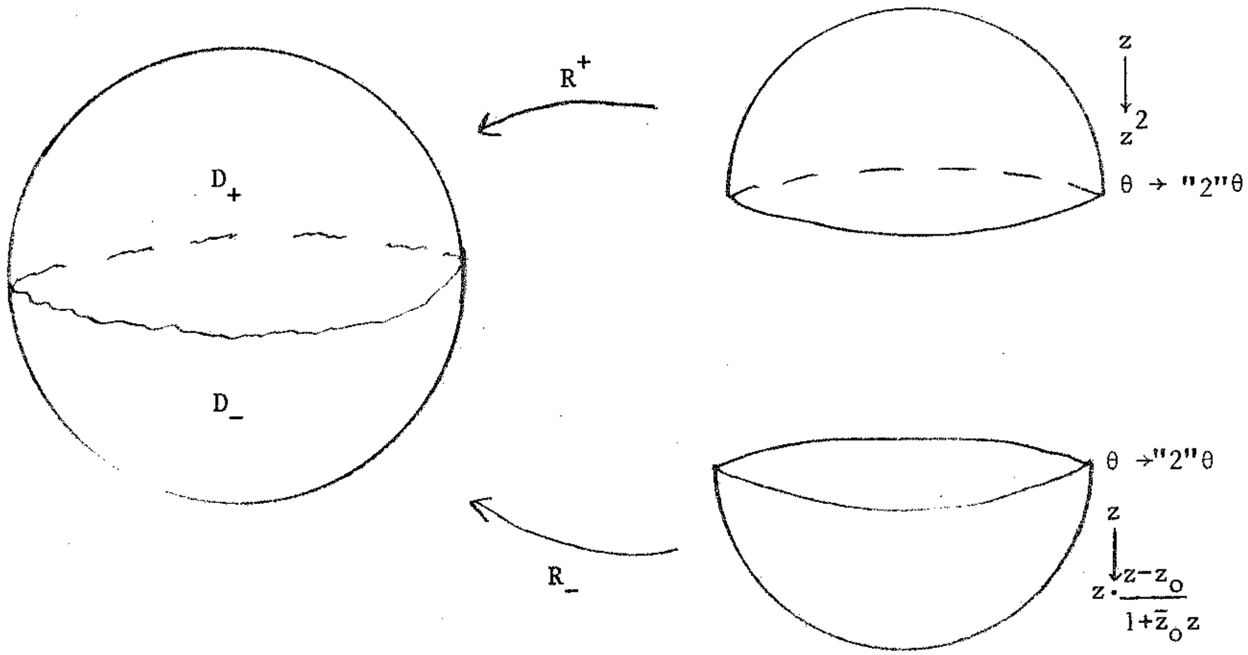
*) Fatou proved this by a function theory argument (1919).

i.e. μ is conformal ($\delta=1$). This curve divides the sphere S^2 into two simply connected domains D_+, D_-. We then have Riemann maps R_+, R_- mapping the unit disc U conformally onto D_+, D_-.



Now, f preserves D_+ and D_-, and R_+ conjugates the self map in D_+ to produce a conformal map f_+ in U which is of the form $z \to z^2$ (Because it has same critical and fixed point). Similarly, the self map in D_- conjugates to $z \to z \cdot \frac{z-z_o}{1+\bar{z}_o z}$. (Because critical point and fixed point are different). The dynamics of the first map on the boundary is $\theta \to 2\theta$, for the second map, we have $\theta \to$ "2"θ (not quite 2).

These Riemanann maps are continuous on the boundary (Caratheodory) and, if the curve is rectifiable, they are also absolutely continuous, by the theorem of F.M. Riesz. Thus, we obtain an absolutely continuous map from S^1 into S^1 which conjugates $\theta \to 2\theta$ into $2 \to$ "2"θ. This is not possible because both maps are strictly expanding, so they are ergodic, and they are ergodically distinct, i.e. there is no Borel map φ on S^1 conjugating 2θ and "2"θ. In fact, both 2θ and "2"θ have invariant measure

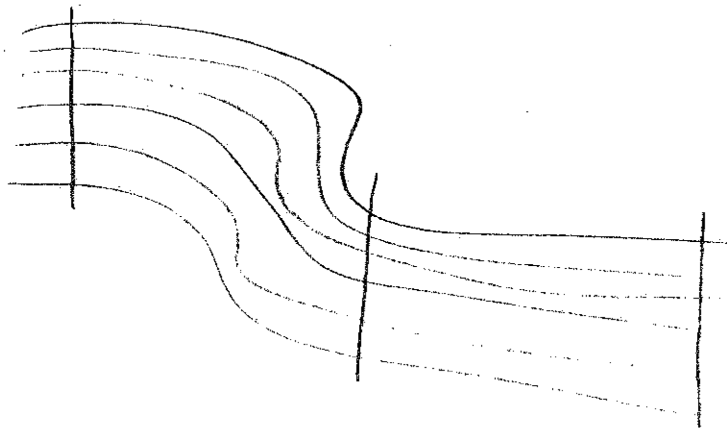
$\nu = d\theta$ and if φ is such a map, then φ has to preserve the invariant measure (by erdodicity). Therefore, φ being continuous is a rigid rotation, but then the conjugate of 2θ under φ is 2θ and not "2"θ .

7. Dynamical systems. Hausdorff dimension.

The proofs of theorems for the Poincaré limit set Λ_Γ and the Fatou-Julia limit set J_f usually have some general arguments in common. These are expressed in the language of conformal dynamical systems.

Definition. A conformal dynamical system consists of a countable collection of partially defined conformal transformations in some Riemannian manifold (usually $\mathbb{R}^n$ or S^n), closed under composition whenever this makes sense.

This definition, as such, does not include flows (uncountable). However, if we are interested in ergodicity, we can look at flows by taking "enough" transversals,



so the flow defines a countable dynamical system.

Examples. 1) Discrete groups of conformal transformations on S^d .

2) Iterates of complex analytic maps $\mathbb{C} \cup \{\infty\}$.

3) Foliations of codimension 1 or with conformal holonomy.

[that is : consider a Riemannian manifold foliated by submanifolds. If L is a leaf and $p(t)$ is a path in L , we can follow $p(t)$ along the nearby leaves. This gives transformations on the transversal space $\mathbb{R}^d$, which are required to be conformal].

All these transformations are manageable because conformal maps have only 1 degree of freedom (conformal = $\lambda \cdot$ orthogonal).

<u>Definition</u>. A closed invariant set is topologically transitive if some full orbit is dense.

For example, Λ_Γ is topologically transitive since every orbit is dense. (Similarly J_f) .

<u>Definition</u>. A conformal dynamical system contains an expanding cover for some compact invariant set X if for all $x \in X$ exists a γ (in the dynamical system) s.t. $|\gamma'(x)| > 1$.

Again, this implies that there is a finite covering of X by open sets V_i and conformal maps γ_i , defined on V_i , s.t. $|\gamma_i'(x)| > 1$ for $x \in V_i$.

The following theorem was stated and proved in §6 for the two special examples Λ_Γ , J_f . The same proof works in general.

<u>Theorem</u>. A conformal measure μ relative to a topologically transitive expanding cover is ergodic.

<u>Corollary</u>. With the above hypothesis, μ is unique given the exponent and the total mass. (Assume $|\mu| = 1$).

<u>Proof</u>. Suppose ν is another one. Then

$$m = \frac{\mu+\nu}{2}$$

also satisfies $\gamma^* m = |\gamma'|^\delta \cdot m$, and $|m| = 1$. Hence μ and ν are absolutely continuous with respect to m. Consider the Radon-Nykodym derivitives $\frac{d\mu}{dm}$, $\frac{d\nu}{dm}$; these are functions invariant under γ (because μ, ν, m have same exponent), so by ergodicity

$$\frac{d\mu}{dm} = 1 = \frac{d\nu}{dm}$$

so $\mu = m = \nu$.

<u>Hausdorff dimension</u>.

A reference is [Rogers], Hausdorff measure, Camb. Univ. Press. Also [Federer], Geometric measure theory, Springer Verlag.

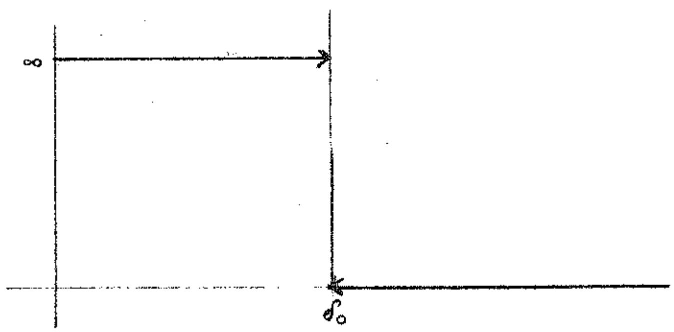
Consider some set X in a metric space, cover it with countably many small balls of radii $r_i < \varepsilon$, and take the infimum of the sums

$$\sum_{|C|<\varepsilon} r_i^\delta$$

over all such coverings, for a fixed $\delta \in \mathbb{R}^+$. Now define a function on δ by

$$H_\delta(X) = \lim_{\varepsilon \to 0} \inf \sum_{|C|<\varepsilon} r_i^\delta$$

this function is the r^δ-<u>Hausdorff measure of</u> X. $H_\delta(X)$ is always 0 or ∞, except at one point δ_0 where it is 0, finite or ∞.



This critical number δ_o is the <u>Hausdorff dimension</u> of X .

Note that if δ is a point where

$$0 < H_\delta(X) < \infty$$

then δ is, necessarily, the Hausdorff dimension.

The Hausdorff dimension is a geometric property. For example, a n-manifold M has Hausdorff dimension n .

