ON THE ERGODIC THEORY AT INFINITY OF AN ARBITRARY DISCRETE GROUP OF HYPERBOLIC MOTIONS

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We will describe an ergodicity phenomenon for the action of an arbitrary discrete group of hyperbolic isometries on the tangent spaces of the sphere at ∞. One application (Section VII) is a maximal extension of Mostow's rigidity theorem "rough isometry $\Longrightarrow$ isometry" from finite volume hyperbolic manifolds to manifolds whose volume grows slower than that of hyperbolic space. Another application (Section V) allows a complete description of the finitely generated Kleinian groups in one quasiconformal conjugacy class in terms of a nice complex manifold Teichmüller space. Along the way to the main theorem we characterize ergodicity of the action of Γ on geodesics in terms of the divergence of a series of solid angles (Sections II, III). We also characterize the conservative part of the action on the sphere at ∞ in terms of the horospherical limit set (Section IV). (See [Su] for extensions to Hausdorff measure.)

To describe the situation in more detail recall that geometry in hyperbolic (n+1)-space tends as we approach ∞ to conformal geometry on the n-sphere at ∞. Thus a discrete group Γ of hyperbolic isometries is equivalent to a discrete group of conformal transformations of S^n. The dynamics of these actions is quite different. In hyperbolic space Γ is essentially permuting convex fundamental domains freely. While on the

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sphere at ∞ one finds the cluster set of an orbit of Γ in hyperbolic space on which one knows Γ acts minimally (each point has a dense orbit) and an open complement where Γ acts discontinuously.

We seek the dynamical picture of Γ in the context of Lebesgue measure or ergodic theory. Thus we heretofore neglect sets on the n-sphere of measure zero. In that case any group action whatsoever breaks into a dissipative piece and a conservative piece. The dissipative piece is the disjoint union of measurable sets permuted by the group. The conservative (or recurrent) piece has the property that for any set A of positive measure $\gamma A \cap A$ has positive measure for infinitely many group elements. We will find this partition for discrete conformal groups and make use of it.

Now in terms of any measure of finite mass on S^n, say the solid angle measure as viewed from a point inside hyperbolic space, we can divide the points of S^n into those for which the sum over the group of area distortions is finite and those for which it is infinite. It follows as in Poincare's famous recurrence theorem that this is the partition into the dissipative and conservative parts. This statement is general and makes no use of our assumption that Γ is a discrete group of conformal transformations. These hypotheses imply more. Namely,

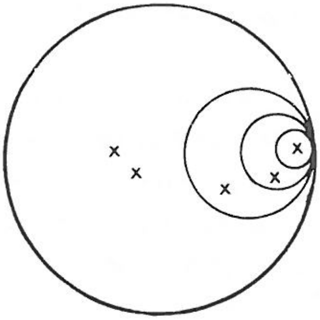
i) the area distortions are unbounded at almost all conservative points (Section IV),

ii) definite variation of the logarithm of area distortion takes place for appropriate individual group elements on small annuli of definite shape anywhere we look in the conservative part (Sections I, III).

The second property of area distortions is the key to the main result. Define a measurable conformal structure on S^n to be a measurable field of similarity structures on the tangent spaces. In Section VII we prove the following theorem. (The case $n = 2$ is treated completely in Section I.)

THEOREM. *Any measurable conformal structure on* S^n *which is kept a.e. invariant by a discrete group of conformal transformations of* S^n *must agree a.e. with the standard conformal structure outside the dissipative part of the action.*

The first property of area distortions above allows us to recognize the conservative part in geometric terms. For example, *the horospherical limit set,* by definition those points of S^n for which an orbit in hyperbolic space enters every horosphere (based at the point, see figure a) has full measure in the conservative part. Or the dissipative part is the union of the parts of the fundamental domains in hyperbolic space on the sphere at ∞ (Section IV). Thus if a fundamental domain has zero area at ∞ (figure b) the action is conservative, the measurable invariant conformal structure is unique and we have the needed step for the generalization of Mostow's theorem mentioned above (Section VII).



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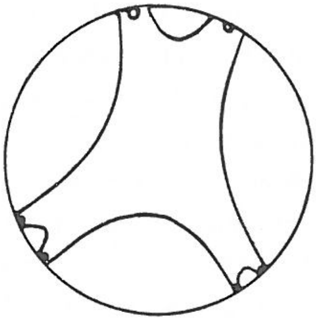

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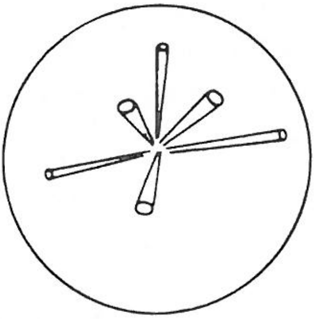

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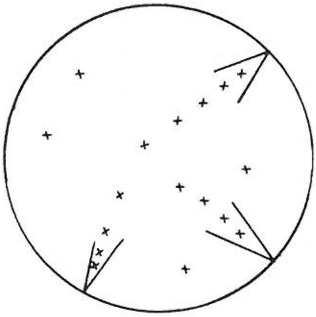

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The argument of Section II uses random walks as in [G] and yields a new geometric characterization of the ergodicity of the geodesic flow in terms of infinite solid angle of the orbit in hyperbolic space (Section II). We recover E. Hopf's (1939) characterization in terms of conical approach (Section III) (figures c and d).

Finally, we turn the discussion to quasi-conformal homeomorphisms. First not only does hyperbolic geometry become conformal geometry at ∞ but quasi-(hyperbolic geometry) becomes quasi-conformal geometry on the sphere at ∞. (This phenomenon plus the ergodicity on tangent directions summarizes Mostow's theorem.)

Second, in dimension 2 quasi-conformal homeomorphisms of S^2 are parametrized [AB] by measurable conformal structures defined a.e. on S^2, and which are a bounded distance away from the standard structure. (Reminiscent of the parametrization of Lipschitz homeomorphisms of S^1 by measures with a bounded density.)

Thus if Γ is a countable group of uniformly quasi-conformal homeomorphisms of S^2 and ν is a bounded measurable conformal structure invariant by Γ one can construct a quasi-conformal conjugacy of Γ to a group of conformal transformations. Furthermore the different "conformal models" of Γ are parametrized by such invariant ν ([AB], [B]).

Now it is remarkable but elementary that one such invariant measurable conformal structure ν always exists for any quasi-conformal group Γ. (One merely forms fiberwise the barycenter of the convex hull of the transforms by Γ of the standard structure. And this works in any dimension.) So in dimension 2, Γ always has at least one conformal realization or model using the measurable Riemann mapping theorem [AB].

COROLLARY. i) *A discrete group* Γ *of uniformly quasi-conformal homeomorphisms of* S^2 *has an invariant measurable conformal structure and this structure is unique on the conservative part.*

ii) *If* Γ *is finitely generated, the dissipative part agrees with the topological domain of discontinuity.*

iii) *A general* Γ *always has conformal models and these are parametrized by varying the invariant measurable conformal structure on the dissipative part.*

iv) *In the finitely generated case, again, this parametrization is by a Teichmüller space associated to a Riemann surface of finite type* (Section V).

In Section VI we discuss further dynamical properties of finitely generated discrete groups of conformal transformations on S^2. The first part of Section I and Sections III, IV, VII also work for S^n $n \geq 1$ as we explain somewhat in Section VII. The proof of the main theorem for $n = 2$ is actually completed in the first section.

