ON LIFTING AUTOMORPHISMS OF MONOTONE σ-COMPLETE C^*-ALGEBRAS

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Introduction

LET $\mathscr{S}$ be the σ-algebra of all Borel subsets of the unit interval. When $\mathscr{N}$ is the σ-ideal of Borel sets of Lebesgue measure zero, it follows from the work of von Neumann [**4**], that each automorphism of $\mathscr{S}/\mathscr{N}$ is induced by a Borel bijection of $[0, 1]$. Answering a question posed by Kakutani, it was shown, recently, by Maharam and Stone [**3**], that when $\mathscr{M}$ is the σ-ideal of meagre Borel subsets of $[0, 1]$ then, given an automorphism β of $\mathscr{S}/\mathscr{M}$, there can be found a dense G_δ-set $X \subset [0, 1]$ such that β is induced by a homeomorphism of X. Their result is more general than this since they replace the unit interval by an arbitrary complete metric space.

In this article we shall be mainly concerned with non-commutative C^*-algebras. When specialized to commutative algebras our results give an alternative proof of the Maharam–Stone Theorem, but only for *separable* complete metric spaces.

Let $\mathscr{A}$ be a separable C^*-algebra with Borel* envelope $\mathscr{A}^\infty$ (see below for definitions). Let $\mathscr{J}$ be a (two-sided) σ-ideal of $\mathscr{A}^\infty$ and, for each $x \in \mathscr{A}^\infty$, let $[x]$ denote the image of x under the quotient map from $\mathscr{A}^\infty$ onto $\mathscr{A}^\infty/\mathscr{J}$. Our main theorem states that, given any automorphism β of $\mathscr{A}^\infty/\mathscr{J}$, we can find an automorphism α of $\mathscr{A}^\infty$ such that

$$\beta([x]) = [\alpha(x)]$$

for all $x \in \mathscr{A}^\infty$.

As an application of this theorem, we show that if $\mathscr{A}_1$ and $\mathscr{A}_2$ are separable C^*-algebras and $\mathscr{J}_1$ and $\mathscr{J}_2$ are σ-ideals of $\mathscr{A}_1^\infty$ and $\mathscr{A}_2^\infty$, respectively, such that $\mathscr{A}_1^\infty/\mathscr{J}_1$ is isomorphic to $\mathscr{A}_2^\infty/\mathscr{J}_2$ then there exist central projections p_1, p_2 in $\mathscr{A}_1^\infty$ and $\mathscr{A}_2^\infty$, respectively, such that $1 - p_1 \in \mathscr{J}_1$, $1 - p_2 \in \mathscr{J}_2$ and $p_1\mathscr{A}_1^\infty$ is isomorphic to $p_2\mathscr{A}_2^\infty$.

We shall also show that every countable group of automorphisms of $\mathscr{A}^\infty/\mathscr{J}$ can be lifted to an isomorphic group of automorphisms of $\mathscr{A}^\infty$.

§1. Preliminaries

Unless we state the contrary, capital script letters $\mathscr{A}$, $\mathscr{B}$ will denote C^*-algebras and capital roman letters A, B their, respective, self-adjoint parts.

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For each C^*-algebra $\mathcal{A}$ we shall use $\mathcal{A}^{**}$ to denote its second dual. We shall identify $\mathcal{A}$ with its image under the universal representation, that is, as an algebra of operators on its universal representation space; we recall that $\mathcal{A}^{**}$ may then be identified with the von Neumann envelope of $\mathcal{A}$. Let A^∞ be the smallest subset of A^{**}, (the self-adjoint part of $\mathcal{A}^{**}$) which contains A and is such that whenever (x_n) $(n=1,2,\ldots)$ is a monotonic sequence in A^∞ with strong limit x, in A^{**}, then $x \in A^\infty$. Let $\mathcal{A}^\infty = A^\infty + iA^\infty$, then by **[5]**, $\mathcal{A}^\infty$ is a C^*-subalgebra of $\mathcal{A}^{**}$. Following Pedersen, we call $\mathcal{A}^\infty$ the *Borel*-envelope* of $\mathcal{A}$. We remark that whenever $\mathcal{A}$ is separable then $\mathcal{A}^\infty$ is unital, because $\mathcal{A}$ has a sequential approximate identity.