<u>Conjecture</u>. If X is quasi-self-similar, then

$$0 < H_\delta(X) < \infty$$

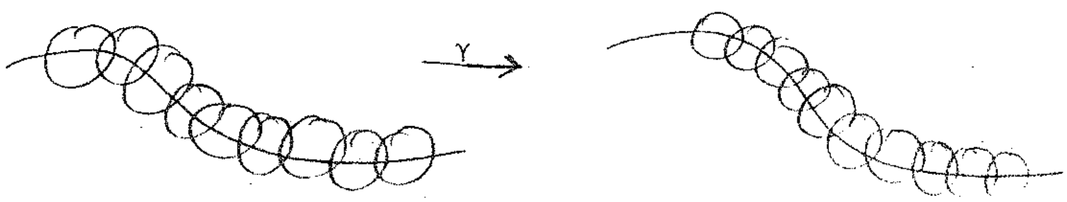
for some δ(= Hausdorff dimension).

(*) Now, consider a conformal dynamical system, a closed invariant set X topologically transitive, with an expanding conformal cover.

<u>Proposition</u>. Suppose the r^δ-Hausdorff measure H_δ of X is finite and positive. Then H_δ is conformal of exponent δ .

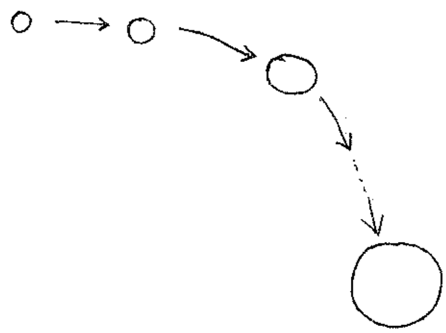
<u>Proof</u>. If γ is any map in our dynamical system, then γ takes small balls into almost small balls, with consistent scaling. In the limit, balls go to balls (of different radii). Hence,

$$\gamma^* H_\delta = |\gamma'|^\delta H_\delta$$



<u>Proposition</u>. Suppose X has a finite conformal measure μ of exponent δ. Then H_δ is finite and positive, and therefore $\mu = H_\delta$ (if normalized). Moreover, $\mu(B(x,r) \cap X)$, $x \in X$, is comparable to r^δ .

<u>Proof</u>. The last statement is clear from the distortion lemma and the transformation law $\gamma^*\mu = |\gamma'|^\delta \mu$.



[In fact, (Marstand) shows that, in general, one cannot get anything better than comparable. As $r \to 0$, we must get oscillation, otherwise $\delta \in \mathbb{Z}$] .

Now, take a covering of X by balls of radii r_i , then

$$\mu(x) \leq \Sigma\mu(x_i,r_i)$$

but also

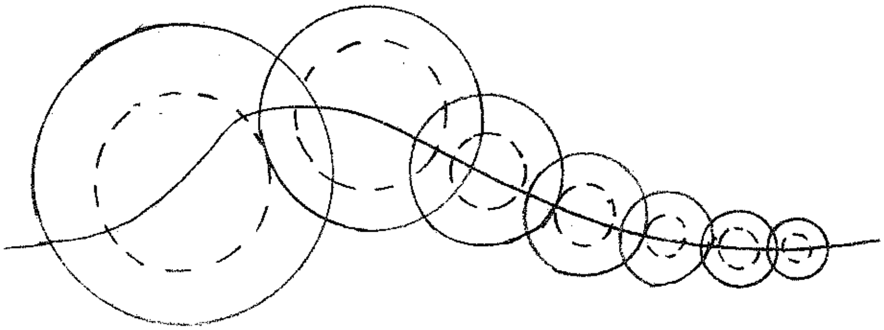
$$\mu(x_i,r_i) \leq r_i^\delta \quad \text{(up to a constant)}$$

because they are comparable. Hence

$$0 < \mu(X,r) \equiv \mu(B(X,r) \cap X) \leq r^\delta$$

this implies $H_\delta(X) > 0$.

It remains to prove $H_\delta(X) < \infty$. For each $\varepsilon > 0$, choose a covering of of X with balls $B_1, B_2, \ldots,$ of decreasing size $\varepsilon \geq r_1 \geq r_2 \geq \ldots,$ and make it an efficient cover, i.e. the center of B_{i+1} is outside $B_1 \cup \ldots \cup B_i$. We will prove $\Sigma r_i^\delta < \infty$, and therefore $H_\delta(x) < \infty$. For this observe that the balls $1/2\, B_i$ are disjoint.



thus,

$$\Sigma r_i^\delta \leq 2^\delta \Sigma (1/2\ r_i)^\delta \approx 2^\delta \Sigma \mu(x_i, 1/2\, r_i) \leq 2^\delta\ \mu(X).$$

<u>Corollary</u>. With the hypothesis of (*) above, for at most one δ, there is a finite positive conformal measure. This δ is the Hausdorff dimension of X and the measure is the Hausdorff measure.

<u>Example</u>. Consider the dynamical system $Z \to Z^2 + \varepsilon$. We know that J_f is non-rectifiable.

<u>Claim</u>. There exists a unique, finite and positive conformal measure μ on J_f, of exponent δ. (So $\mu = H_\delta$ and δ is the Hausdorff dimension).

The analogue following result on the Kleinian side of the dictionary is due to Bowen.

Proposition. For $\varepsilon \neq 0$ of small modulus, the Hausdorff dimension δ of J_f is $1 < \delta < 2$.

Proof. J_f is topologically a curve (see below), so $\delta \geq 1$. If $\delta = 1$ then the curve is rectifiable, so $\delta > 1$. Also, $\delta \leq 2$ because J_f is sitting in $\mathbb{C}$; suppose $\delta = 2$, then the 2-dimensional Lebesgue measure of J_f is finite and positive, which is not possible. [In fact, the Lebesgue measure of a nowhere dense closed invariant set X of a conformal dynamical system with expanding cover, is always 0] .

We now study the Kleinian case by characterizing a single Möbius transformation γ on $\mathbb{R}^{d+1}$, (or on S^d). The idea is to think of Mostow's rigidity theorem to prove something in dimension 2, where the theorem is false.

Proposition. γ satisfies the following "mean value" formula

$$|\gamma(x) - \gamma(y)|^2 = |\gamma'(x)|\,|\gamma'(y)|\,|x-y|^2$$

Proof. This can be proved by a direct computation for $d = 1$, i.e. on $\mathbb{C}$. In general, look at the plane containing x,y and do the same.

Corollary. γ preserves the cross ratio :

$$\frac{|x-y|\,|z-w|}{|x-z|\,|y-w|} = \frac{|\gamma(x) - \gamma(y)|\,|\gamma(z) - \gamma(w)|}{|\gamma(x) - \gamma(z)|\,|\gamma(y) - \gamma(w)|} .$$

This formula shows that the Lebesgue measure is not preserved by γ , and also says how to correct the distortion. We want to have a measure invariant under Möbius transformations. For this, we let γ act on $S^d \times S^d$

by $\gamma\cdot(x,y) = (\gamma x,\gamma y)$; then γ preserves the measure

$$\nu = \frac{dx \times dy}{|x-y|^{2d}}$$

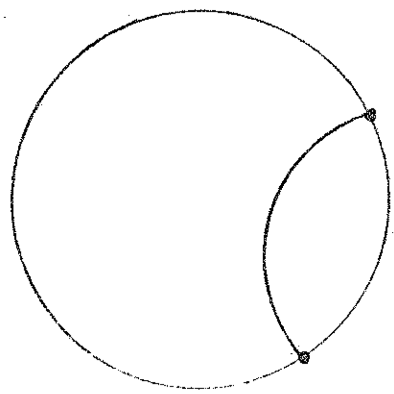
(ν is the Euclidean chordal distance between x,y) . This ν is the unique invariant metric on the product.

In fact, if μ is a measure on the limit set Λ_Γ which satisfies $\gamma^*\mu = |\gamma'|^\delta$ for some $\gamma \in \Gamma$, then the measure

$$\nu = \frac{\mu \times \mu}{|x-y|^{2\delta}}$$

on $\Lambda_\Gamma \times \Lambda_\Gamma$ is invariant under γ , so ν is Γ'-invariant, for $\Gamma' = \{\gamma \in \Gamma : \gamma^*\mu = |\gamma'|^\delta \mu\}$. [This is a measure theoretic form of invariance of cross ratio under Möbius transformations].

<u>Remark</u>. The above measure ν is related to the Liouville measure on the tangent bundle $T(H^{d+1}/\Gamma)$: the action of Γ on $S^d \times S^d$ is ergodic with respect to ν if and only if the geodesic flow on $T(H^{d+1}/\Gamma)$ is ergodic . A geodesic in $\mathbb{H}^{d+1}$ is determined by a pair of distinct points in the sphere at infinity S^d ,



so the geodesics in $\mathbb{H}^{d+1}$ are parametrized by $S^d \times S^d - \Delta$. (Note that the diagonal Δ is irrelevant for measure theory). Now, a point in the unit tangent bundle $T_1(H)$ of H^{d+1} determines a geodesic. This gives a map

$$T_1(H) \longrightarrow S^d \times S^d - \Delta$$

which defines a foliation of $T_1(H)$ by lines. So, we have a natural measure on $T_1(H)$ given by the cross ratio measure on $S^d \times S^d$ and the arc length upstairs. Since ν is Γ-invariant, this leads to the invariant measure of the goedesic flow on $T(\mathbb{H}^{d+1}/\Gamma)$.

Example. (E. Hopf, 1930's). If Γ is cocompact, then the geodesic flow is ergodic. (i.e. the action of Γ on $S^d \times S^d$ is ergodic).

Proposition. Suppose $\varphi : S^d \to S^d$ is a Borel bijection almost everywhere (a.e.), such that $A > 0 \Leftrightarrow \mu(A) > 0$, (i.e. non singular with respect to Lebesgue measure), and such that $\varphi \times \varphi : S^d \times S^d \circlearrowleft$ preserves ν, up to a constant, then φ agrees a.e. with a Möbius transformation.