We record our debt of motivation to Ahlfors' papers "Remarks on the limit point set of a finitely generated Kleinian groups," Annals of Math. Studies 66, and especially "Some Remarks on Kleinian Groups" from the unpublished Tulane proceedings on Kleinian groups. In the latter paper Ahlfors establishes the topological limit set is conservative for finitely generated Kleinian groups (with a domain of discontinuity). In his well-known finiteness paper (1965) he establishes the (domain of discontinuity)/Γ is a Riemann surface of finite type.

For me these two theorems of Ahlfors are almost the axioms for a good theory of Kleinian groups. Together they impose a tight structure on the situation.

Finally, we dedicate this paper to Lucy Garnett who supplied the random walk idea for Theorem II, pointed out a non-obvious point about horocycle limit points, Section IV, and generally sustained the work in this paper.

Section I. *The variation of area distortion lemma and groups with finite solid angle,*[1] $\sum_{\Gamma} 1/\lambda^2 < \infty$

THEOREM I. *For groups* Γ *of finite solid angle*[1] *in* H^3 *there is on the conservative part of the action of* Γ *on* S^2 *no measurable tangent line field invariant a.e. by* Γ.

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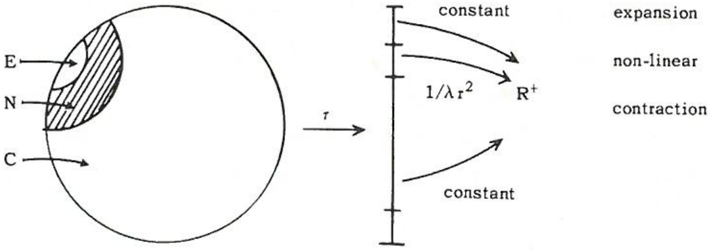
[1] This finite solid angle condition can be dropped in two ways, see addendum to Section I or Section II.

Proof. Any conformal transformation γ of the sphere is the composition of a rotation followed by a hyperbolic transformation with antipodal fixed points. Thus the linear distortion d_γ of the spherical metric varies in an interval $[1/\lambda, \lambda]$ with $\lambda \geq 1$. The extreme values of the distortion are taken at antipodal points. The intermediate values are taken on concentric circles interpolating between these points. We need a uniform picture of the distortion for λ large.

Let r be the natural radial parameter for these circles with $r = 0$ corresponding to the value of distortion λ. For any given $k > 1$ and $\lambda^2 > k$ let D_γ be the unique continuous function on the sphere with values $1/\lambda r^2$ on the circles of radius r with $k/\lambda \leq r \leq 1/k$ and which is constant otherwise. Thus



$$D_\gamma = \begin{cases} \lambda/k^2 & \text{on} \quad \{0 \leq r \leq k/\lambda\} \quad = E(\gamma) \\ 1/\lambda r^2 & \text{on} \quad \{k/\lambda \leq r \leq 1/k\} = N(\gamma) \\ k^2/\lambda & \text{on} \quad \{1/k \leq r \leq 1\} \quad = C(\gamma)\,. \end{cases}$$

LEMMA 1. *The ratio of the actual distortion* d_γ *to the approximation* D_γ *is bounded in terms of* k *for all conformal transformations* γ *with* $\lambda(\gamma) \geq k^2$. *Moreover on the nonlinear part* $N(\gamma)$ *this ratio is arbitrarily close to* 1 *for* k *sufficiently large.*

Proof. i) We compute for the affine transformation $x \to \lambda x$ on the line the distortion of the measure coming via stereographic projection from the uniform measure on the circle of diameter 1. This transposed measure is

$\frac{dx}{1+x^2}$. Thus the distortion of $x \to \lambda x$ is $d(x) = \frac{\lambda(1+x^2)}{1+(\lambda x)^2}$, which is monotone decreasing in x.

ii) For $0 \le x \le k \cdot 1/\lambda$, $d(x)$ lies in $\left[\frac{\lambda(1+k^2/\lambda^2)}{1+k^2}, \lambda\right] \subset \lambda[1/1+k^2, 1]$.

For $1/k \le x \le \infty$, $d(x)$ lies in $\left[1/\lambda, \frac{\lambda(1+1/k^2)}{1+\lambda^2/k^2}\right] \subset 1/\lambda[1, 1+k^2]$.

For $k/\lambda \le x \le 1/k$, $d(x) \cdot \lambda x^2$ lies in

$$\left[\frac{(\lambda x)^2}{1+(\lambda x)^2}, \frac{\lambda(1+1/k^2) \cdot \lambda x^2}{(\lambda x)^2}\right] \subset \left[\frac{1}{1+1/k^2}, 1+1/k^2\right] .$$

iii) Now we reinterpret these inequalities replacing the variable x on the line by the variable r on the circle. The assertions for the parts $C(\gamma)$ and $E(\gamma)$ follow. For $N(\gamma)$ note that for k large stereographic projection is almost an isometry between $0 \le r \le 1/k$ and $0 \le x \le 1/k$.

COROLLARY 2. *Area* $E(\gamma) \le 1/\lambda^2 \cdot a_k$, *area* $\gamma C(\gamma) \le 1/\lambda^2 \cdot a_k$, *diameter* $N(\gamma) \le 2/k \cdot c_k$, *diameter* $\gamma N(\gamma) \le 2/k \cdot c_k$, *where* a_k *is a constant depending only on* k *and* c_k *is arbitrarily close to* 1 *for* k *large.*

Proof. The first three follow from Lemma 1 and the definitions. The last follows because the circles of radius r filling up $N(\gamma)$ map (by conformality) to circles of radius $r \cdot 1/\lambda r^2$ (times a factor near 1 for k large). Since $k/\lambda \le r \le 1/k$ in $N(\gamma)$ these radii lie in the interval $[k/\lambda, 1/k]$ (times c_k).

REMARK. If "$\sim$" denotes an equality up to a factor (or discrepancy) which is near 1 (or negligible) when k is chosen large, then

$$\gamma(N(\gamma) \sim N(\gamma^{-1}),$$

$$\gamma E(\gamma) \sim C(\gamma^{-1}) \quad \text{and} \quad \gamma C(\gamma) \sim E(\gamma^{-1})$$

although we don't use these facts explicitly.

Now suppose R is a *recurrent* (conservative) set on S^2 for the discrete group Γ of conformal transformations. Thus $X \subset R$ of positive measure implies $X \cap \gamma X$ has positive measure for infinitely many γ. Suppose also that Γ has "finite solid angle": $\sum_{\Gamma} 1/\lambda^2 < \infty$. The *modulus* of a concentric annulus on the sphere is by definition the logarithm of the ratio of radii.

LEMMA 3. *Let* $X \subset R$ *of positive measure and two positive constants* δ *and* Δ *be given. Then there is a point* x, *an element* γ *and a concentric annulus* A *so that,*

i) $x \in X$, $x \in A$, *and* $\gamma(x) \in X$.

ii) A *and* $\gamma(A)$ *have diameter* $< \delta$.

iii) A *and* $\gamma(A)$ *have modulus as close as we like to* Δ.

iv) *The distortion of* γ *is constant on the concentric circles of* A *and the* $\log$ *of the distortion varies in* A *by an amount as close as we like to* 2Δ.