A C^*-algebra $\mathcal{B}$ is said to be *monotone σ-complete* if, whenever (b_n) $(n=1,2,\ldots)$ is a norm-bounded, monotone increasing sequence in B, then this sequence has a least upper bound in B. Let $\mathcal{J}$ be a closed (two-sided) ideal of a monotone σ-complete C^*-algebra $\mathcal{B}$; $\mathcal{J}$ is said to be a *σ-ideal* if, whenever (b_n) $(n=1,2,\ldots)$ is a monotone increasing sequence in $\mathcal{J} \cap B$, which has supremum b in B then $b \in \mathcal{J}$.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be monotone σ-complete C^*-algebras. An homomorphism $\phi\colon \mathcal{B}_1 \to \mathcal{B}_2$ is said to be a *σ-homomorphism* if, whenever (b_n) $(n=1,2,\ldots)$ is a monotone increasing sequence in B_1, with supremum b, then $\phi(b)$ is the supremum of $(\phi(b_n))$ $(n=1,2,\ldots)$ in B_2. Clearly the kernel of a σ-homomorphism is a σ-ideal, conversely, see Lemma 2.13 **[1]**, the quotient of a monotone σ-complete C^*-algebra by a σ-ideal is always monotone σ-complete and the quotient homomorphism is a σ-homomorphism.

We remark that whenever $\mathcal{B}$ is monotone σ-complete then there exists a σ-ideal $\mathcal{M}$ in $\mathcal{B}^\infty$ such that $\mathcal{B}$ is isomorphic to $\mathcal{B}^\infty/\mathcal{M}$ **[13]**.

We shall lean heavily on the methods and results of Pedersen **[6, 7]** on Borel*-algebras. A particularly lucid account of their theory is given in **[9]**. In particular, in the special case where $\mathcal{B}$ is a von Neumann algebra, Theorem 2.2 already follows from the results of Pedersen **[6]**. We remark that whenever $\mathcal{A}$ is simple, unital, separable and infinite-dimensional then there exists a σ-ideal $\mathcal{M}$ in $\mathcal{A}^\infty$ such that $\mathcal{A}^\infty/\mathcal{M}$ is a monotone complete AW^*-factor which is never a von Neumann algebra **[11]**.

Let $\mathcal{A}$ be an arbitrary C^*-algebra. Let $Z(\mathcal{A}^{**})$ be the self-adjoint part of the centre of $\mathcal{A}^{**}$. For each $x \in \mathcal{A}^{**}$, the *central cover* of x is defined to be the infimum, in $Z(\mathcal{A}^{**})$, of $\{a \in Z(\mathcal{A}^{**})\colon\ a \geq x\}$. Since the self-adjoint part of a commutative von Neumann algebra is a conditionally complete lattice, $c(x)$ is well-defined. When p is a projection then $c(p)$ is also a projection and is the supremum of all projections in the set $\{u^*pu\colon u$ is a unitary in $\mathcal{A}^{**}\}$.

LEMMA 1.1. (Pedersen). *Let $\mathcal{A}$ be a separable C^*-algebra. Let $\mathcal{J}$ be a σ-ideal in $\mathcal{A}^\infty$ and let x be a positive self-adjoint element of $\mathcal{J}$. Then $c(x) \in \mathcal{J}$.*

The proof of Proposition 4.5.8 **[9]**, shows that $c(x)$ is the strong limit of a monotone increasing sequence of elements of $\mathcal{A}^\infty$. When $x \in \mathcal{I}$ then the terms of this sequence are in $\mathcal{I}$.

LEMMA 1.2. *Let $\mathcal{A}$ be any C^*-algebra and let $\mathcal{B}$ be a monotone σ-complete C^*-algebra. Let ϕ: $\mathcal{A} \to \mathcal{B}$ be an homomorphism. Then ϕ has a unique extension to a σ-homomorphism $\tilde{\phi}$: $\mathcal{A}^\infty \to \mathcal{B}$.*

This is proved in Proposition 1.1 **[12]** by combining Proposition 4.2 **[8]** with the main result of **[13]**.

The following lemma can be proved in much greater generality but this simple result is all we shall need. Since we do not know of a convenient reference we sketch a proof.

LEMMA 1.3. *Let $\mathcal{S}$ be the σ-field of all Borel subsets of a Polish space X. Let α be a Boolean automorphism of $\mathcal{S}$. Then there exists a Borel measurable bijection θ from X onto X which induces α, that is, such that*

$$\alpha(A) = \theta^{-1}[A]$$

for all $A \in \mathcal{S}$.

Let S be the Stone structure space of $\mathcal{S}$. Since $\mathcal{S}$ is atomic and separates the points of X, we can regard X as densely embedded in S and can identify X with the set of all isolated points of S. Thus any homeomorphism of S maps X onto X.

Let Φ be the homeomorphism of S which corresponds to the automorphism α. Let θ be the restriction of Φ to X. Then θ is a bijection from X onto X and it is straightforward to show that

$$\alpha(A) = \theta^{-1}[A]$$

for each $A \in \mathcal{S}$. Clearly θ is Borel measurable.