Proof. Since $\varphi \times \varphi$ preserves ν, for almost all points $x,y \in S^d$,

$$\frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^2}$$

is a product of a function on x and a function on y. (The Radon-Nykodym derivatives $\frac{d\varphi}{dx}, \frac{d\varphi}{dy}$). Therefore, for almost all 4-tuples, the cross ratio

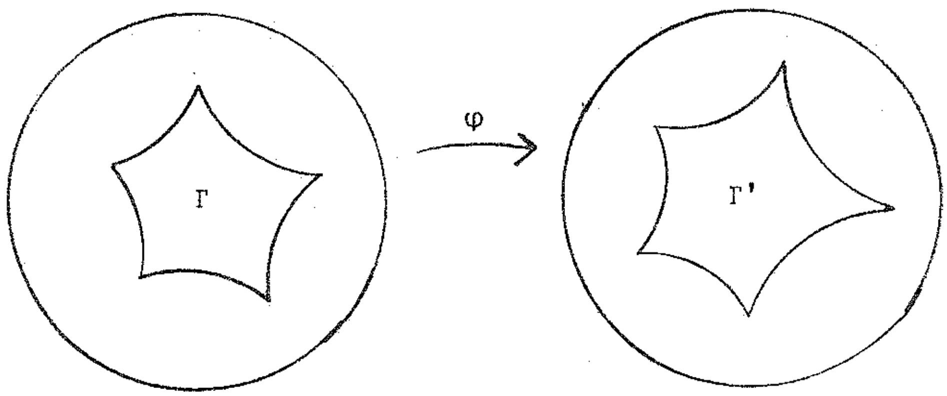
$$\frac{|x-y|\,|z-w|}{|x-z|\,|y-w|}$$

is preserved. If we change coordinates and we put w at infinity, then for almost all triples in $\mathbb{R}^d$ the ratio $|x-y|/|x-z|$ is preserved. Permuting x,y,z we see that triangles go into similar triangles, with a proper scaling. Therefore, we can scale φ to make it an isometry a.e., so a Möbius transformation a.e.

Problem. Do this for other simple Lie groups. That is, try to characterize elements of a Lie group G by looking at measure theoretic properties on G/K, K = maximal solvable subgroup.

As a corollary, we have something like a Mostow's rigidity theorem in dimension 2.

Corollary. Let $\varphi : S^1 \to S^1$ be a Borel bijection, absolutely continuous with respect to Lebesgue measure. If for some cocompact Fushsian group Γ, the conjugate $\varphi\Gamma\varphi^{-1} = \Gamma'$ is again a Fuchsian group, then Γ and Γ' are conjugate in $SL_2(\mathbb{R})$. In fact, $\varphi \in SL_2(\mathbb{R})$.



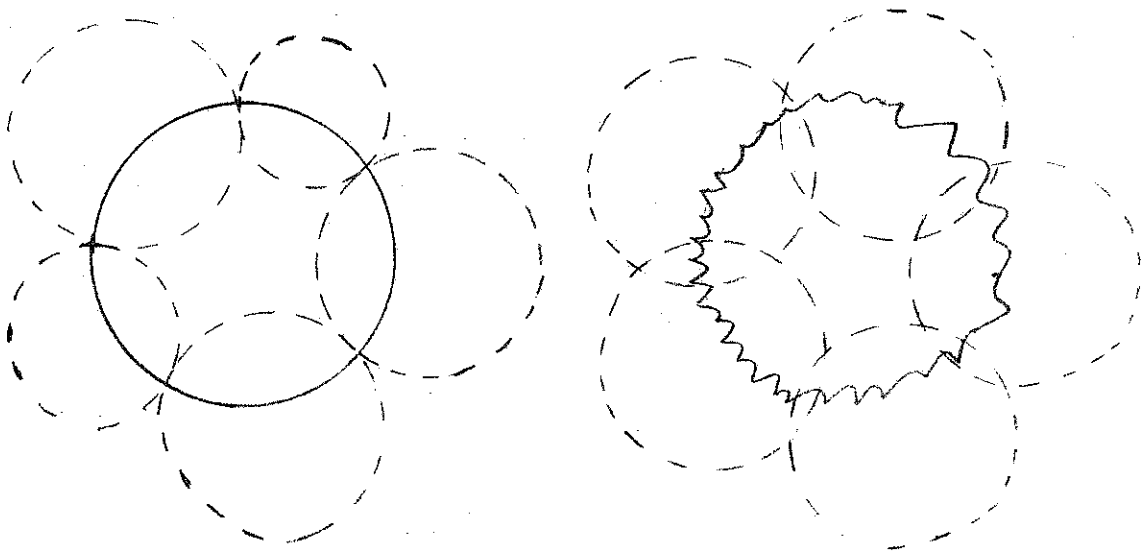
Proof. Because Γ is cocompact, the geodesic flow on $T(\mathbb{H}/\Gamma)$ is ergodic (Hopf), so Γ acting on $S^1 \times S^1$ is ergodic; similarly, Γ' acting on $S^1 \times S^1$ is ergodic. Now, φ is a conjugacy between ergodic actions, so φ takes the Γ-invariant measure on $S^1 \times S^1$ into the Γ'-invariant measure on $S^1 \times S^1$; but in both cases, this measure is ν. So φ is a Möbius transformation.

In constrast with higher dimensions, φ here has to be assumed to be absolutely continuous. In higher dimensions it is quasi-conformal, so automatically absolutely continuous.

The theorem says that the Teichmüller space injects into abstract ergodic theory.

<u>Problem</u>. Define the moduli of Riemann surfaces as invariants in ergodic theory. (Perhaps with suitable restrictions on the systems considered).

Now, we use ergodic theory to prove that Λ_Γ is non-rectifiable. (The other side of $z \to z^2+\varepsilon$ in the dictionary). Start with a Fuchsian group with limit set a geometric circle,

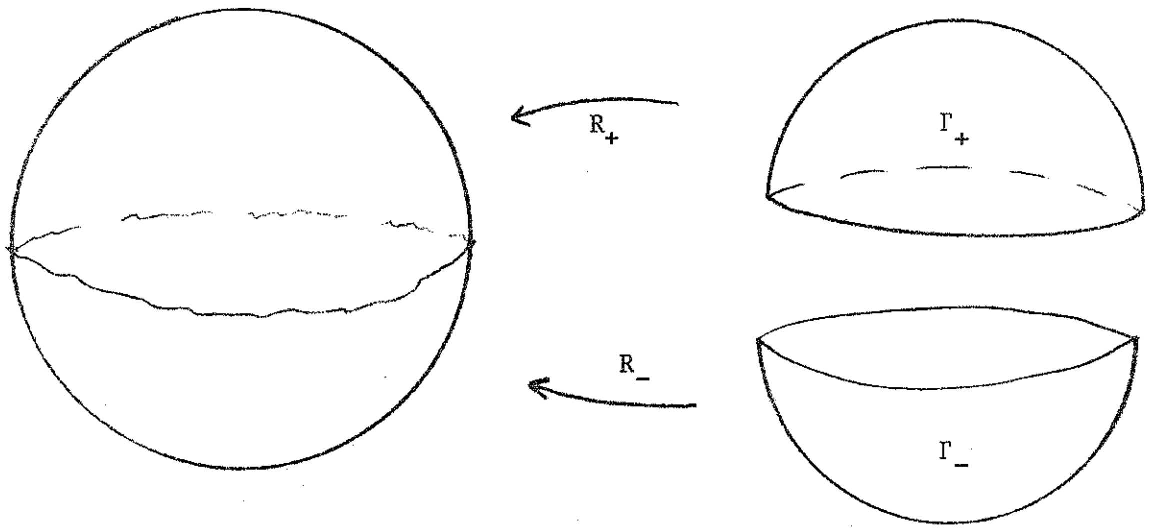


and deform this slightly, (by varying the parameters a,b,c,d), destroying that there is an invariant circle. By topological stability, the limit set Λ_Γ of the new Fuchsian group is still a topological circle (see §8).

<u>Theorem</u>. (Bowen). Λ_Γ is either a geometric circle or it is non-rectifiable.

<u>Proof</u>. The curve Λ_Γ divides S^2 into two discs D_+, D_-, and we have

Riemann maps R_+, R_-



these conjugate Γ to Fuchsian groups Γ_+, Γ_- acting on the unit disc. If they were the same (up to a reflexion), they could be fused together in 1 Kleinian group and Λ_Γ would be a geometric circle.

Now, suppose the curve is rectifiable, then the Riemann maps restricted to the boundary are absolutely continuous (F.M. Riesz). Hence the map $\varphi : S^1 \to S^1$ that conjugates Γ_+, Γ_- is absolutely continuous, so it is a Möbius transformation.

Here, we used ergodicity on the Möbius transformations with respect to pairs of points. In J_f, we used ergodicity on $z \to z^2$.

Again, if we have a conformal measure,

Theorem (Bowen). The Hausdorff dimension D of Λ_Γ is $1 < D < 2$, except for the round circle.

The proof is essentially the same as for J_f. By deforming the group, we can actually cover the whole range between 1 and 2. (See Sullivan's paper in celebration of Nico Kuiper's sixtieth birthday).

Observe that D is a function on the Teichmüller space. Is it continuous, analytic ?