Proof. i) Fix k so large that $2/k \cdot c_k < \delta$ (see Lemma 1).

ii) Remove finitely many elements from the group to make $\log \lambda/k^2 > \Delta$ for the rest (possible since Γ is discrete).

iii) Remove finitely many elements so the sum of area $(E(\gamma) \cup \gamma C(\gamma))$ for the rest is less than area X (possible since $\sum_{\Gamma} 1/\lambda^2 < \infty$ using Lemma 1).[2] Remove this infinite union from X to obtain X' which still has positive measure.

iv) Find a γ outside the finite sets above so that $\gamma^{-1}X' \cap X'$ has positive measure (using recurrence of the action). If $x \in \gamma^{-1}X' \cap X'$, by construction $x \notin E(\gamma)$ and $\gamma(x) \notin \gamma C(\gamma)$. Thus $x \in N(\gamma)$.[2]

v) On $N(\gamma)$ the distortion d_γ varies between k^2/λ and λ/k^2 (Lemma 1), so $d_\gamma(x)$ is somewhere in this interval. On the $\log$ scale we

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[2] See the addendum to Section I for an alternative to this step.

can fit an interval of length 2Δ about the value $\log d_\gamma(x)$, and stay in $\log d_\gamma N(\gamma)$, because $\log \lambda/k^2 > \Delta$ by ii) above. Let such an interval define the annulus A in question, so that $x \epsilon A \subset N(\gamma)$ and $\log(d_\gamma)$ varies through $\sim 2\Delta$ in A, by construction.

vi) Since $A \subset N(\gamma)$, diameter A and diameter $\gamma A < \delta$ by i) and Corollary 2.

vii) Since $d_\gamma \sim 1/\lambda r^2$ on $N(\gamma)$ and thus on A, variation $\log d_\gamma \sim 2\Delta$ on A implies modulus $A \sim \Delta (\log r_2^2/r_1{}^2 = 2\Delta$ iff $\log r_2/r_1 = \Delta)$. Similarly γA has radii $dr_1 e^{2\Delta}$ and dr_2 (for some d) because we know d_γ on A. Thus the modulus of γA is

$$\log \frac{dr_1 e^{2\Delta}}{dr_2} \sim 2\Delta - \log r_1/r_2 \sim \Delta .$$

Now we pass from the sphere to the plane.

LEMMA 4. *If* e *is an absolutely continuous isomorphism of the plane (relative to Lebesgue measure) carrying* B *to* B′, *then a subset* $A \subset B$ *with a proportion* η *of area is carried to a subset* $e(A) = A' \subset B'$ *of proportion at least* η' *of area where* $\eta' = 1 - d(1-\eta)$ *and* d *is the maximum ratio of area distortion at various points of* B.

Proof. By an affine scaling we can assume area $B = 1$, the low value of area distortion is 1, and the high value is d. The worst case occurs when 1 occurs on all of A and d occurs on all of the complement of A. Then,

$$\eta' = \text{area } A'/\text{area } B' = \eta/\eta + d(1-\eta) \geq 1 - d(1-\eta) .$$

LEMMA 5. *Let* X *be a set in the plane of positive measure and let* η *and* Δ *be given positive numbers. Consider sectorial boxes of shape* Δ,

$$\{(r,\theta): r_0 \leq r \leq e^{\Delta} r_0 , \theta_0 \leq \theta \leq \theta_0 + \Delta\} .$$

Then there is a $\delta > 0$ *and a subset* X′ *of* X *of positive measure so that each box of shape* Δ *and diameter* $< \delta$ *containing a point of* X′ *also contains at least the proportion* η *of* X.

Proof. The class of sectorial boxes of shape Δ are generated by similarity transformations from one of them. Thus Lebesgue's theorem concerning density points is true using these instead of round disks. (See E. Stein "Singular Integrals ..." pp. 11, 12.)

So for almost all x there is a largest positive δ_x such that the proportion of X in boxes containing x of diameter $\leq \delta_x$ is at least η. But then $x \to \delta_x$ is a positive measurable function which has to be greater than some $\delta > 0$ on a set $X' \subset X$ of positive measure. This proves the lemma.

Now consider a conformal transformation of the plane

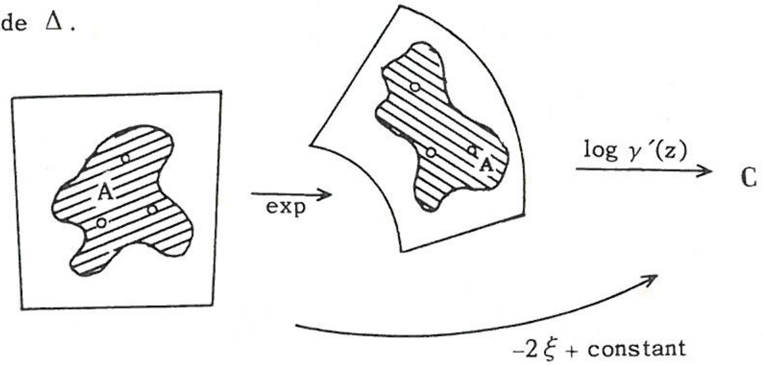
$$\gamma : z \to \frac{az+b}{cz+d} \qquad ad - bc = 1 , \qquad c \neq 0$$

and sectorial boxes B_Δ of shape $\Delta < \pi/2$ centered at $-d/c$.

LEMMA 6. *If* $A \subset B_\Delta$ *is any subset with the proportion* η *of area, then the variation on* A *of the real and imaginary parts of* $\log \gamma' z$ *is at least* $2\Delta(1 - e^{2\Delta}(1-\eta))$.

Proof. i) On a unit square the function $(x,y) \to x$ has variation at least η' on any subset whose proportion of area is at least η'.

ii) Introduce the variable $e^{\xi} = z + d/c$ so that the variation of $\log \gamma' z = \log \dfrac{1}{(cz+d)^2} = -2 \log(z+d/c) + \text{constant}$ on $A \subset B_\Delta$ is just the variation of -2ξ on a corresponding subset A' of a square in the ξ-plane of side Δ.



iii) The ratio of area distortion of exp at different points of the square is at most $e^{2\Delta}$. By Lemma 4 the proportion of $A' = \exp^{-1}(A)$ in the square is at least $\eta' = 1 - e^{2\Delta}(1-\eta)$.

iv) Applying i) the result follows.

Now we are ready to prove the nonexistence of invariant measurable line fields for groups of finite solid angle.

i) Choose a small number, $\pi/2 > \Delta > 0$ and a set of positive measure X in the plane where the hypothetical invariant line field varies only in an interval of inclinations of length $\frac{1}{2}\Delta$.

ii) Choose $0 < \eta < 1$ so that $1 - e^{2\Delta}(1 - \eta + e^{2\Delta}(1-\eta)) > 1/2$.

iii) Find $X' \subset X$ of positive measure satisfying a $\delta' > 0$ uniform density relative to X, η, and sectorial boxes of shape Δ (as in Lemma 5).

iv) Choose a point of density of X' and stereographically project the action of Γ on the plane to a sphere resting on this point.

v) Let Y denote the intersection of X' with a ball B' about this point sufficiently small so that the distortion of stereographic projection on $2B$ is as close to 1 as we need for the following. Let $\delta = \min(\delta',$ radius $B')$.

vi) Relative to δ, Δ and Y (put Y on the sphere) find the element γ and the concentric annulus A satisfying Lemma 3 (and put A back on the plane). In A choose a sectorial box B of shape $\sim\Delta$ containing x and centered at the pole of γ (possible because we know the variation of $\log d_\gamma$ on A).

vii) Since the diameters of B' and $\gamma B'$ are less than δ' (even δ) they each contain the proportion η (at least) of X. ($x \epsilon B \subset A$, $\gamma x \,\epsilon\, \gamma B' \subset A$, and x and γx belong to X'.)