§2. Lifting automorphisms

Let $\mathcal{A}^\infty$ be the Borel*-envelope of a C^*-algebra $\mathcal{A}$. A function γ: $\mathcal{A}^\infty \to \mathcal{A}^\infty$ is said to be a *σ-normal map* if γ is a linear map such that, whenever (x_n) $(n = 1, 2, \ldots)$ is a monotone increasing sequence in A^∞, with strong limit x, then $\gamma(x)$ is the strong limit of $(\gamma(x_n))$ $(n = 1, 2, \ldots)$.

LEMMA 2.1. *Let $\mathcal{A}$ be a separable C^*-algebra with Borel*-envelope $\mathcal{A}^\infty$. Let $\mathcal{I}$ be a (two-sided) proper, σ-ideal in $\mathcal{A}^\infty$. For each $x \in \mathcal{A}^\infty$, let $[x]$ be the quotient-class containing x. Let γ: $\mathcal{A}^\infty \to \mathcal{A}^\infty$ and δ: $\mathcal{A}^\infty \to \mathcal{A}^\infty$ be positive σ-normal maps such that*

$$[\gamma(x)] = [\delta(x)]$$

for every $x \in \mathscr{A}^\infty$. Then there exists a central projection d in $\mathscr{A}^\infty$ such that $1-d \in \mathscr{J}$ and

$$d\gamma(x) = d\delta(x)$$

for each $x \in \mathscr{A}^\infty$.

For notational convenience we shall assume that $\|\gamma\| \leq 1$ and $\|\delta\| \leq 1$.

Let (x_n) $(n = 1, 2, \ldots)$ be dense in the unit ball of A. For each n, let $y_n = \frac{1}{2}\{\gamma(x_n) - \delta(x_n)\}$. Then $y_n^2 \in \mathscr{J}$ and $\|y_n^2\| \leq 1$. Thus $((y_n^2)^{1/k})$ $(k = 1, 2, \ldots)$ is a monotone increasing sequence in $\mathscr{J}$ whose supremum, in $\mathscr{A}^\infty$, is a projection q_n. Then $q_n \in \mathscr{J}$ and so, by Lemma 1.1, $c(q_n) \in \mathscr{J}$.

Let $1 - d = \bigvee_{n=1}^{\infty} c(q_n)$. Then d is a central projection in $\mathscr{A}^\infty$ and $1-d \in \mathscr{J}$. Moreover, $dy_n^2 = 0$ for each n. So $d\gamma(x_n) = d\delta(x_n)$ for each n. Hence $d\gamma(x) = d\delta(x)$ for all $x \in A$.

Since A^∞ is σ-generated by A and since γ and δ are σ-normal it follows that $d\gamma(x) = d\delta(x)$ for all $x \in A^\infty$ and hence all $x \in \mathscr{A}^\infty$.

We now come to our main theorem. We have already remarked that when $\mathscr{A}$ is a separable C^*-algebra then $\mathscr{A}^\infty$ is unital. Hence, whenever $\mathscr{J}$ is a proper ideal of $\mathscr{A}^\infty$ then $\mathscr{A}^\infty/\mathscr{J}$ is also unital. Thus if $\mathscr{B}$ is a surjective image of $\mathscr{A}^\infty$ then $\mathscr{B}$ is necessarily unital.

THEOREM 2.2. *Let $\mathscr{B}$ be a monotone σ-complete C^*-algebra and let β be an automorphism of $\mathscr{B}$. Let $\mathscr{A}$ be a unital separable C^*-algebra with Borel*-envelope $\mathscr{A}^\infty$. Let π be a surjective σ-homomorphism from $\mathscr{A}^\infty$ onto $\mathscr{B}$. Then there exists an automorphism α of $\mathscr{A}^\infty$ such that*

$$\beta\pi = \pi\alpha.$$

In other words, the automorphism β of $\mathscr{B}$ can be lifted to an automorphism α of $\mathscr{A}^\infty$.

Furthermore, if $\gamma: \mathscr{A}^\infty \to \mathscr{A}^\infty$ is a positive σ-normal map such that $\pi\gamma = \beta\pi$ and p is a central projection of $\mathscr{A}^\infty$ for which $\pi(1-p) = 0$ then there exists a central projection q in $\mathscr{A}^\infty$, with $q \leq p$ and $\pi(1-q) = 0$, such that the restriction of α to $q\mathscr{A}^\infty$ is an automorphism of $q\mathscr{A}^\infty$ and $\alpha(x) = q\gamma(x)$ for all $x \in q\mathscr{A}^\infty$.

Since $\mathscr{A}$ is separable, $\beta\pi[\mathscr{A}]$ is also separable. So we can find $\mathscr{C}$, a separable unital C^*-subalgebra of $\mathscr{A}^\infty$, such that $\pi[\mathscr{C}] \supset \beta\pi[\mathscr{A}]$.