In order to actually compute the Hausdorff dimension D of a given limit set $\Lambda_\Gamma \subset S^2$ one can do as follows : define a function

$$\varphi : \mathbb{H}^3 \longrightarrow \mathbb{R}$$

by looking at the mass at infinity viewed from variable point. That is, take the usual metric on the sphere at ∞ ; given a point x in $\mathbb{H}^3$, the geodesics through x identify the unit tangent sphere $T_1(\mathbb{H}^3)_x$ with the sphere at ∞. This gives a metric on $T_1(\mathbb{H}^3)_x$, and the function $\varphi(x)$ is defined to be the Hausdorff measure of the image in $T_1(\mathbb{H}^3)_x$ of the limit set Λ_Γ. Now, this function satisfies

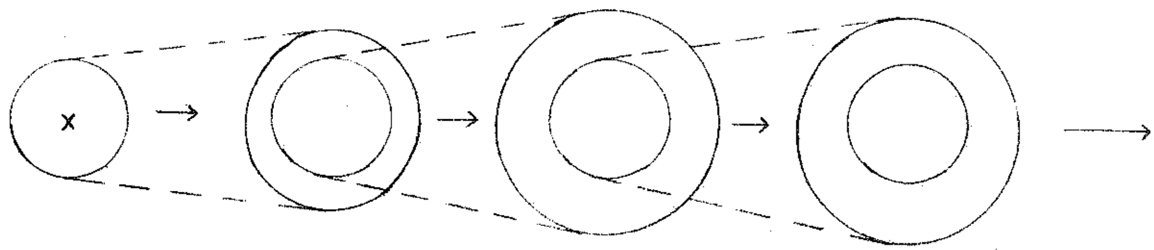
$$\Delta\varphi = D(D-2)\,\varphi$$

and it is Γ-invariant, so φ is a (positive) eigenfunction of the Laplacian on $\mathbb{H}^3/\Gamma$ with respect to the hyperbolic metric. This is enough to characterize φ : it is the "smallest" possible eigenfunction, i.e. for the smallest possible eigenvalue, so if we know the spectrum of Δ on $\mathbb{H}^3/\Gamma$ we know the Hausdorff dimension of the limit set of Γ .

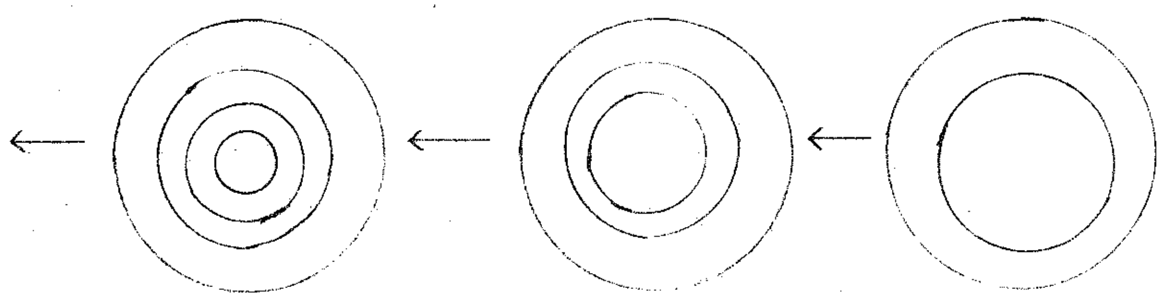
8. Topological Stability of J_f , .

We want to show that in the cases of J_f , Λ_Γ , if we have the expansion property, then a slight deformation of the coefficients preserves these sets topologically; i.e. there is a homeomorphism commuting with the dynamics. The idea is to use telescopes.

Let X be a closed invariant set of an expanding system, and cover X with balls having an expanding map on each. Now take a point $x \in X$ and an ε-disc about it, with ε smaller than the Lebesgue number of the cover. Thus the disc is in at least one member of the cover, use the corresponding map to blow the disc up; now take the image point of x , and a new ε-disc about it, and blow it up again, and so on.



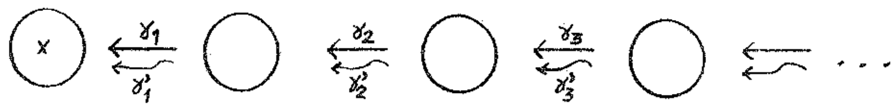
So we get maps



with a definite amount of compression at each step. After infinitely many steps, one single point is determined.

For the Julia set case the "telescope" is unique, because our cover consists of one set. In the limit set case Λ_Γ we get a tree of possibilities depending on how the ε-disc is related to the cover. This causes difficulties in the proof below.

Now suppose that you deform your dynamical system. That is, you deform the generating maps slightly. If we have a telescope



then the images of the deformed γ_i's will have approximately same image. If the deformation is of class C^1, then we will have a definite amount of compression, so we get a new telescope converging to a nearby point x'. This defines a correspondence φ by $x \longrightarrow x'$. In the Julia set case this is a well defined function, and we will prove that φ is actually a homeomorphism onto its image. In the case of Λ_Γ we have to prove first that φ is well defined, i.e. that it does not depend on the choice of telescope.

Julia set case.

Let us assume that our expanding maps are expanding in a neighbourhood of J_f. It is clear that if g is the perturbed function, then $\varphi f = g\varphi$, so φ commutes with the dynamics. Now, φ is continuous because it is determined arbitrarily accurately by finite pieces of the telescope. Moreover, φ is 1-1 because it only moves points by less than ε, so if two points were identified, then by expanding a little we would have that φ is identifying two points whose initial distance is more than ε. [This argument works for all "expansive" dynamical systems]. By point set topology, this implies that φ is a homeomorphism onto its image.

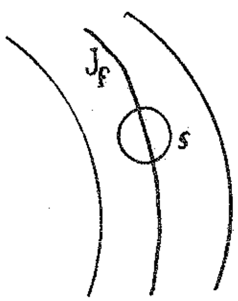
Now look at a neighbourhood N of J_f. The Julia set is the only set in N that remains in N after all iterations of f. The perturbed set $\varphi(J_f)$ will be the set in N that remains there after iterations of the perturbed function g. That is, $\varphi(J_f)$ is a closed invariant subset of J_g. From the fact that the inverse orbit of any point in J_f is dense in J_f, one deduces $\varphi(J_f) = J_g$.

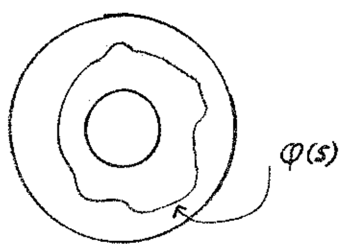
Example. If f is $z \longrightarrow z^2$, then g is $z \longrightarrow z^2 + \varepsilon$, and φ is a homeomorphism between J_f and J_g.

Note that φ is not Lipschitz, since it does not preserve the Hausdorff dimension, however φ is "nice" :

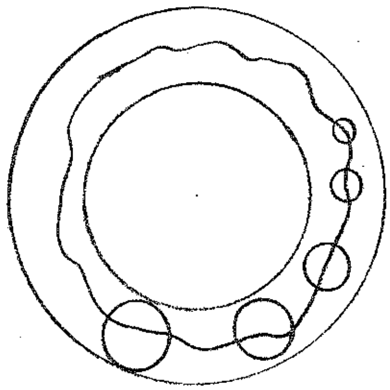
Property. φ is quasi-conformal, in the sense that it does not distort shapes too much. (The argument will make this statement clear).

If S is a little circle intersecting J_f, then $\varphi(S)$ is not too eccentric,





$\varphi(S)$ is contained in an annulus of bounded ratio of radii. Look at all circles within the annulus, of a fixed size and touching J_f; they form a compact set, so φ of these circles has bounded distortion. If you start with small circles, then you can first expand them until they have "full" size, and the distortion is controlled by the distortion lemma of §5. What φ does to small circles is controlled by what it does to large ones.



Now take a shape, if small first expand (distortion lemma). So, we only have to consider large shapes, and here we can apply the compactness principle. Thus, φ does not distort shapes too much.

Example. On the unit disc, consider the map

$$(r, \theta) \longrightarrow (r^{1/2}, \theta)$$

this is quasi-conformal since it takes circles into ellipses of eccentricity 2. However, the distortion of distance goes to ∞ near 0.

The Kleinian group case.

Here we must cope with problem that telescopes are not unique. However, they are "essentially" unique (see below).

Start with a Kleinian group Γ with the expanding property, so Γ is finitely presented (see §6). Now deform the generators slightly, keeping the relations fixed. (A priori, more relations might occur, but we will see that this does not happen if the deformation is sufficiently small).

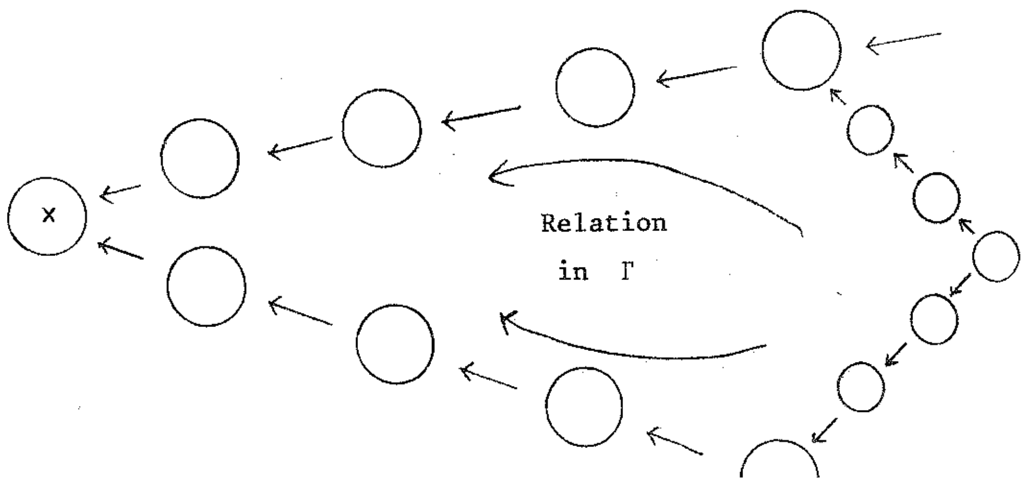
Theorem. If Γ' is a sufficiently small perturbation of Γ , then there exists a homeomorphism

$$\varphi : \Lambda_\Gamma \longrightarrow \Lambda_{\Gamma'}$$

commuting with the actions, and φ does not distort shapes too much.

We construct φ by the telescoping technique. Note that if φ is well defined, then it is also continuous and 1-1 (same proof as for J_f) , so automatically onto, because it commutes with the dynamics (this implies that its image is a closed invariant subset of $\Lambda_{\Gamma'}$, which is minimal). Also, φ does not distort shapes too much (same arguments as for J_f).

We have to see that φ is well defined. For this it is enough to prove that if you have two telescopes converging to a point x , then you can go as far as you please in either telescope, and you can always connect them up by little discs and group actions, so that the resulting circuit is a relation is the group (i.e. the identity).