Since the ratio of area distortion of $B' \to \gamma B'$ is at most $e^{2\Delta}$ the proportion of y in B' so that $\gamma y \,\epsilon\, X$ is at least $\eta' = 1 - e^{2\Delta}(1-\eta)$ by Lemma 4 again. Thus the proportion of $y \,\epsilon\, B'$ so that $y \,\epsilon\, X$ and $\gamma y \,\epsilon\, X$ is at least $1 - [(1-\eta) + (1-\eta')] = \eta + \eta' - 1 = \eta''$.

viii) By Lemma 6 the variation of the argument of $\gamma'(z)$ on this subset $B \cap X \cap \gamma^{-1}X$ of B is at least $2\Delta(1 - e^{2\Delta}(1-\eta''))$. This variation is greater than $\sim \Delta$ by ii) which contradicts the definition of X i) and the invariance of the line field under γ. This proves the Theorem I for discrete groups of finite solid angle, $\sum_{\Gamma} 1/\lambda^2 < \infty$.

Addendum

After writing this paper I learned from Klaus Schmidt's notes "Cocycles of ergodic transformation groups" Warwick 1976 §4, that for any conservative group action, any set A of positive measure, and any $\varepsilon > 0$ there are infinitely many group elements γ so that $\{x \in A \cap \gamma^{-1}A$ and the area distortion of γ at x is ε-close to $1\}$ has positive measure. For Kleinian groups such x belong to N_γ by the formula above for D_γ.

Section II. *Ergodicity of the geodesic flow and groups with infinite solid angle,* $\sum_{\Gamma} 1/\lambda^2 = \infty$

THEOREM II. *If* $\sum_{\Gamma} 1/\lambda^2 = \infty$ *for a discrete group of conformal transformations on* S^2 *the action of* Γ *on* $S^2 \times S^2$ *is ergodic.* ($\phi(x,y) = \phi(\gamma x, \gamma y)$ *for all* $\gamma \in \Gamma$ *and* ϕ *measurable implies* ϕ *is constant a.e.*)[1]

COROLLARY.[2] *For groups* Γ *of infinite solid angle in* H^3 *there is on* S^2 *no measurable tangent line field invariant by* Γ *a.e.*

Proof. If there were such, for almost all pairs of points on S^2 we would have a measurable angle difference function (measured along connecting geodesic). By ergodicity one deduces this angle is constant a.e. The line field is seen to be the restriction of a continuous line field invariant by Γ. This is absurd.

Proof of Theorem. First we gather some facts from hyperbolic geometry. What we need follows from the complete symmetry of hyperbolic space

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[1] This theorem holds for S^n using $\sum 1/\lambda^n$, $n \geq 1$, and the proof is the same.

[2] This proof is independent of Section I and is akin to Mostow's original discussion (see point iii), Section VII).

together with the positive lower bound on the convexity of spheres of arbitrarily large radius. Let $\theta(x, y)$ be the reciprocal of the area of the sphere passing through y with center x. So $\theta(x, y)$ is the (density of) solid angle of one point as viewed from the other.

LEMMA 1. $\sum_{\Gamma} 1/\lambda^2 = \infty$ *if and only if the total solid angle of an orbit,* $\sum_{\Gamma} \theta(x, \gamma y)$, *viewed from any point* x *not on the orbit is infinite.*

Proof. Consider a homothety γ in the upper half space model with fixed point at zero and a very small linear derivative $1/\lambda$. If x and y_0 are on the z axis above zero, then clearly $\theta(x, y_0)$ and $\theta(x, \gamma y_0)$ are in the approximate ratio $1/\lambda^2$. Thus in general the order of the term $\theta(x, \gamma y)$ for $\lambda(\gamma)$ large and x fixed is in a bounded ratio to $1/\lambda^2$. This proves the lemma.

EXTRA REMARK. If $g_x(y)$ denotes the Green's function of hyperbolic space with pole at x ($g_x(y)$ is positive, harmonic, tending to zero at ∞, and symmetric about x), there is the exact formula

$$g_x(y) = \int_{\infty}^{y} \theta(x, y')\, dy'$$

obtained by integrating in along a radius from ∞. Since $\theta(x, y)$ is exponentially decreasing in the distance (x, y), the integral is approximately the upper limit so $g_x(y) \sim \theta(x, y)$, at large distances.

Thus in case hyperbolic space mod Γ is a manifold V one sees the condition $\sum_{\Gamma} 1/\lambda^2 < \infty$ is equivalent to V has a finite Green's function $g_x(\bar{y}) = \sum_{\gamma \epsilon \Gamma} g_x(\gamma y)$. Or in other words $\sum_{\Gamma} 1/\lambda^2 = \infty$ if and only if random motion on V is recurrent. This idea is the point of the ensuing discrete time proof.

Now to each point p in hyperbolic space associate the *Poisson measure* μ_p on the sphere S^2 at ∞. If $A \subset S^2$ then $\mu_p(A)$ is the *solid angle* of A viewed from p.

LEMMA 2. *The measure* μ_p *is the spherical average of the measure* μ_q *where* q *ranges over a sphere with center* p.

Proof. Each of the two measures on the sphere at ∞ is invariant by the full rotation group about p. Thus they are equal.

We can fill in bounded measurable functions ϕ on the sphere at ∞ to bounded harmonic functions on hyperbolic space. Namely define

$$h(p) = \int \phi \, d\mu_p \equiv <\phi, \mu_p>, \quad \text{“Poisson formula”} .$$

LEMMA 3. ϕ *non-constant a.e. implies* h *non-constant.*

Proof. Take density points x and y in S^2 of sets where ϕ has values in disjoint intervals. For points p in hyperbolic space near x, μ_p sees mostly the values in the interval associated, so h(p) lies nearly in this interval. Similarly for points q near y, h(q) nearly lies in the disjoint interval.

Let P denote the (averaging) operator on functions and measures on hyperbolic space $f \mapsto Pf$ and $\mu \mapsto \mu P$, where $Pf(x)$ is the average of f over a ball of radius η centered at x, $B(x,\eta)$ and if δ_x denotes the dirac mass at x, $\delta_x P$ is the uniform measure on $B(x,\eta)$ of total mass 1. Note $<\mu P, f> = <\mu, Pf>$ when both make sense.

By Lemma 2 the functions h constructed by the Poisson formula are P-harmonic, namely $Ph = h$. Also P clearly commutes with isometries.

LEMMA 4. *The density at the point* y *of the measure* $\sum_{n=1}^{N} \delta_x P^n$ *is for* N *large at least a fixed constant times* $\theta(x,y)$.

Proof. It is clear that the sequence of measures $\delta_x P^n$ $n = 0, 1, 2, \cdots$ begins at x, spreads out symmetrically by steps of length at most η and converges to ∞ in the sense that almost all the mass is eventually outside any sphere centered at x. The last part follows since a ball of radius η centered on a sphere which is itself centered at x has a proportion of area definitely more than $1/2$ outside the sphere.

Thus all the mass of $\delta_x P^n$ as $n \to \infty$ passes through any spherical shell of thickness η centered at x. Since the density function of the measure $\sum_{n=1}^{\infty} \delta_x P^n$ is *decreasing*, only depends on the radial coordinate from x, and puts at least mass 1 in each shell of width η whose volume is proportional to $1/\theta(x, y)$ when y lies in the shell, the result follows. Q.E.D.

Now form the space T from the product of hyperbolic space with the sphere at ∞ by dividing by the diagonal action of Γ, $T = H \times S^2/\mathrm{mod}\ \Gamma$. Provide T with an averaging operator $\tilde{P}$ on measures and functions which is defined using P in each H-level. Provide T with a natural smooth measure dm using the family of Poisson measures μ_p on the factors $(p \times S^2)$ and the natural measure on H to obtain a Γ invariant volume element dm' on $H \times S^2$. Let μ_p also denote the image measure resting on a sphere of T. Now we come to the key lemma.