Let $\mathscr{J}$ be the kernel of π, so that $\mathscr{J}$ is a σ-ideal of $\mathscr{A}^\infty$. Because $\mathscr{C} \cap \mathscr{J}$ is a separable C^*-algebra, there exists a monotone increasing sequence (f_n) $(n = 1, 2, \ldots)$, in the positive part of the unit ball of $\mathscr{C} \cap \mathscr{J}$, which is an approximate identity for $\mathscr{C} \cap \mathscr{J}$. Let f be the strong limit of this sequence in $\mathscr{A}^\infty$. Then f is a projection such that

$$fx = x = xf \tag{1}$$

for all $x \in \mathscr{C} \cap \mathscr{J}$. Moreover, since $\mathscr{J}$ is a σ-ideal, $f \in \mathscr{J}$.

Since f need not be an element of $\mathscr{C}$, $(1-f)\mathscr{C}(1-f)$ need not be a subalgebra of $\mathscr{A}^\infty$. Let $e = c(f)$, the central cover of f in $\mathscr{A}^\infty$. Thus e is a central projection and, by Lemma 1.1, $e \in \mathscr{I}$. Clearly $(1-e)\mathscr{C}$ is a C^*-subalgebra of $\mathscr{A}^\infty$.

Let $y \in (1-e)\mathscr{C}$ such that $\pi(y) = 0$. Then $y = (1-e)x$, for some $x \in \mathscr{C}$. Since $e \in \mathscr{I}$ we have,

$$\pi(x) = \pi(ex) + \pi((1-e)x) = \pi(e)\pi(x) + \pi(y) = 0.$$

So $x \in \mathscr{C} \cap \mathscr{I}$ and thus, by (1),

$$0 = \|(1-f)x\|^2 = \|x^*(1-f)x\| \geq \|x^*(1-e)x\| = \|(1-e)x\|^2 = \|y\|^2$$

Hence the restriction of π to $(1-e)\mathscr{C}$ is an injective homomorphism into $\mathscr{A}^\infty$. Thus $(1-e)\mathscr{C}$ is isometrically and algebraically *-isomorphic to

$$\pi[(1-e)\mathscr{C}] = \pi[\mathscr{C}] \supset \beta\pi[\mathscr{A}]$$

So there exists an homomorphism ε: $\beta\pi[\mathscr{A}] \to (1-e)\mathscr{C}$ such that, for every $a \in \mathscr{A}$,

$$\pi\varepsilon(\beta\pi(a)) = \beta\pi(a).$$

Let ϕ_0: $\mathscr{A} \to \mathscr{A}^\infty$ be the homomorphism defined on $\mathscr{A}$ by

$$\phi_0(a) = \varepsilon\beta\pi(a) + ea$$

By Lemma 1.2, ϕ_0 has a unique extension to a σ-homomorphism ϕ: $\mathscr{A}^\infty \to \mathscr{A}^\infty$.

Let $V = \{a \in A^\infty$: $\pi\phi(a) = \beta\pi(a)\}$. Since ϕ is an extension of ϕ_0, we have that $A \subset V$. Since π, ϕ and β are all σ-homomorphisms, we have $V = A^\infty$. Thus $\pi\phi = \beta\pi$.

Similarly, we can find a σ-homomorphism ψ: $\mathscr{A}^\infty \to \mathscr{A}^\infty$ such that

$$\pi\psi = \beta^{-1}\pi.$$

So

$$\pi\phi\psi = \beta\pi\psi = \beta\beta^{-1}\pi = \pi$$

and

$$\pi\psi\phi = \beta^{-1}\pi\phi = \beta^{-1}\beta\pi = \pi.$$

By Lemma 2.1, there must exist central projections d_0 and d_1 in $\mathscr{A}^\infty$ such that $\pi(d_0) = \pi(d_1) = 0$ and, for all $x \in \mathscr{A}^\infty$,

$$(1-d_0)\phi\psi(x) = (1-d_0)x$$

and

$$(1-d_1)\psi\phi(x) = (1-d_1)x.$$

We define a sequence of central projections in $\mathscr{A}^\infty$, recursively, as follows. Let $g_1 = d_0 \vee d_1$.

$$g_{n+1} = g_n \vee c(\phi(g_n)) \vee c(\psi(g_n)).$$

Clearly, $g_1 \in \mathcal{I}$. Suppose that $g_n \in \mathcal{I}$. Thus

$$0 = \pi(g_n) = \beta\pi(g_n) = \pi\phi(g_n).$$

So $\phi(g_n) \in \mathcal{I}$ and hence $c(\phi(g_n)) \in \mathcal{I}$. Similarly, $c(\psi(g_n)) \in \mathcal{I}$. So $g_{n+1} \in \mathcal{I}$.