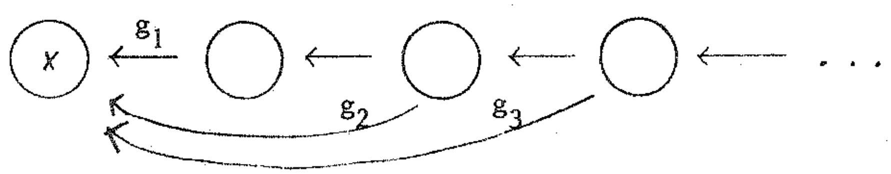


In fact, we do not need the "connection" to be telescoping, we only need to get a commutative diagram, then the two constructions will agree to all orders and φ will be well defined.

Actually, the following proof shows something stronger. It shows that the limit set Λ_Γ can be constructed in abstract, just from the group Γ itself. The proof is really a construction in the group which has been interpreted geometrically (ideas of Margulis, Gromov, etc.).

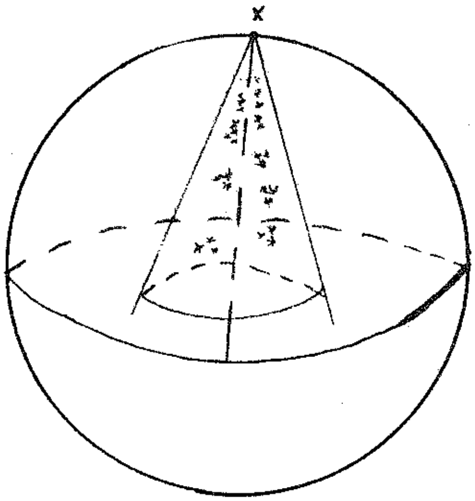
For example, if Γ is free in two generators, then we may let Γ act on S^2 by reflections on four disjoint circles, and Λ_Γ is the set of end points of the tree of fundamental domains, i.e. the infinite words in Γ.

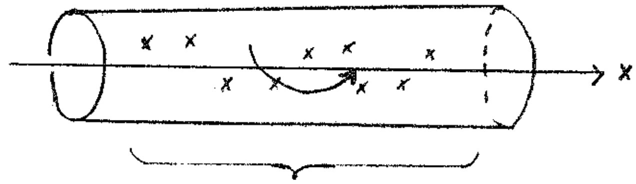
Now, start with a telescope converging to a point $x \in \Lambda_\Gamma \subset S^2$. This gives a sequence of maps $g_1, g_2, g_3, \ldots$ onto the ε-disc about x;



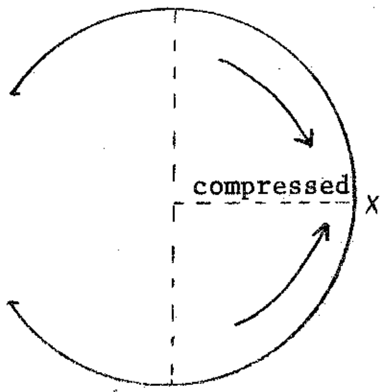
think of these group elements as words $W_1, W_2, \ldots$ of shortest length in the group (written in terms of the generators). Now choose a "centre" $c \in \mathbb{H}^3$, and let the W_n's act on c.

Proposition. The sequence $W_n(c)$ converges to x conically (i.e. contained in cylinder about a geodesic) and regularly distributed along a geodesic.





Proof. We know that any hyperbolic motion is a rotation and translation along a geodesic. Thus, near ∞ it is compression and rotation.



Now observe that there are two natural metrics on the set $\Gamma(c)$, one is the hyperbolic metric, coming from $\mathbb{H}^3$, the other is the word metric, i.e. we define a metric $|\ |$ on Γ by $|\gamma|$ = minimal length of $\gamma \in \Gamma$ written in terms of the generator, and we translate this metric over to $\Gamma(c)$. (Note that this is a construction in Γ) .

Proposition. These two metrics are equivalent, i.e. they are Lipschitz quasi-isometric, and both are left invariant.

Proof. We have to show that there are constants K_0, K_1 such that

$$K_0|\gamma| \leq \rho(c, \gamma(c)) \leq K_1|\gamma|$$

for all $\gamma \in \Gamma$, where $\rho(,)$ is the hyperbolic distance. For Γ cocompact, this follows because we have a tiling of $\mathbb{H}^3$ by a compact set. In general, we consider the convex hull N of Λ_Γ, then any geodesic joining c with a point in its Γ-orbit is contained in N (see Thurston's notes), and N is tiled by $N \cap$ (fundamental domain), which is compact. (see [Floyd], Inv. Math. 57, 1980, p.213, for a detailed proof).

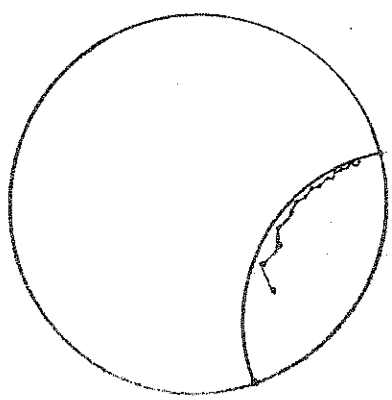
Therefore, the sequence $W_n(c)$ goes towards x at a rather uniform rate, i.e. each step in the sequence is given by a generator,

$$* \longrightarrow * \longrightarrow * \longrightarrow * \longrightarrow \cdots$$

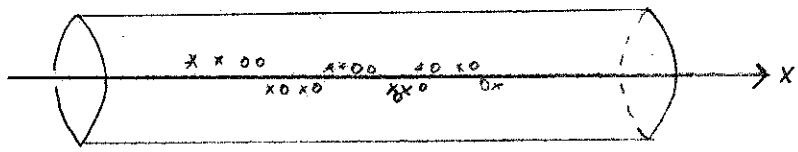
the length is $\sim n$ (the number of steps), and the hyperbolic distance grows less than a constant times n and more than a constant times n . That is, the sequence $W_n(c)$ forms a quasi-geodesic.

Lemma. Every quasi-geodesic is a bounded distance away from a unique geodesic.

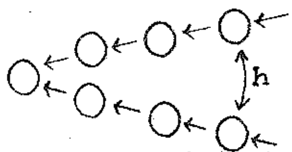
See Thurston's notes (Ch. 5, §9) for the proof of this lemma.



Now suppose you have another telescope converging to x , i.e. another quasi-geodesic $\tilde{W}_n(c)$ going to x , staying in a cylinder a regularly distributed along a geodesic,



then infinitely often we have pairs of points $W_n(c)$, $\tilde{W}_m(c)$ right next to each other, i.e. $h = W_n^{-1}\tilde{W}_m$ is a short word. So, there is a subsequence with $W_n^{-1}\tilde{W}_m = h$. This gives a relation in Γ (not telescoping, but this is irrelevant for uniqueness).



Now, given two quasi-geodesics, we say that they are equivalent if they are a bounded distance apart. Then the limit set of Γ is

$$\Lambda_\Gamma = \left\{\begin{array}{l}\text{Equivalence classes of}\\ \text{quasi-geodesics in } \Gamma\end{array}\right\}$$

This correspondence is given by the telescoping technique.

Remarks. 1) This shows that the topology of Λ_Γ is determined by Γ alone, but not the geometry (e.g. Hausdorf dimension).

2) φ is unique. In fact, a hyperbolic element in Γ has a compressing fixed point and an expanding fixed point. If $\tilde{\varphi}$ is any homeomorphism commuting with the dynamics, then $\tilde{\varphi}$ has to preserve fixed points. But the orbits are dense in the limit set, so $\tilde{\varphi}$ is determined on a dense set of Λ_Γ .

3) Note that the above construction of Λ_Γ out of Γ works for all $\mathbb{H}^n$.

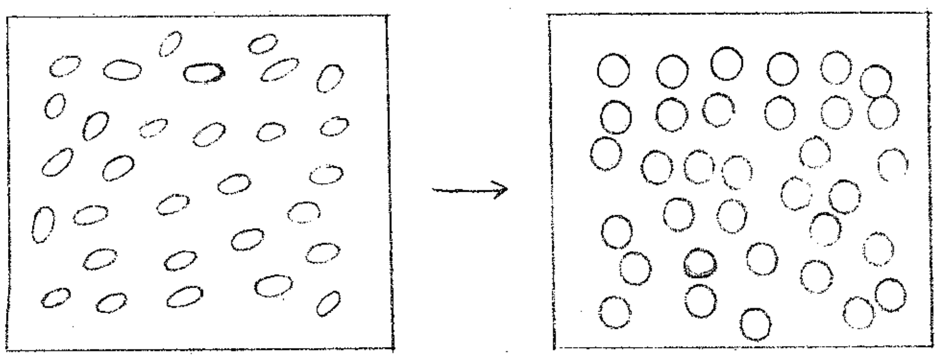
In the case of $\mathbb{H}^3$ there is the reverse theory of constructing deformations of Γ from deformations of the Riemann surfaces (Alfhors, etc.). An important tool for this is the measurable Riemann mapping theorem (Gauss, Morrey, Ahlfors, Bers, etc.). The following local theorem was proved first by Gauss in the real analytic case.

<u>Local Theorem</u>. Given any smooth metric g_{ij} in $\mathbb{R}^2$, there exists a diffeomorphism of the plane taking g_{ij} into the usual flat metric δ_{ij}.

So now, if we have any oriented 2-manifold M, we can use the theorem above to define a complex structure on M from a given metric. (The metric defines a multiplication by i in each tangent space). Taking $M = S^2$, we then have that given any smooth metric g_{ij} on S^2, there is a diffeomorphism of the sphere carrying g_{ij} into the usual metric δ_{ij}. Moreover, we have the following theorem :

<u>Theorem</u>. (Measurable Riemann mapping). Given any measurable metric in $\mathbb{R}^2$ (or in S^2) with a.e. bounded conformal distortion, there exists a quasiconformal homeomorphism taking this metric into the usual one.