LEMMA 5. *The density of the measure* $\sum_{n=1}^{N} \mu_p P^n$ *relative to* dm *converges to* $+\infty$ *at almost all points of* T *as* $N \to \infty$.

Proof. We compute the density of $\nu_N = \sum_{n=1}^{N} \mu_p (P \times \mathrm{id})^n$ in $H \times S^2$ relative to dm and add these densities up along an orbit of Γ: $(y, s), (\gamma y, \gamma s), \cdots$.

Let $g_x(y)$ denote the density of the Green's measure $\delta_x + \delta_x P + \delta_x P^2 + \cdots$ in terms of the natural volume dh on hyperbolic space. In terms of the product measure $dh \times \mu_p$ on $H \times S$ (which is not Γ invariant) the measure $\lim_{N\to\infty} \nu_N$ has density $g_p(y)$ at each point (y,s) of the product space. Rewriting in terms of dm' (which is Γ invariant) introduces the Radon Nikodym factor $\frac{d\mu_p}{d\mu_y}(s)$ at each point (y,s). The desired density is thus $\sum_\Gamma g_p(\gamma y)\frac{d\mu_p}{d\mu_{\gamma y}}(\gamma s)$.

Using γ^{-1} to transform $(p,\gamma y,\gamma s)$ into $(\gamma p^{-1}, y, s)$ and the symmetry $g_x(y) = g_y(x)$ changes the sum to one of the form $\sum_\Gamma g_y(\gamma p)\frac{d\mu_{\gamma p}}{d\mu_y}(s)$. By Lemmas 1 and 4, $\sum_\Gamma g_y(\gamma p) = \infty$ so Lemma 5 results from the following lemma.

LEMMA 6. $\sum_{\gamma\epsilon\Gamma} g_y(\gamma p) = \infty$ *implies that for almost all* s *on the sphere* $\sum_{\gamma\epsilon\Gamma} g_y(\gamma p)\frac{d\mu_{\gamma p}}{d\mu_y}(s) = \infty$.

Proof. i) Denote by B a small ball centered at p, and by ΓB the disjoint union $\bigcup_{\gamma\epsilon\Gamma} \gamma B$. Define a Γ invariant function $\pi_B(y)$ to be the probability that a random walk, whose transition operator is P, starting at y hits ΓB. Clearly $\pi_B(y) \leq 1$ and $P\pi_B \leq \pi_B$. If π_B is not identically 1, the Γ invariant function $\pi_B - P\pi_B$ is greater than $\varepsilon > 0$ on the Γ orbit of some smaller ball. Now consider the identity

$$<\delta_y + \delta_y P + \cdots \delta_y P^N, \pi_B - P\pi_B> \;=\; <\delta_y, \pi_B> - <\delta_y P^{N+1}, \pi_B> .$$

The right-hand side is uniformly bounded wrt N. The left-hand side is at least $\varepsilon \sum_\Gamma g_y(\gamma p)$ as $N \to \infty$. We conclude that $\pi_B = P\pi_B$ or $\pi_B(y) = 1$. Thus almost all paths starting at y hit ΓB when $\sum_{\gamma\epsilon\Gamma} g_y(\gamma p) = \infty$.

ii) For fixed s_0 and x_0 the function of x, $\frac{d\mu_x}{d\mu_{x_0}}(s_0)$ is a P-harmonic function (Lemma 2), whose boundary values are $+\infty$ at s_0

and zero at other points of the sphere. One sees by a standard limiting procedure [K] that $g_x(y)\,\frac{d\mu_y}{d\mu_x}(s_0)$ is the corresponding Green's density for the random walk conditioned so that the limit at ∞ is s_0. By Fubini's theorem and i) for almost all s_0, almost all paths starting at y and conditioned to end up at s_0 must also hit ΓB.

Now if for one of these s_0 $\sum\limits_{\gamma\epsilon\Gamma} g_y(\gamma p)\,\frac{d\mu_{\gamma p}}{d\mu_y}(s_0) < K < \infty$, the same inequality would hold for p in a small ball B. It would follow that the conditioned Green's measure would give finite measure to ΓB. This contradicts the fact that almost all conditioned paths hit ΓB starting from any point and thus hit it infinitely often by the Markov property. This completes the proof of Lemma 6.

Proof of Theorem II. Let $\phi(x,y)$ be a non-constant characteristic function on $S^2 \times S^2$ invariant by Γ and suppose $\sum\limits_{\Gamma} 1/\lambda^2 = \infty$. We can suppose by Fubini (after interchanging x and y if necessary) that for a set of y of positive measure $\phi(x,y)$ is a non-constant a.e. function of x.

Fill in each $\phi(x,y)$ to a P-harmonic function on $H \times y$ by the Poisson formula above. We obtain a function $h(x,y)$ on T by invariance of ϕ, harmonic on each level (Lemma 2) which implies $\tilde{P}h = h$, and finally $h(x,y)$ is non-constant on a set of levels of positive measure (Lemma 3).

Break $h-1/2$ into positive and negative parts h_+ and h_- and add the inequality $\tilde{P}|h-1/2| \geq |\tilde{P}(h-1/2)| = |h-1/2|$ to the equality $\tilde{P}(h-1/2) = h-1/2$ to obtain $Ph_+ \geq h_+$. Using Lemma 5 and the cancellation argument of part i) Lemma 6 deduce $Ph_+ = h_+$ a.e. The latter implies the subregions of leaves (levels) where h_+ is zero don't communicate via posers of P with their complements. This is absurd and the theorem is proved.

NOTE. This method of proving measurable functions harmonic along the leaves of a foliation are constant was borrowed from Lucy Garnett's thesis

[G] which contains a very simple proof of the ergodicity of the geodesic flow in the finite volume case.

Section III. *Conical approach and E. Hopf's theorem*[1]

Say that Γ has *conical approach* at a point s on the sphere at infinity in hyperbolic space if an orbit of Γ has infinitely many points in a circular cone with vertex at s. E. Hopf proved in 1939 that Γ acts ergodically on the (sphere × sphere) if and only if Γ has conical approach at almost all points of the sphere. We study conical approach and derive E. Hopf's theorem (cf. [BM]).

LEMMA 1. *If Γ has conical approach to a density point of a Γ-invariant set on the sphere, this set has full measure.*

Proof. We integrate the characteristic function χ_A of the set A against the Poisson measure of p inside hyperbolic space. Choose a sequence of group elements γ_i so that $\gamma_i p$ has conical approach to a density point of A.

$$\text{area } A = \mu_p A = \int \chi_A d\mu_p = \int \gamma_i \chi_A d\mu_p = \int \chi_A d\mu_{\gamma_i p} .$$

The easy classical estimate shows the right hand side approaches 1 since a is a density point of A.

COROLLARY. *If Γ has conical approach to a set of positive measure on the sphere then Γ has conical approach to a set of full measure on the sphere.*

Proof. Apply Lemma 1 to the set where Γ has conical approach.

LEMMA 2. *If Γ only has conical approach to a set of measure zero, the action of Γ on $S^2 \times S^2$ is dissipative (i.e. it has a fundamental domain a.e.).*

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[1] This section works on S^n, $n \geq 1$, without change.