It follows by induction that (g_n) $(n = 1, 2, \ldots)$ is a monotone increasing sequence (of central projections) in $\mathcal{I}$. Let g be the strong limit of this sequence. Then g is a central projection of $\mathcal{A}^\infty$ and $g \in \mathcal{I}$. Furthermore,

$$(1-g)\phi\psi(x) = (1-g)\psi\phi(x) = (1-g)x$$

for all $x \in \mathcal{A}^\infty$.

We have

$$g \geq g_{n+1} \geq \phi(g_n).$$

Since ϕ is a σ-homomorphism, this implies that

$$g \geq \phi(g).$$

Similarly

$$g \geq \psi(g).$$

Thus

$$(1-g)\phi(1-g) = 1-g = (1-g)\psi(1-g).$$

We define α and α' on $\mathcal{A}^\infty$ by

$$\alpha(x) = gx + (1-g)\phi((1-g)x) = gx + (1-g)\phi(x)$$
$$\alpha'(x) = gx + (1-g)\psi(x).$$

For each $x \in \mathcal{A}^\infty$,

$$\begin{aligned}\pi\alpha(x) &= \pi(g)\pi(x) + \pi(1-g)\pi\phi(x)\\ &= \pi\phi(x)\\ &= \beta\pi(x)\end{aligned}$$

Thus

$$\pi\alpha = \beta\pi.$$

Straightforward calculations show that α is a homomorphism from $\mathcal{A}^\infty$ to $\mathcal{A}^\infty$ with $\alpha(1) = 1$ and that $\alpha\alpha'(x) = x = \alpha'\alpha(x)$ for all $x \in \mathcal{A}^\infty$. Thus α is an automorphism of $\mathcal{A}^\infty$.

We now show that α is "almost" unique. By hypothesis $\gamma\colon \mathcal{A}^\infty \to \mathcal{A}^\infty$ is such that $\pi\gamma = \beta\pi = \pi\alpha$. We are given that p is a central projection in $\mathcal{A}^\infty$

such that $\pi(1-p)=0$. By Lemma 2.1, there is a central projection $d \in \mathscr{A}^\infty$, such that $\pi(d)=0$ and

$$(1-d)\gamma(x)=(1-d)\alpha(x) \quad \text{for all} \quad x \in \mathscr{A}^\infty.$$

Since we could replace d by $d \vee (1-p)$, we can suppose, without loss of generality, that $d \geq 1-p$.

The identity $\pi\alpha=\beta\pi$ implies that α maps $\mathscr{I}$ onto $\mathscr{I}$. Hence $\alpha^k(d) \in \mathscr{I}$ for $k=0, \pm 1, \pm 2, \ldots$.

Also, since α is an automorphism of $\mathscr{A}^\infty$, $\alpha^k(d)$ is a central projection of $\mathscr{A}^\infty$ for each k. Let e be the central projection in $\mathscr{A}^\infty$ which is the supremum of all the projections $\{\alpha^k(d)\colon k=0, \pm 1, \pm 2 \ldots\}$. Then, because $\mathscr{I}$ is a σ-ideal, $e \in \mathscr{I}$.

For each j, $\alpha^j(e)$ is an upper bound of $\{\alpha^{k+j}(d)\colon k=0, \pm 1, \pm 2, \ldots\}$. So $\alpha(e) \geq e$ and $\alpha^{-1}(e) \geq e$. Hence $\alpha(e)=e$.

Let $q=1-e$. Then $q=1-e \leq 1-d$. So $q\gamma(x)=q\alpha(x)=\alpha(x)$ for all $x \in q\mathscr{A}^\infty$.

COROLLARY 2.3. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be separable unital C^*-algebras with Borel*-envelopes $\mathscr{A}_1^\infty$ and $\mathscr{A}_2^\infty$, respectively. Let $\mathscr{B}$ be a monotone σ-complete C^*-algebra and, for $j=1, 2$, let π_j be a surjective σ-homomorphism from $\mathscr{A}_j^\infty$ onto $\mathscr{B}$. Then there exist central projections q_1 and q_2 in $\mathscr{A}_1^\infty$ and $\mathscr{A}_2^\infty$, respectively, for which $\pi_1(1-q_1)=0$ and $\pi_2(1-q_2)=0$. Furthermore there exists an isomorphism γ from $q_2\mathscr{A}_2^\infty$ onto $q_1\mathscr{A}_1^\infty$ such that,*

$$\pi_1\gamma(y)=\pi_2(y)$$

for each $y \in q_2\mathscr{A}_2^\infty$.