In other words, given any field of infinitesimal ellipses on $\mathbb{R}^2$ with bounded eccentricity a.e., there exists a quasi-conformal map taking these ellipses into infinitesimal circles.



If we fix 0,1 and ∞, then the q.c-map is unique. So, if we identify the field of ellipses with an L^∞-function by $z \longrightarrow (e,\theta)$ where e is the eccentricity and θ is the orientation of the main axis, this gives rise to a correspondence

$$L^\infty(\text{Plane}) \longleftrightarrow \left\{ \begin{matrix} \text{quasi-conformal maps} \\ \text{fixing } 0,1,\infty \end{matrix} \right\}$$

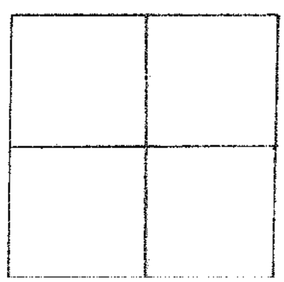
The measurable Riemann mapping theorem can be deduced from the the smooth version as follows. First, we note a compactness principle : If we have a set of K-quasi-conformal homeomorphisms on $\mathbb{C}$, fixing $0,1,\infty$, they form a compact family. Now, if we have a distortion μ_∞ in the metric, we can approximate μ_∞ by a sequence

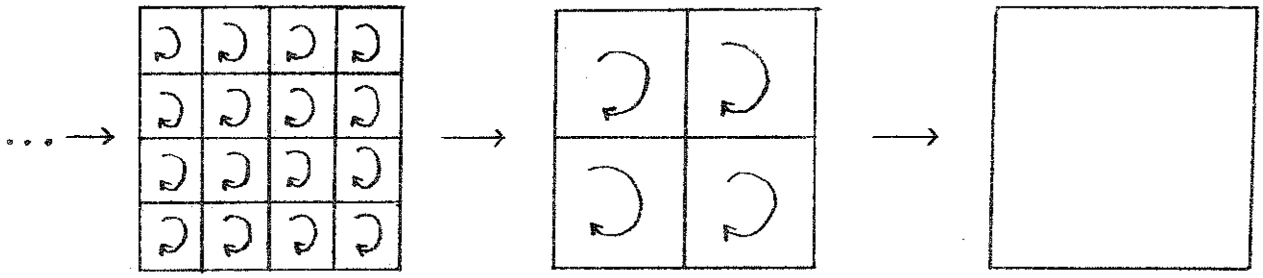
$$\mu_1, \mu_2, \dots \longrightarrow \mu_\infty$$

of smooth μ_i's, where the convergence is pointwise a.e. For each μ_i there is a quasi-conformal map φ_i taking the distorted metric into the usual one. By the compactness principle, the φ_i's converge to a quasi-conformal map φ_∞. Since distortion of φ_i converges pointwise to μ_∞ it follows μ_∞ = distortion φ_∞ (see Lehto et al,"Quasi conformal mappings of the plane" Springer).

<u>Remark</u>. Here we used that the distortion of the limit is the limit of the distortions. For this, in general, we need that both exist.

<u>Example</u>. Take a square I^2 and divide it into four small squares.





Now, make a smooth twist in the interior of each little square, and map everything back into I^2 by the identity. If we carry this construction to ∞, in the limit we get the identity map on I^2, but the distortion does not converge to zero.

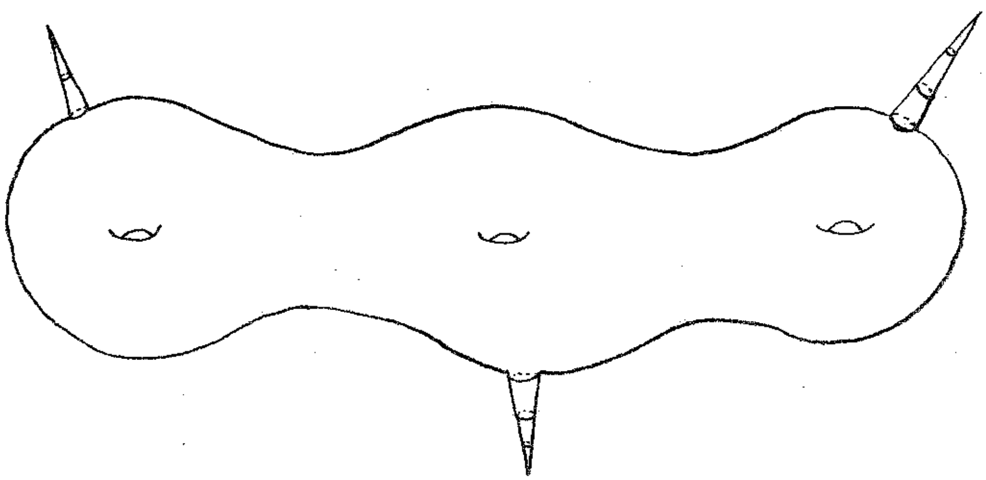
9. Ahlfors finiteness theorem.

There was an important result of Ahlfors in 1965 that aroused new interest in Kleinian groups. The proof of this theorem that we give below is technically different from that of Ahlfors, and it can be adapted to prove a similar result in the Julia set case (§10).

Let Γ be a finitely generated Kleinian group and assume for simplicity that Γ has no torsion. (If it does, then take a subgroup of finite index with no torsion). The theorem has two statements.

Ahlfors theorem. I) If $\Lambda_\Gamma = S^2$, or if Λ_Γ has positive 2-dimensional Lebesgue measure(*), then there is no set $A \subset \Lambda_\Gamma$ of positive measure, with $A \cap \gamma(A) = \phi$ for all $\gamma \in \Gamma - e$. (i.e. there is no A which wanders around).

II) The associated Riemann surface of Γ , $S_\Gamma = \Omega(\Gamma)/\Gamma$, is of finite type. That is, if $\mathbb{C} \cup \{\infty\} - \Lambda_\Gamma$ is the region of discontinuity of Γ , then S_Γ is obtained from a compact Riemann surface by removing (at most) finitely many points.



*) This statement is hypothetical. There is no known example of such a Kleinian group. In fact, it is conjecture that no such Kleinian group exists.

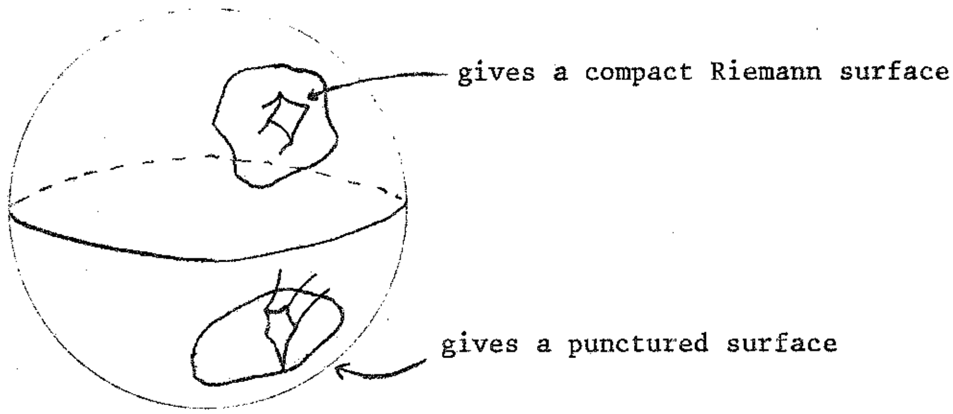
This theorem is only for dimension 3. In fact, it is not even clear what the 2nd statement should be in higher dimensions.

Problem. Is (I) true in $\mathbb{H}^n$?

Note that $\Omega(\Gamma)$ can have ∞ many connected components. However, an immediate consequence of the theorem is,

Corollary. The connected components of $\Omega(\Gamma)$ fall into finitely many Γ-orbits.

This follows because you can pass to the quotient and have a Riemann surface of finite type.



Proof of theorem. Consider the space

$$\mathrm{Hom}(\Gamma, PSL_2(\mathbb{C}))$$

of representations of Γ in $PSL_2(\mathbb{C})$. This is clearly finite dimensional since a homomorphism is defined by the image of the (finitely many) generators of Γ and $PSL_2(\mathbb{C})$ has complex dimension 3. The idea of the proof is that if (I) or (II) were false, this would imply that $\mathrm{Hom}(\Gamma, PSL_2(\mathbb{C}))$ is infinite

-dimensional. This contradiction will come from the measurable Riemann mapping theorem of §8.

Definition. A homeomorphism φ of S^2 is compatible with Γ iff $\varphi\Gamma\varphi^{-1}$ is again a Kleinian group.

Let Homeo_Γ be the set of all such homeomorphisms. By definition, we have a map

$$\text{Homeo}_\Gamma \longrightarrow \text{Hom}(\Gamma, PSL_2(\mathbb{C}))$$

If $\varphi_1, \varphi_2 \in \text{Homeo}_\Gamma$ have the same image, then $\varphi_1\varphi_2^{-1}$ commutes with Γ, so it preserves the fixed points of Γ. Since orbits are dense in Λ_Γ this implies that $\varphi_1\varphi_2^{-1}$ is the identity in Λ_Γ. So the idea is that if the action of Γ "did not cooperate", then we would have an infinite family in Homeo_Γ consisting of maps that disagree on Λ_Γ.

To use the measurable Riemann mapping theorem (MRMT), we need a bounded eccentricity field of ellipses defined a.e., and we just put circles where this field is not defined. (This is fine because circles are invariant under conformal maps, and we want to have Γ-invariance). We call such a field a bounded measurable conformal structure. We will say that such a field is Γ-invariant if the direction of the distortion is Γ-invariant and if any two points which are related by an element in Γ have the same eccentricity.

It is clear that a Γ-invaraint bounded measurable conformal structure on S^2 gives rise, via the MRMT, to a homeomorphism φ such that for every $\gamma \in \Gamma$, $\varphi\gamma\varphi^{-1}$ is a quasi-conformal map that preserves a.e. the standard structure on S^2. Thus $\varphi\gamma\varphi^{-1}$ is in $PSL_2(\mathbb{C})$, i.e. $\varphi \in \text{Homeo}_\Gamma$.