Proof. To each point q of the orbit of p under Γ associate A_q, those pairs of points on the sphere so that the connecting geodesic is closer to q than to any other point of the orbit. For almost all pairs there is a q because given a pair we merely swell up a geodesic tubular nghd of the connecting geodesic until it first meets the orbit. There is a first time because of the no conical approach assumption. The point q is unique after we throw out from the pairs countably many lower dimensional submanifolds. The A_q provide the desired partition of almost all the pairs into sets permuted freely by Γ.

COROLLARY. *If Γ has infinite solid angle then Γ has conical approach at a set of full measure.*[2]

Proof. Infinite solid angle implies the action on pairs is ergodic (Theorem II) which would be contrary to the conclusion of Lemma 2 if Γ did not have full conical approach.

LEMMA 3. *If Γ has conical approach at a set of positive measure on the sphere then Γ has infinite solid angle.*

Proof. The best angle of approach is a positive measurable function so is at least α for a set of positive measure which by Lemma 1 can be assumed to be the entire sphere.

α determines the size of a ball B appropriate for what follows. If the total solid angle of the orbit of B were finite, we could cast out finitely many balls so the remaining solid angle would be arbitrarily small. But the α conical approach implies the balls near infinity block infinity from view.

COROLLARY (E. Hopf). *A group Γ has conical approach either at a set of measure zero or at a set of full measure. In the first case the action on pairs of points on the sphere is dissipative. In the second the action on pairs is ergodic.*

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[2] Proven by Tsuji (1944) for the case of Fuchsian groups using complex function theory.

HISTORICAL REMARK:

Theorem II (for hyperbolic 2-space) has an interesting history in function theory and the theory of Fuchsian groups. There were the implications

$$\begin{tikzcd}
\text{infinite angle} \arrow[r, "\text{Tsuji (1944)}"] & \text{conical approach} \arrow[d, "\text{E. Hopf (1939)}"] \\
& \text{geodesics ergodic} \arrow[ul, "\text{Tsuji (1945)}"]
\end{tikzcd}$$

However, the implication

$$\text{infinite angle} \longrightarrow \text{geodesics ergodic}$$

was called *Tsuji's problem,* Shimada (1960). Tsuji apparently only knew the earlier Hopf theorem

$$\underset{\text{E. Hopf (1937)}}{\text{finite area} \longrightarrow \text{geodesics ergodic}}$$

which he reproved using potential theory rather than Birkhoff's ergodic theorem. Hopf's stronger (1939) theorem became more accessible after his Gibbs Lecture on the topic, AMS Bull 1971. P. J. Nicholls connected up Hopf (1939) and Tsuji (1944), corrected by Yujobo (1949), and Tsuji (1951), in a 1976 paper (see $[N_1]$ and p. 531 of Tsuji's book).

REMARK. The action of Γ on pairs is orbit equivalent to the geodesic flow on H/Γ. E. Hopf used the latter model, the associated asymptotic foliations, and his extension of Birkhoff's individual ergodic theorem to the case of an infinite invariant measure for the proof of the 1939 theorem.

For Fuchsian groups the condition infinite angle is just the divergence of the Blaschke product associated to an orbit. For the 1944 result Tsuji used complex function theory in an essential way, in particular the Riemann mapping theorem and the theorem of F. and M. Riesz, (Stockholm 1925).

Recall that our path around the triangle was different,

$$\begin{tikzcd}
\text{infinite angle} \arrow[dr] & \text{conical approach} \arrow[l] \\
 & \text{geodesics ergodic} \arrow[u]
\end{tikzcd}$$

Only the slating arrow using random walks was not obvious.

Perhaps it is worth mentioning a conceptual anatomy of our proof. One rearrangement is

i) There is a general theorem that a discrete group Γ of symmetries of the state space E underlying a symmetric Markov process acts ergodically on pairs of points of the boundary of the process if and only if the induced random walk on E/Γ is not transient [G].

ii) This abstract theorem applies to random motion on an infinite regular covering of any Riemannian manifold complete for random motion.

iii) For a complete constant negatively curved manifold the boundary of the natural random process can be identified because a random path hits a definite point on the sphere at ∞ with probability 1. Also transience can be identified with finite solid angle.

Section IV. *Horospherical approach and recurrence*

Say that Γ has *horospherical approach* to a point s on the sphere at ∞ if the Γ orbit of a point in hyperbolic space enters every horosphere based at s. We call such points of the sphere the "horospherical limit set of Γ." It is geometrically clear the horospherical limit set is independent of the choice of reference orbit. These points were studied by Hedlund and later Eberlein [H], [E].

If s is not a horospherical limit point we can try to swell up horospheres at s until one first hits a point q of a fixed reference orbit. If there is a unique first hit we define this q to be the "closest orbit point to s."

THEOREM III. *For any group Γ and any choice of reference orbit in hyperbolic space the horospherical limit set union the points of the sphere which have a closest orbit point form a set of full measure. Furthermore this dichotomy is precisely the partition of the action of Γ on the sphere into its conservative part and its dissipative part.*

Proof. Assuming the first part for the moment, let us prove the second. To each orbit point q, associate the points A_q of the sphere to which it is closest. The A_q are freely permuted by Γ, so their union is a dissipative set.

On the other hand the horospherical limit set is easily seen to be characterized as those points s at which the derivatives of elements of Γ evaluated at s are unbounded. It follows there is no dissipative part in the horospherical limit set because along almost all dissipative orbits the sum of Jacobians is finite (the union of areas is finite).

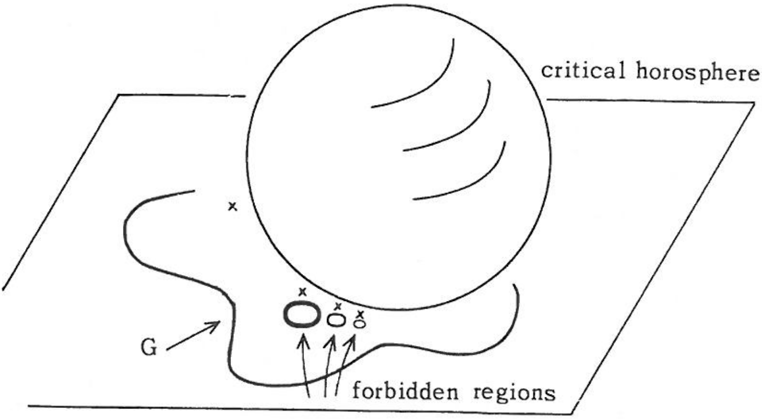
Let us return to the first part. Throw out countably many lower dimensional submanifolds of the sphere which are bases of horospheres containing more than one point of the reference orbit. Now a point ℓ on the sphere not in the horospherical limit set fails to have a unique closest orbit point only if there is a critical horosphere $h(\ell)$ so that every larger horosphere based at ℓ contains infinitely many orbit points. We call such points "Garnett points."[1] The proof of the theorem is completed by the following lemma.

LEMMA 1. *The Garnett points have measure zero.*

Proof. Let s be a point of density in a set G of Garnett points of positive measure where the radius of the critical horosphere is a continuous function. Now there are infinitely many orbit points x_i outside the critical horosphere converging to s but entering every larger horosphere. For

---

[1] Lucy Garnett pointed out that these points have to be considered. Later John Garnett independently found a simple proof they have measure zero. In [N_2] this possibility is overlooked (line 5, p. 310) invalidating the proof of Theorem 1, i) $\Longrightarrow$ ii), there.



each x_i there is an associated forbidden region on the sphere at ∞ which does not contain Garnett points of G. For a point of G is the base of a horosphere whose radius is approximately that of $h(s)$ by continuity and *which contains no orbit points*.

One calculates easily the forbidden region associated to x_i is a definite proportion of the area of the smallest disk centered at s containing the vertical projection of x_i. But then this picture contradicts the fact that s is a Lebesgue density point of G. Q.E.D.