Let $\mathscr{A}$ be the direct sum $\mathscr{A}_1 \oplus \mathscr{A}_2$. Then $\mathscr{A}^\infty$ may be identified with $\mathscr{A}_1^\infty \oplus \mathscr{A}_2^\infty$. Let $\pi=\pi_1 \oplus \pi_2$, so that π is a surjective σ-homomorphism from $\mathscr{A}^\infty$ onto $\mathscr{B} \oplus \mathscr{B}$. Let β be the automorphism of $\mathscr{B} \oplus \mathscr{B}$ which interchanges the first and second coordinates, that is

$$\beta(x, y)=(y, x).$$

By Theorem 2.2, there exists an automorphism α of $\mathscr{A}^\infty$ such that $\pi\alpha=\beta\pi$. For $i=1,2$ and $j=1,2$, there exist homomorphisms $\phi_{i,j}$ from $\mathscr{A}_i^\infty$ into $\mathscr{A}_j^\infty$ such that, for $x \in \mathscr{A}_1^\infty$ and $y \in \mathscr{A}_2^\infty$,

$$\alpha(x \oplus y)=(\phi_{11}(x)+\phi_{21}(y)) \oplus (\phi_{12}(x)+\phi_{22}(y)).$$

So

$$\pi\alpha(x \oplus y)=(\pi_1\phi_{11}(x)+\pi_1\phi_{21}(y)) \oplus (\pi_2\phi_{12}(x)+\pi_2\phi_{22}(y)).$$

The left-hand side of this equation is $\beta\pi(x \oplus y)=\pi_2(y) \oplus \pi_1(x)$. Thus

$$\pi_2=\pi_1\phi_{21} \quad \text{and} \quad \pi_1=\pi_2\phi_{12}.$$

Let $\psi(x\oplus y)=\phi_{21}(y)\oplus\phi_{12}(x)$. Then, by the "almost" uniqueness part of Theorem 2.2, there exists a central projection $q\in\mathcal{A}^\infty$ for which $\pi(1-q)=0$, the restriction of α to $q\mathcal{A}^\infty$ is an automorphism and, for each $x\oplus y\in q\mathcal{A}^\infty$,

$$\alpha(x\oplus y)=q\psi(x\oplus y).$$

There exist central projections q_1 and q_2 in $\mathcal{A}_1^\infty$ and $\mathcal{A}_2^\infty$, respectively, such that $q=q_1\oplus q_2$. Then, for $x\in q_1\mathcal{A}_1^\infty$ and $y\in q_2\mathcal{A}_2^\infty$,

$$\alpha(x\oplus y)=q_1\phi_{21}(y)\oplus q_2\phi_{12}(x).$$

Since α, when restricted to $q\mathcal{A}^\infty$, is an automorphism of $q\mathcal{A}^\infty$ it follows that if we define $\gamma(y)=q_1\phi_{21}(y)$ for all $y\in q_2\mathcal{A}_2^\infty$ then γ is an isomorphism of $q_2\mathcal{A}_2^\infty$ onto $q_1\mathcal{A}_1^\infty$. Moreover, for $y\in q_2\mathcal{A}_2^\infty$,

$$\pi_1\gamma(y)=\pi_1(q_1\phi_{21}(y))=\pi_1\phi_{21}(y)-\pi_1(1-q_1)\pi_1\phi_{21}(y)$$

So

$$\pi_1\gamma(y)=\pi_2(y)\quad\text{for all}\quad y\in q_2\mathcal{A}_2^\infty.$$

The next two corollaries show that, when specialized to commutative algebras, Theorem 2.2 gives some known results for Boolean algebras. The first is due to Sikorski, see Theorem 32.5 **[10]**, and generalizes a result of von Neumann **[4]**.

We recall that a topological space X is said to be a *standard Borel space* if it is Borel isomorphic to a Borel subset of a Polish space, in particular, all Polish spaces are standard Borel spaces.

COROLLARY 2.4. *Let X be a standard Borel space. Let $\mathcal{S}$ be the σ-field of all Borel subsets of X, let $\mathcal{J}$ be a proper σ-ideal of $\mathcal{S}$ and let π be the quotient homomorphism from $\mathcal{S}$ onto $\mathcal{S}/\mathcal{J}$. Let β be an automorphism of $\mathcal{S}/\mathcal{J}$. Then there exists a Borel measurable bijection θ from X onto X such that*

$$\beta(\pi A)=\pi(\theta^{-1}[A])$$

for all Borel sets $A\subset X$.

Suppose X is not countable. Then, by the general theory of standard Borel spaces, X is Borel isomorphic to $[0, 1]$. Hence we may assume that X is a compact separable metric space. Let $\mathcal{A}$ be the commutative, separable C^*-algebra $C(X)$. Then $\mathcal{A}^\infty$ is isomorphic to $B^\infty(X)$, the algebra of all bounded Borel functions on X. Let $\mathcal{I}$ be the set of all f in $B^\infty(X)$ for which the set $\{x\in X\colon f(x)\neq 0\}$ is in $\mathcal{J}$. Then $\mathcal{I}$ is a σ-ideal of $B^\infty(X)$.