Now, given a quasi-conformal homeomorphism φ of S^2, we call $\varphi\Gamma\varphi^{-1} \subset PSL_2(\mathbb{C})$ a quasi-conformal deformation of Γ. We then have a surjective map

$$\left\{\begin{matrix}\text{quasi-conformal}\\ \text{deformations of } \Gamma\end{matrix}\right\} \to \left\{\begin{matrix}\text{quasi-conformal}\\ \text{deformations of } \Omega(\Gamma)/\Gamma\end{matrix}\right\}$$

(In fact, this map is also 1-1, but we do not need this here).

Proof of (I). Suppose there is an $A \subset \Lambda_\Gamma$ of positive measure that wanders. Any L-function, $A \to \mathbb{C}$, gives a field of ellipses on A, and so on the translates of A by elements in Γ (by specifying Γ-invariance); elsewhere, we take the field to be circles. This yields an ∞-dimensional family of Γ-invariant bounded measurable conformal structures, which gives an ∞-dimensional family in Homeo_Γ, and they disagree on Λ_Γ (because we are specifying the derivative by the bounded measurable conformal structure).

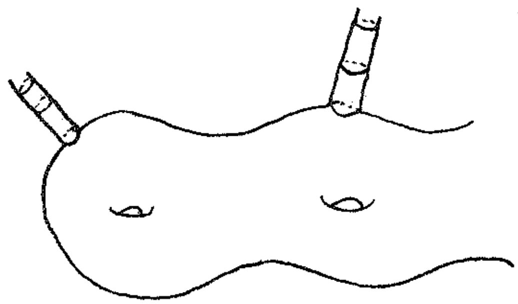
Proof of II. Suppose first that there is a disc Δ which is a connected component of $\Delta(\Gamma)$, and suppose Δ wanders. Note that the Banach space

$$\{\text{q.c. homeomorphisms of } \Delta\}/\text{agreement on } \partial\Delta$$

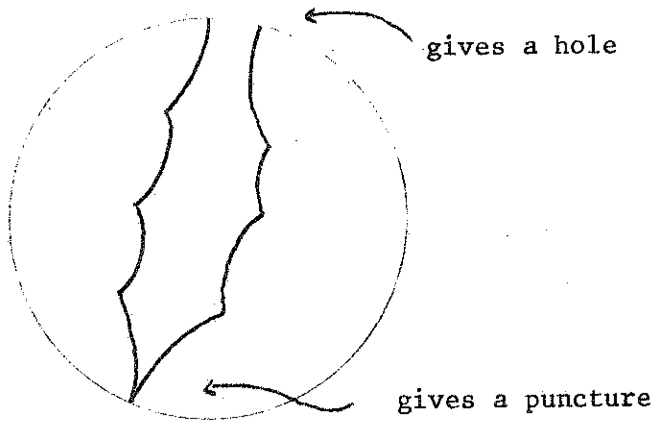
is an ∞-dimensional manifold. (It looks like $SL_2(\mathbb{R}) \times \infty$-dimensional space). For example, it contains all diffeomorphisms on $\partial\Delta \simeq S^1$ with a fixed bound in the derivative.

Now, as in the proof of (I) above, consider all fields of ellipses on Δ with bounded eccentricity, and move them around by elements in Γ. We then have an ∞-family of homeomorphisms on Δ defining, as above, an infinite family in Homeo_Γ, and they disagree on Λ_Γ. (Since $\partial\Delta \subset \Lambda_\Gamma$).

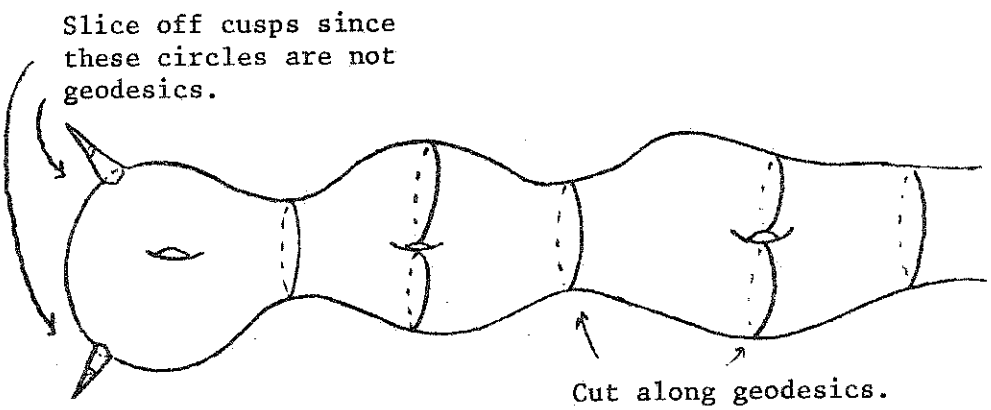
Similarly, the disc argument shows that the Riemann surface $S_\Gamma = \Omega(\Gamma)/\Gamma$ has no "holes", these correspond to funnels in S going to ∞ .



In other words, if we restrict to a component U of $\Omega(\Gamma)$, then the stabilizer of U must have dense orbits in ∂U . [Thus, if U is a disc, then the Fuchsian group corresponding to the Riemann surface is of the 1st kind].

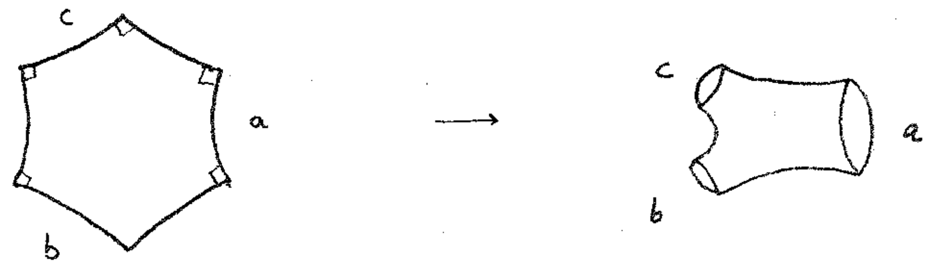


To finish (modulo the technicality that the boundaries of domains are not always Jordan curves), we must count the number of parameters in the conformal structure on such a (punctured) Riemann surface S_Γ .



Remark (method of Ahlfors). The number of punctures is bounded by 3 . N, where N = number of generators. (see Ahlfors paper in Tulance Congress or Sullivan "Finiteness of cusps Acta Math. 1982).

Now, cut the Riemann surface along geodesics, in such a way that the basic units have three holes (i.e. they are pantalones). Then, the number of pantalones is bounded by the number of generators. This follows because a pantalon is the same as having a right angle hexagon,



Therefore, the Riemann surfaces S_Γ has finite genus, bounded by the number of generators.

Remarks. It is clear that we are proving something quantitative, i.e. bounds in the number of cusps, genus, etc., in terms of the number of parameters. In §10 , we will prove similar things for the Julia set J_f .

10. On the iteration of complex analytic maps.

If f is a complex analytic self-mapping of the Riemann sphere, Fatou (1918) defined the domain (of equicontinuity) where points have neighborhoods so that the restrictions of f, $f \circ f$, $f \circ f \circ f, \ldots$ form an equicontinuous family. Julia (1918) in effect defined the complement J_f of the domain of equicontinuity F_f as the closure of the expanding periodic points. In this paper, we add certain techniques to the general study in the memoires of Fatou and Julia, using the anology with the modern study of discrete groups of hyperbolic motions.

A component Ω of the domain of equicontinuity is called cyclic (of order k) if for some $k > 0$ $f^k\Omega = \Omega$.

Theorem 1. Every component of the domain of equicontinuity has an image under f^n, some $n > 0$, which is cyclic. (There are no wandering components).

The method of proof combines ideas of orbit equivalence (to study the equivalence relations $\{x \approx y$ if $\exists\, n \geq 0$ so that $f^n x = f^n y\}$ and $\{x \sim y$ if $\exists\, n,m \geq 0$ so that $f^n x = f^m y\}$ and hyperbolic geometry (analytic self maps of general Riemann surfaces are distance decreasing for the hyperbolic metric, and discrete subgroups of $PSL(2,R)$ can be characterized by nature of elliptic elements) via the Riemann mapping theorem for measurable Riemannian metrics on $\overline{\mathbb{C}}$.

Theorem 2. There are only finitely many cycles of domains. There are Fatou domains which contain non-expanding periodic points either in the interior or on the frontier to which all points of the domain tend. And there are Siegel-Arnold-Herman domains, discs or annuli where the k^{th} power of f

is conjugate to an irrational rotation. There are no others).

Theorem 3. There is a cyclic covering of the domain of equicontinuity*) so that the quotient by the large orbits of f (the equivalence classes of $x \sim y \leftrightarrow f^n x = f^m y$ some n, m) is a Riemann surface, S_f.

Theorem 4. All the conformal structures on S_f compatible with the cyclic cover appear for rational maps quasi-uniformally homeomorphic to f. Thus, we have a finiteness theorem for S_f (the analogue of the Ahlfors finiteness theorem for finitely generated Kleinian groups).

Theorem 5. There is no measurable set $A \subset J_f$ in the Julia set which wanders ($A, fA, f^2A, \ldots$ are all disjoint) and which has positive Lebesgue 2-dimensional measure.

A) Hyperbolic preliminaries.

1) An analytic transformation of an open Riemann surface R covered by the disc, is either an isometry or strictly distance decreasing for the unique conformally equivalent complete metric of curvature -1. By easy arguments one can then show

a) If f has a fixed point x then f is either a rotation of a disc or for all y $f^n y \to x$ as $n \to \infty$. If f has no fixed point then for all y, $f^n y \to x$ as $n \to \infty$.

b) If R is a domain on the sphere and f extends continuously to ∂R where it has only finitely many points fixed (at most), then there is a unique fixed point p in $R \cup \partial R$ to which all orbits of f tend ($f^n y \to p$ for y in R $n \to +\infty$). This uses a) and the fact the hyperbolic metric and the

*) Minus the fixed point if any and all its inverse images under f.... .

spherical metric are related by factor tending to $+\infty$ at ∂R).