Associated to the reference orbit is the partition of hyperbolic space into convex fundamental domains. To each orbit point q one assigns all the points D_q closer to it than to any other orbit point. The above discussion has extended this notion of closest point to the sphere at ∞. One has the

COROLLARY. *The dissipative part of the action of the discrete group* Γ *on the sphere is just the union of the "fundamental domains intersect the sphere at* ∞.*"* ($s \in D_q \cap S$ *if there is a geodesic in* D_q *hitting* s.)

COROLLARY. *The action of* Γ *on the sphere is conservative if the area of one fundamental domain intersect the sphere at* ∞ *is zero.*

REMARK. The condition of the last corollary may be reformulated in terms of the growth of volume of the quotient manifold $V = H/\Gamma$. Let $V(r)$

denote the volume of the points of V within a distance r of some base point p. Let $H(r)$ denote the volume of the ball of radius r in hyperbolic space.

THEOREM IV. *The following are equivalent*

i) *The fundamental domain has zero area at* ∞.

ii) *The ratio* $V(r)/H(r) \to 0$ *as* $r \to \infty$.

iii) *The action of* Γ *on the sphere at* ∞ *is conservative.*

iv) *The horospherical limit set of* Γ *has full measure on the sphere.*

Proof. Only i) $\Longrightarrow$ ii) is not obvious or already proven. Consider the solid angle $\omega(r)$ of the sphere of radius r centered at q within the fundamental domain D_q. By convexity of D_q $\omega(r)$ is decreasing. By i) the limit at $r = \infty$ is zero. If $a(r)$ denotes the area of the sphere of radius r we have

$$V(r)/H(r) = \frac{\displaystyle\int_0^r a(r)\,\omega(r)\,dr}{\displaystyle\int_0^r a(r)\,dr}\,.$$

Since $\omega(r) < \varepsilon$ for $r > r_0$ we have

$$V(r)/H(r) \leq \frac{\displaystyle\int_0^{r_0} a(r)\,\omega(r)\,dr}{\displaystyle\int_0^r a(r)\,dr} + \varepsilon\,\frac{\displaystyle\int_{r_0}^r a(r)\,dr}{\displaystyle\int_0^r a(r)\,dr}\,.$$

The first term is zero in the limit of $r \to \infty$ and the second is at most ε.

NOTE. In the 1939 treatise E. Hopf asked whether condition ii) or others like it might imply the ergodicity of the geodesic flow. (Equivalently the action of Γ on $S \times S$). In the fifties a Fuchsian group Γ was constructed[2] (after a long search) whose Riemann surface carried a Green's function but no bounded harmonic function. Thus the action of this Γ on $S^1 \times S^1$ is not ergodic but the action of Γ on S^1 is ergodic. In particular condition ii) is satisfied but there is no sectorial approach. By Theorem IV there is horocycle approach almost everywhere.

Section V. *Ahlfors-Bers quasi-conformal deformations of discrete subgroups of* $PS\ell(2, \mathbb{C})$

If Γ is a discrete group of isometries of hyperbolic space $\mathbb{H}^3$, one knows how to deform Γ in a nice way using a measurable line field invariant by Γ and a positive bounded measurable function invariant by Γ. By Ahlfors-Bers [AB] there is a quasi-conformal homeomorphism ϕ of the sphere whose conformal distortion lies a.e. in the direction of the line field with strength given by the function.

One may view the measurable data consisting of the line field and function as a new measurable conformal structure invariant by Γ (which is a bounded distance away from the smooth conformal structure). The homeomorphism ϕ converts the measurable structure back to the smooth conformal structure on S^2.

Bers [B] conjugates the conformal transformations of Γ by ϕ to obtain new conformal transformations, $\{\gamma\} \to \{\phi\gamma\phi^{-1}\}$, and a new discrete group Γ'. If Γ and Γ' are also conjugate by a conformal transformation it follows the measurable data we started with vanishes a.e. on the topological limit set (because fixed points are carried to fixed points by any conjugacy, so any topological conjugacy between Γ and Γ' is determined uniquely on the topological limit set). $\Gamma' = \phi\Gamma\phi^{-1}$ is called a quasi-conformal deformation of Γ.

From Section I we find there can only exist invariant line fields on the dissipative part of the action of Γ on S^2.

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[2] See Ahlfors-Sario, p. 256-257.

Now the dissipative part of the action splits into two pieces: the domain of discontinuity of Γ (if any) whose quotient by Γ is called the Riemann surface associated to Γ, and the dissipative part (if any) of the action on the topological limit set.

Any dissipative piece in the limit set will lead to an infinite dimensional space of inequivalent quasi-conformal deformations by the uniqueness of the conjugacy remark above. The domain of discontinuity leads to a Teichmüller space of deformations which can be nicely described if the Riemann surface of Γ is obtained from a compact by removing at most finitely many points, [B] and [M].

For *finitely generated* groups Γ one has Ahlfor's result [A] that the associated Riemann surface is obtained from a compact surface by removing at most finitely many points. One also knows the embeddings of the abstract group into $PS\ell(2,\mathbb{C})$ is a finite dimensional space.

Thus the infinite dimensional linear space is not present for finitely generated groups and the Teichmüller space in question is a nice complex manifold about which much is known, [B].

THEOREM V. *For a finitely generated discrete subgroup* Γ *of* $PS\ell(2,\mathbb{C})$ *the classes of quasi-conformal deformations consists of a Teichmüller space associated to the uniformized Riemann surface of finite type. For example, if* Γ *has a dense orbit on* S^2, *then* Γ *is quasi-conformally rigid.*

Proof. One only needs to read [B] and [M] and forget their hypothesis that the deformation data be zero on the limit set. For it must be by the uniqueness theorem and the above deformation remark.

Section VI. *Dynamical properties of finitely generated Kleinian groups*

Let us collect here the information about finitely generated groups following from the foregoing general discussion. The main point of the finite generation of Γ for us is the deformation remark of Theorem V which is mostly due to Ahlfors.

THEOREM VI. *For a finitely generated discrete group of isometries of* H^3, *the horospherical limit set has full measure in the topological limit set.*

Proof. The deformation remark of Section V shows the topological limit set contains no dissipative piece.[1] Thus by Theorem III, Section IV the horospherical set has full measure in the topological limit set.

We record the proof as a

COROLLARY. *The action of a finitely generated* Γ *on the topological limit set is conservative.*[1]

COROLLARY. *There is no measurable field of regular tangent* n-crosses *defined on a positive area subset of the limit set and invariant by* Γ.

Proof. We have proved this result for tangent line fields (= tangent 2 - cross) in Section I. The same proof works for n-gons.

REMARK. A formal ergodic consequence of the previous corollary is that the "angular ratio set" is the entire circle. Namely given θ on the circle, a positive number ε, and a subset A of the limit set of positive measure there is an element $\gamma \in \Gamma$ so the subset of $A \cap \gamma^{-1}A$ where the angular part of the derivative of γ is within ε of θ has positive measure. In other words one sees every twist (θ) up to any approximation (ε) anywhere one looks (A).

COROLLARY. *There is no absolutely continuous measure on the limit set invariant by* Γ.

Proof. If ρ were the density function of such a measure, work in a subset where ρ is approximately constant using the variation of distortion

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[1] Ahlfors (1967) gave a "Cauchy transform" proof of this fact when there was a non-trivial domain of discontinuity. See "Remarks on Kleinian Groups" Tulane Conference, 1967.

produced by Lemma 3 of Section I to derive a contradiction, as in the last part of Section I.