It follows by applying Theorem 2.2 that there exists a Boolean automorphism α of $\mathcal{S}$ such that $\pi\alpha(A)=\beta\pi(A)$ for all $A\in\mathcal{J}$. By Lemma 1.3, α is induced by a Borel bijection θ on X.

Now suppose X is countable. We ignore the trivial situation when X is finite and identify X with $\mathbb{N}$, the set of natural numbers. Let $\mathscr{A}$ be the commutative C^*-algebra c_0, the algebra of all complex sequences which converge to 0. Then $\mathscr{A}^\infty$ is isomorphic to l^∞, the algebra of all bounded sequences. On applying Theorem 2.2 to $\mathscr{A}$, the required result follows.

The next corollary shows that the Maharam–Stone Theorem [3] for Polish spaces is an easy consequence of Theorem 2.2. We emphasise that the argument below does not establish the Maharam–Stone Theorem in full generality, since their theorem holds for arbitrary complete metric spaces, not just separable spaces.

COROLLARY 2.5. *Let X be a Polish space. Let $\mathscr{S}$ be the σ-field of all Borel subsets of X, let $\mathscr{M}$ be the σ-ideal of all meagre Borel sets and let π be the quotient homomorphism from $\mathscr{S}$ onto $\mathscr{S}/\mathscr{M}$. Let β be an automorphism of $\mathscr{S}/\mathscr{M}$. Then there exists a dense G_δ-set $Y \subset X$ and a homeomorphism ϕ from Y onto Y such that*

$$\beta\pi(A) = \pi\phi^{-1}(A \cap Y),$$

that is, the homeomorphism ϕ induces the automorphism β.

Let θ: $X \to X$ be the Borel isomorphism whose existence was established in Corollary 2.4. Since X is a Polish space, by applying a theorem of Kuratowski [2; page 400] to θ and θ^{-1}, there exists a co-meagre G_δ-set G_0 such that θ and θ^{-1} are continuous when restricted to G_0. Let us recall that by the Baire Category Theorem, each co-meagre subset of X is dense, and, conversely, each dense G_δ-subset of X is co-meagre.

Since θ is a bijection of X which maps $\mathscr{M}$ onto $\mathscr{M}$, whenever C is a co-meagre subset of X then $\theta[C]$ is a co-meagre set and hence $\theta[C]$ contains a dense G_δ-set. So we can choose a dense G_δ-set $G_1 \subset G_0 \cap \theta[G_0] \cap \theta^{-1}[G_0]$. By repeating this process we can choose a decreasing sequence of dense G_δ-sets (G_n) $(n = 0, 1, 2, \ldots)$ such that

$$G_{n+1} \subset G_n \cap \theta[G_n] \cap \theta^{-1}[G_n].$$

Let $Y = \bigcap_1^\infty G_n$. Then, by the Baire Category Theorem, Y is a dense G_δ-set. Let ϕ be the restriction of θ to Y. Then ϕ is a homeomorphism of Y onto Y with the required properties.

We now return to the non-commutative situation. Let $\mathscr{A}$ be a separable C^*-algebra and $\mathscr{J}$ a proper σ-ideal of $\mathscr{A}^\infty$. If $\alpha \in \operatorname{Aut} \mathscr{A}^\infty$ and $\alpha[\mathscr{J}] = \mathscr{J}$ then α induces an automorphism $\Phi(\alpha)$ of $\mathscr{A}^\infty/\mathscr{J}$ by

$$\Phi(\alpha)([x]) = [\alpha(x)].$$

Let $\operatorname{Aut}_{\mathscr{J}} \mathscr{A}^\infty$ be the set of all $\alpha \in \operatorname{Aut} \mathscr{A}^\infty$ for which $\alpha[\mathscr{J}] = \mathscr{J}$. Then, by

Theorem 2.2, Φ is a surjective group homomorphism from $\mathrm{Aut}_{\mathscr{I}}\,\mathscr{A}^\infty$ onto $\mathrm{Aut}\,(\mathscr{A}^\infty/\mathscr{I})$.

COROLLARY 2.6. *Let $\mathscr{A}$ be a separable unital C^*-algebra and let $\mathscr{I}$ be a proper σ-ideal of $\mathscr{A}^\infty$. Let Γ be a countable group of automorphisms of $\mathscr{A}^\infty/\mathscr{I}$. Let Φ be the canonical group homomorphism from $\mathrm{Aut}_{\mathscr{I}}\,\mathscr{A}^\infty$ onto $\mathrm{Aut}\,(\mathscr{A}^\infty/\mathscr{I})$. Then there exists a countable subgroup G of $\mathrm{Aut}_{\mathscr{I}}\,\mathscr{A}^\infty$ such that Φ induces an isomorphism from G onto Γ.*

For each $\gamma\in\Gamma$ we choose $\rho_\gamma\in\mathrm{Aut}\,\mathscr{A}^\infty_{\mathscr{I}}$ such that $\Phi(\rho_\gamma)=\gamma$. Let 1 be the neutral element of Γ. Then we choose ρ_1 to be the identity operator on $\mathscr{A}^\infty$.