2) (Siegel). A non-elementary subgroup of PSL(2,R) is discrete iff it contains no irrational elliptic.

B) The proofs and the construction of the Riemann surface S_f .

1) (Wandering domains). If $\Omega_1 \xrightarrow{f} \Omega_2 \xrightarrow{f} \dots$ are all disjoint, the relations $x \sim y$ and $x \approx y$ are identical in Ω_1 . We may discard finitely many to have no branched points.

a) If Ω_1 is a disc, the f are all injective and $\Omega_1/\sim = \Omega_1/\approx = \Omega_1$ is a Riemann surface with ideal boundary in Julia set.

b) If Ω_1 is an annulus one can show that f is eventually injective (see Appendix (wandering annulus)), so $\Omega_1/\sim$ is a Riemann surface with ideal boundary in the Julia set.

c) If Ω_1 has higher connectivity then $\Omega_1 \xrightarrow{f} \Omega_2 \xrightarrow{f} \dots$ determines an increasing union of discrete subgroups of PSL(2,R) whose union is discrete by A)2). Thus, either the f are eventually injective and we have $\Omega_1/\sim$ is a Riemann surface with boundary in the Julia set or $\Omega_1/\sim$ is a Riemann surface with a non-finitely generated fundamental group.

Proof of theorem 1. In all the cases a) b) c), we use the measurable Riemann mapping theorem to construct (appendix-infinite parameters) an infinite dimension space of rational maps homeomorphic to f . (In a), b), and the first part of c), we use the fact that small conjugacies are unique on the Julia set (Appendix - conjugacy on Julia set). We also use the theory of prime ends to relate the frontier of a domain and the boundary of the standard disc). (Appendix-infinite parameters).

2) (Invariant domains). Suppose $f : \Omega \to \Omega$. If there is a fixed point in Ω remove it and its full orbit from Ω (keep the name Ω) . (This is a discrete set because the inverse orbit $\to$ Julia set). We say that (in a cyclic cover), the full orbit of any critical point is now discrete (Appendix (critical point).

Construct a function $m : \Omega \to \{1,2,3,\ldots\}$ satisfying $m(f(x)) =$ (local degree of f) $(x)\cdot m(x)$ which is identically 1 outside the full orbits of critical points. Then (Ω,m) is a Riemann surface with branch points and $(\Omega,m) \xrightarrow{f} (\Omega,m)$ is a geometric covering.

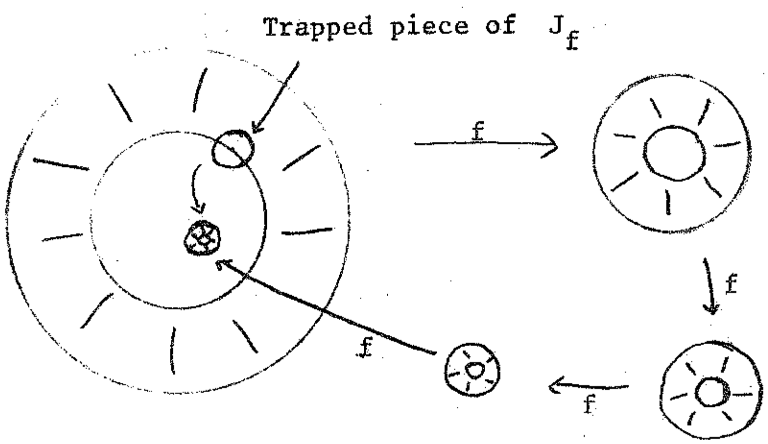
Then $(\Omega,m) \xrightarrow{t} (\Omega,m) \xrightarrow{t} (\Omega,m) \quad \ldots$ determines an increasing sequence of discrete groups whose union is either elementary or discrete by A2). Leaving the elementary cases aside, we have then a branched Riemann surface (in the cyclic cover) representing the $\approx$ equivalence classes on which f becomes an analytic isomorphism. Now f acts discontinuously here and we can form a further quotient which is a Riemann surface and classifies the $\sim$ equivalence classes (in the cyclic cover).

This analysis proves theorem 3. Theorem 4 uses this and the measurable Riemann mapping theorem. Theorem 5 uses the uniqueness of small conjugacies on the Julia set, Appendix (conjugacies on the Julia set) and the measurable Riemann mapping theorem. Theorem 2 follows from A) 1).

Appendix. (Wandering Annuli).

Case ii). If Ω_1 is an annulus, we need to rule our an infinite amount of winding in the sequence $\Omega_1 \to \Omega_2 \to \dots$. If this is such, the modul s of $\Omega_i \to \infty$. Since disjointness implies area $(\Omega_i) \to 0$, conformal geometry implies the diameter of one component of $\partial\Omega_i \to 0$ *). Since f has a bounded Identity distortion and is an open mapping, this implies eventually the small side of Ω_i is mapped to that of Ω_{i+1} .

However, each time there is winding $\Omega_i \to \Omega_{i+1}$, each complementary disc of Ω_i contains a branch point. This forces resting in a subsequence of Ω_i . Then a high power of f takes one annulus into its small side a piece of the Julia set is trapped.



This completes the proof of Proposition 2.

Appendix. (Uniqueness of small conjugacy on Julia Set).

We take advantage of non-trivial topological dynamics on the Julia set in the following proposition.

Proposition 1. A homeomorphism φ of the Riemann sphere commuting with a rational map f must be of finite order on the Julia set of f .

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*) See last section.

Proof. We take a power of φ so that it fixes a periodic point p in J_f of lowest period and its 1st inverse image under f . We suppose also there is no critical point in the backwards orbit of p . We construct a contractible infinite tree over the backwards orbit of p avoiding all critical points. We have d local branches of f^{-1} commuting with the power of φ near p . These equations hold by analytic continuation along the tree and we deduce this power of φ fixes the backward orbit of p which is dense in J_f . Q.E.D.

Remark. For the subsequent arguemnent, we could get by with the more immediate fact that the homeomorphismsof J_f commuting with f from a Cantor group.

A corollary of the proposition is that such homeomorphisms close to the identity must be the identity on J_f .

Appendix. (Infinite parameters).

Proof of theorem 1 special case. Suppose some component Ω is a simply connected Jordan domain which wanders. Then $\Omega/\sim$ is a disc*). There is an infinite dimensional (linear) space of quasi-conformal homeomorphisms of a disc modulo those which are the identity on the boundary**). Thus taking into account all measurable conformal structures on $\Omega/\sim$ leads to an infinite dimensional (linear) space**) of quasi-conformal conjugations $\emptyset$ which are different on the frontier of $\Omega \subset J_f$. All these near (zero) lead to different rational map $\emptyset f \emptyset^{-1}$ by the proposition. Q.E.D.

We formulate in theorem 4 what this type of argument proves.

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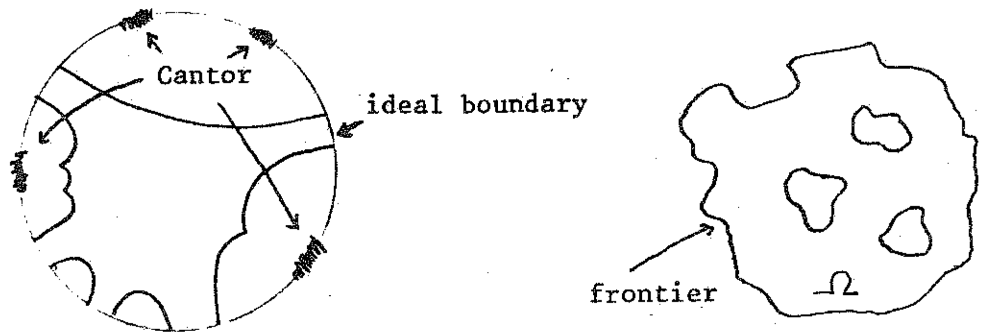
*) Actually a bounded open set of an infinite dimension linear space.

**) We suppose here f is injective on $f^n\Omega$ for $n > 0$. Then no points of $\Omega_\sim$ are identified by $\sim$.

Theorem 4. It cannot happen that a Borel part of $\Omega_f/\sim$ be a Riemann surface with an ideal boundary (i.e. defined by a Fuchsian group Γ whose limit set is not all of S^1) corresponding to a part of the Julia set.

Proof. We proceed as in the case of the wandering disc to get infinitely many parameters. There is one technical point . There is a topological difference between



the ideal boundary (mod Γ) in the universal cover of $\Omega_f/\sim$ and the frontier. This is surmounted using prime ends.

Namely, one knows there is a totally disconnected set of prime ends (or ideal boundary points) associated to a prime point in the frontier. Thus a deformation of the identity through quasi-conformal homeomorphisms of Ω which are the identity in the frontier lift to maps of the disc which become the identity on the boundary of the disc.

Appendix. (Critical point class is discrete).

If Ω has a fixed point, we remove it and its full orbit. This is a discrete set. If the fixed point was also a critical point, we pass to the logarithmic cover (around it). In all cases, we have a map on a domain in the plane and all points tend toward a frontier point p (near which (say I) the map is 1-1).

Thus, for any point x there is an n so that $f^m(x)$ is in the injective region I for $m \geq n$. There is a disc D around $f^n(x)$ so that D is disjoint from $\bigcup_{i \geq 1} f^i D$ since $f^m(x) \to p$. The disc must be pairwise disjoint because if $f^i D \cap f^{i+j} D \neq \phi$ f^{-j} (intersection) $\subseteq D \cap f^i D$ which is various by construction. Now, it is easy to construct a disc about x so that all the forward images are pairwise disjoint. In particular, there is a neighborhood of any point x which intersects any full orbit in only finitely many points (because there are only finitely many branch points and because of the cyclic cover f is eventually injective on $f^i D$).

Thus, any full orbit is discrete. In particular the full orbit of a critical point is discrete.