NOTE. The density function ρ is not assumed to be locally integrable. For it is elementary that for an arbitrary group Γ there is no invariant probability measure. The corollary means for finitely generated groups there is never a σ-finite invariant measure in the smooth measure class. This fact means that the measurable equivalence relation induced by Γ on the limit set is type III in the sense of Murray and Von Neumann. One can hope to classify the equivalence relation up to Lebesgue isomorphism by a further study of the derivatives (going beyond Lemma 3 of Section I).[2] For example when are there non-trivial elements in the "ratio set", defined as in the previous remark using the area distortion of the derivative rather than the angular part?[2]

Recall that we can pretend to understand the ergodic theory of groups where $\sum_{\Gamma} 1/\lambda^2 = \infty$. If then $\sum_{\Gamma} 1/\lambda^2 < \infty$, what is the significance of the of the function $\sum_{\Gamma} 1/\lambda^s$ for other values of s? For $s = 1$ one has the

COROLLARY. *If the topological limit set has positive area then* $\sum_{\Gamma} 1/\lambda = \infty$.

Proof. The area of the isometric disk of γ union the image of its complement has the order $1/\lambda$ using Lemma 1 of Section I. Using recurrence as in the final argument of Section I after casting out finitely many elements to make $\Sigma\, 1/\lambda$ small we arrive at a contradiction. Namely, $\Sigma\, 1/\lambda < \infty \Rightarrow$ the action of any Γ on S^2 is totally dissipative.

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[2] Actually the addendum to Section I and Lemma 3 implies the ratio set for the area distortion is all the positive reals. Thus any Kleinian group has type III_1 on its conservative part. This theorem will be explained in more detail in a future publication.

Section VII. *Rigidity in the sense of Mostow for infinite volume discrete groups in* O(n, 1)

Say that a complete hyperbolic n-manifold $V = \mathbf{H}^n/\Gamma$ is Mostow-rigid if any pseudo-isometry between V and another hyperbolic manifold V' is homotopic to an isometry.

A pseudo-isometry is by definition a continuous map $V \to V'$ inducing an isomorphism between the fundamental groups and keeping the distances between sufficiently farlying pairs of points in a bounded ratio.

Mostow showed [Mo] that

i) a pseudo-isometry of hyperbolic n-space is onto and has a continuous extension to the sphere at ∞ which is a quasi-conformal homeomorphism;

ii) a quasi-conformal homeomorphism of S^{n-1} $(n \geq 3)$ whose derivative is a.e. a similarity is a conformal transformation;

iii) for discrete groups Γ whose fundamental domains have finite volume there is only one a.e. measurable conformal structure on the sphere invariant by Γ.

MOSTOW'S THEOREM. *Complete hyperbolic* n-*manifolds of finite volume are rigid in the above sense,* $n \geq 3$.

Proof. Mostow lifts the pseudo isometry to hyperbolic space, extends to the boundary by i) to find a quasi-conformal homeomorphism which must be conformal by iii) then ii), and obtains an equivariant isometry of hyperbolic space with these boundary values.

Celebrated corollaries are: compact hyperbolic n-manifolds M even complete finite volume hyperbolic n-manifolds are determined up to isometry $(n \geq 3)$ by their fundamental groups.

Proof. The fundamental group determines the pseudo-isometry type in these cases. This is clear for the compact case and follows from an analysis of the cusps in the finite volume case (Margulis, Prasad).

Using Theorem VI below to extend point iii) we obtain by the same proof an extension of Mostow's theorem to the infinite volume case.

THEOREM VI. *If* V *is a complete hyperbolic* n*-manifold* $(n \geq 3)$ *and satisfies any of the equivalent conditions of Theorem IV (for example if the volume of the part of* V *out to distance* r *grows slower than the ball of radius* r *in hyperbolic space), then* V *is rigid in the sense of Mostow.*

COROLLARY. *If* V^3 *is defined by any discrete finitely generated group of isometries of* $\mathbf{H}^3$ *with a dense orbit on* S^2, *then* V^3 *is rigid in the sense of Mostow.*

Proof of Corollary. Section VI reduces the corollary to Theorem VI.

Proof of Theorem VI. Same as above proof using Theorem VII.

THEOREM VII. *Let* Γ *be a discrete group of conformal transformations of* S^n, $n \geq 2$ *and let* ν *be a measurable conformal structure (on the tangent spaces), which is a.e. invariant by* Γ. *Then* ν *agrees a.e. with the standard conformal structure on the conservative part of action of* Γ *on* S^n.

Proof. By comparing ν with the standard structure we find a field of ellipsoids, each defined up to similarity. We can work on an invariant set of positive measure where the smallest axes form a proper subspace of constant dimension. We find then an invariant tangent k-plane field invariant by Γ, $k < n$.

We ruled out this possibility for $k = 1$, $n = 2$ in Section I by an argument based on the variation of distortion (Lemma 3). We will deduce the general case from this same argument. Also note that Sections I, II, III go word for word the same for general n, using $\sum_{\Gamma} 1/\lambda^n$ instead of $\sum_{\Gamma} 1/\lambda^2$.

Now working in the R^n-model choose a set of positive measure X where the k-plane field $\{P\}$ is within ε of being parallel. Now consider any transformation γ keeping $\{P\}$ invariant. We can factor γ into a composition

$$\gamma = (\text{inversion in circle}) \cdot (\text{Euclidean isometry}) = I_\gamma R_\gamma .$$

Then $\{P'\} = R_\gamma\{P\}$ is another almost parallel field (on $X' = R_\gamma(X)$) which is carried by the inversion I_γ to $\{P\}$, $I_\gamma\{P'\} = \{P\}$.

Now the tangent maps to an inversion are just the reflections in the tangent planes to spheres concentric about the center of inversion. Near to the center of inversion these tangent planes turn rapidly and it is difficult for the inversion to carry an almost parallel field on X' to an almost parallel field on X (if X' has many points sufficiently close to the center which are also carried to X).

Now if $\gamma \in \Gamma$ we can determine where X' is relative to the center of inversion of I_γ by knowing the distortion of γ on X' (since R_γ is an isometry, and in polar coordinates based at the center the distortion of the inversion $(r, \theta) \to (1/\lambda r, \theta)$ is $1/\lambda r^2$). What we will know from conservativity and the argument of Section I is that for any $Y \subset X$ of positive measure there is a $\gamma \in \Gamma$ so that $\gamma^{-1}Y \cap Y$ has a point y, log(planar distortion γ) varies in an interval of length $\sim \Delta$ over an $\sim$ concentric annular shell A containing y (being constant on $\sim$ concentric spheres), A and γA have diameter $\leq \delta'$, and "$\sim$", Δ, and δ' are at our disposal.

Writing $A' = R_\gamma A$ we see from these values of distortion that A' is a nearly concentric annular shell about the center of inversion of I_γ of diameter δ'. Since the shape of A is determined by Δ and "$\sim$" we can use such sets to describe (η, δ) uniform density points of X as in Lemma 3, Section I.

We choose η so close to 1 that a contradiction results from the following chain of considerations. Y is chosen, using Lemma 3, Section I for (X, η, δ). Then γ is chosen as above for Δ, δ', Y with $\delta' < \delta$. A will be almost filled with points of X which map to X by γ (as in the argument of Section I). So the same holds for I_γ, namely A' is almost filled with points of X' which map to X (note R_γ alters nothing being an isometry).

The above description of the tangent map forces a contradiction, I_γ cannot carry the ε-almost parallel filled on X' to the ε-almost parallel field on X.

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