For each $\gamma\in\Gamma$ and each $\mu\in\Gamma$,

$$\Phi(\rho_\gamma\rho_\mu)=\Phi(\rho_{\gamma\mu}).$$

So, by Lemma 2.1, there exists a central projection $e_{\gamma,\mu}$ in $\mathscr{A}^\infty$ such that $(1-e_{\gamma,\mu})\in\mathscr{I}$ and $e_{\gamma,\mu}(\rho_\gamma\rho_\mu-\rho_{\gamma\mu})(x)=0$ for all $x\in\mathscr{A}^\infty$.

Let e be the infimum, in the centre of $\mathscr{A}^\infty$, of $(e_{\gamma,\mu}\colon\ (\gamma,\mu)\in\Gamma\times\Gamma\}$. Then e is a central projection such that $e(\rho_\gamma\rho_\mu-\rho_{\gamma\mu})=0$. Furthermore, since $\mathscr{I}$ is a σ-ideal, $1-e\in\mathscr{I}$.

Since, for each γ, ρ_γ is an automorphism of $\mathscr{A}^\infty$, it maps the centre of $\mathscr{A}^\infty$ onto itself. Let f be the infimum, in the centre of $\mathscr{A}^\infty$, of $\{\rho_\gamma(e)\colon\ \gamma\in\Gamma\}$. Then f is a central projection. Since, ρ_γ maps $\mathscr{I}$ to $\mathscr{I}$, $\rho_\gamma(1-e)\in\mathscr{I}$. So $1-f\in\mathscr{I}$.

We have, since $f\leq e$,

$$f\rho_\gamma\rho_\mu(x)=f\rho_{\gamma\mu}(x)$$

for all $x\in\mathscr{A}^\infty$ and each γ, μ in Γ. In particular,

$$f\rho_\gamma\rho_\mu(e)=f\rho_{\gamma\mu}(e).$$

Since ρ_γ is an isomorphism of $\mathscr{A}^\infty$, it is a σ-homomorphism, and so

$$f\rho_\gamma(f)=\bigwedge_{\mu\in\Gamma}f\rho_\gamma\rho_\mu(e)=f\bigwedge_{\mu\in\Gamma}\rho_{\gamma\mu}(e)=f.$$

We define α_γ on $\mathscr{A}^\infty$ by

$$\alpha_\gamma(x)=(1-f)x+f\rho_\gamma(x).$$

Let $G=\{\alpha_\gamma\colon\ \gamma\in\Gamma\}$. Then G is a subgroup of $\mathrm{Aut}\,\mathscr{A}^\infty_{\mathscr{I}}$ and the restriction of Φ to G is an isomorphism from G onto Γ.

REFERENCES

1. E. Christensen, 'Non commutative integration for monotone sequentially closed C^*-algebras', *Math. Scand.* 31 (1978), 171–190.
2. K. Kuratowski, *Topology*, volume I, Academic Press (1966).

3. Dorothy Maharem and A. H. Stone, 'Realizing isomorphisms of category algebras', *Bull. Australian Math. Soc.* 19 (1978), 5–10.

4. J. von Neumann, 'Einige Sätze über messbare Abbildungen', *Ann. of Math.* (2) 33 (1932), 574–586.

5. G. K. Pedersen, 'On weak and monotone σ-closures of C^*-algebras', *Commun. Math. Phys.* 11 (1969), 221–226.

6. G. K. Pedersen, 'Borel structure in operator algebras', *Danske Vid. Selsk. Mat. Fys. Medd.* 39, 5 (1974).

7. G. K. Pedersen, 'A non-commutative version of Souslin's theorem', *Bull. London Math. Soc.* 8 (1976), 87–90.

8. G. K. Pedersen, 'Measure theory for C^*-algebras III', *Math. Scand.* 25 (1969), 71–93.

9. G. K. Pedersen, *C^*-algebras and their automorphism groups, London Math. Soc. Monographs* 14, Academic Press (1979).

10. R. Sikorski, *Boolean Algebras*, second edition, Springer (1964).

11. J. D. M. Wright, 'Wild AW*-factors and Kaplansky–Rickart algebras', *J. London Math. Soc.* (2) 13 (1976), 83–89.

12. J. D. M. Wright, 'Regular σ-completions of C^*-algebras', *J. London Math. Soc.* (2) 12 (1976), 299–309.

13. J. D. M. Wright, 'Every monotone σ-complete C*-algebra is the quotient of its Baire* envelope by a two-sided σ-ideal', *J. London Math. Soc.* (2) 6 (1973), 210–214.

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