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DO NOT REMOVE

A homological characterization of foliations consisting of minimal surfaces

by Dennis Sullivan

Say that p-dimensional foliation of a compact n-manifold is *geometrically taut* if there is a Riemann metric for which the leaves become minimal surfaces. Say that an oriented p-dimensional foliation is *homologically taut* if no foliation cycle [S_1] constructed from an invariant transverse measure [P] and [RS] is approximately the boundary of a $(p+1)$-chain tangent to the foliation.

THEOREM. For an orientable foliation to be geometrically taut it is necessary and sufficient that it be homologically taut.

Proof. The ingredients are Rummler's calculation [R], an algebraic operation "purification" preserving a differential condition, Stokes theorem, and the Hahn Banach theorem in the general set-up of [S_1].

Rummler's calculation. A p-dimensional foliation $\mathcal{F}$ of a piece of Riemannian manifold has leaves which are minimal surfaces if and only if the characteristic p-form is relatively $\mathcal{F}$-closed. The characteristic p-form is obtained from the oriented volume form of $\mathcal{F}$ by orthogonal projection of p-vectors onto the tangent planes of $\mathcal{F}$. A p-form is relatively $\mathcal{F}$-closed if its restriction to every $(p+1)$-manifold tangent to $\mathcal{F}$ is closed.

Purification. To each p-form ω on a vector space positive on an oriented p-dimensional subspace F associates the pair

$$(P_\omega, \omega/F) = (\text{projection onto } F, \text{ volume form on } F).$$

Here "/" means restriction and P_ω is defined by the equation

$$P_\omega(v) \wedge (\omega/F) = (v \wedge \omega)/F$$

where "$\wedge$" means contraction. The pure form $\tilde{\omega} = P_\omega^*(\omega/F)$ is called the purification of ω.

Hahn-Banach. An orientation of a foliation $\mathscr{F}$ alllows one to regard a transversal invariant measure [RS] and [S_1] as a p-current and these form precisely the intersection of the closed p-currents with the "compact cone" of foliation currents (convex combinations of oriented tangent p-vectors) Theorem I.13 [S_1]. Homological tautness means the closed subspace S generated by boundaries of $(p+1)$ – chains tangent to $\mathscr{F}$ strictly supports the intersection cone of foliation cycles. Hahn–Banach then applied as in Theorem I.7 [S_1] allows us to construct a p-form ω positive on $\mathscr{F}$ and which annihilates this space S of boundaries. Such a form is relatively $\mathscr{F}$-closed by the obvious local argument.

Homological tautness complies *geometrical tautness*: take the form ω just constructed using Hahn–Banach and homological tautness. Now construct pointwise projections $\{P_\omega\}$ onto the tangent planes $\{F\}$ of $\mathscr{F}$ using purification. Happily, purification is natural and equal to the identity on $(p+1)$ subspaces containing F. So the purified form $\tilde{\omega}$ is *still* relatively $\mathscr{F}$ – closed. Now construct any metric on the family of subspaces {kernel P_ω} orthogonal direct sum any metric on the family *of tangent planes* $\{F\}$ giving the volume forms $\{\omega/F\}$. For this metric $\tilde{\omega}$ is the characteristic form, and by Rummler's calculation the leaves of $\mathscr{F}$ are minimal surfaces.

Stoke's theorem implies the converse. If c_n is a sequence of $(p+1)$ chains tangent to $\mathscr{F}$ so that ∂c_n approaches a foliation cycle z and ω is a p-form positive of $\mathscr{F}$ which is relatively $\mathscr{F}$-closed, we arrive at the contradiction

$$0=\int_{c_n} d\omega=\int_{\partial c_n} \omega \to \int_z \omega>0.$$

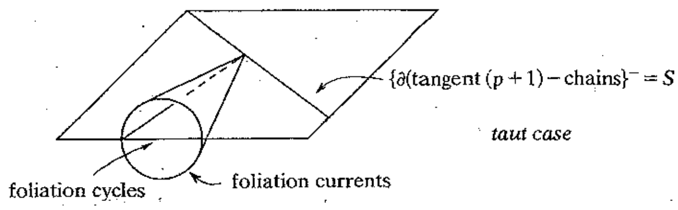
The proof of this theorem is now complete.

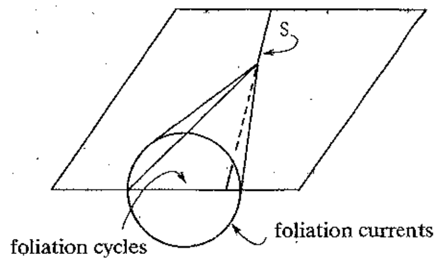
Remark. The proof shows that tautness of a foliation (geometrical or homological) is equivalent to either of the following conditions expressed by differential forms:

(i) there is a p-form ω positive on the oriented leaves of the foliation so that $d\omega$ is zero on any $(p+1)$ manifold tangent to the foliation.

(ii) there is a pure p-form ω satisfying the conditions of (i).

The equivalence of (ii) with geometrical tautness is Rummler's calculation [R] while the algebraic operation of purification $\omega \to \tilde{\omega}$ shows (i) and (ii) are equivalent. Purification converts the *non-linear* problem (ii) into a *linear* problem (i) which may be treated as in [S_1] by Hahn–Banach. Thus we arrive at the necessary and sufficient homological condition of the theorem. The theorem is a generalization of the case $p=1$ treated in [S_2] which was in turn motivated by an interesting open letter from Hermann Gluck.





EXAMPLES AND COROLLARIES

COROLLARY 1. *A foliation of a compact manifold has either a transversal invariant measure or for some Riemann metric all the leaves are minimal surfaces (of course, both can happen).*

COROLLARY 2. *A foliation is geometrically taut if no transversal invariant measure determines a trivial homology class.*

Proof. The boundaries form a closed subspace of currents.

EXAMPLE. Corollary 2 is illustrated by foliations which admit an immersed cross-section. (An immersed transversal submanifold cutting every leaf).

COROLLARY 3. *A codimension one oriented foliation is geometrically taut, if and only if every compact leaf is cut by a closed transversal curve.*

Proof. Transversal invariant measures carried on non-compact leaves intersect closed transversal curves and are thus determine foliation cycles essential in homology. By our assumption the same is true for the rest. (cf. Theorem II.20 [S_1] where this curve condition was shown to be equivalent to the existence of a transversal volume preserving flow.)

COROLLARY 4. *Foliations by geodesics are characterized by the condition that no flow cycle approximately bounds a tangent homology.* (*cf.* [S_2]).

Two general classes of foliations relevant to this discussion are "self-linking foliations" and "horospherical foliations."

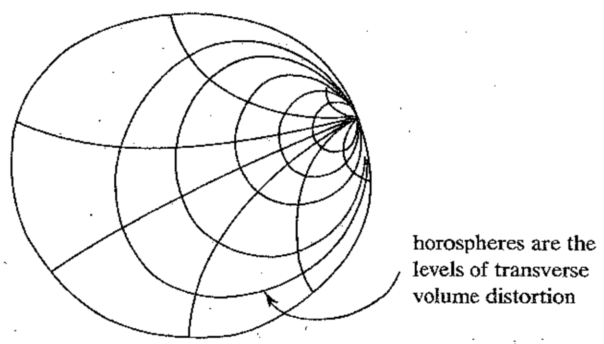
A "*self-linking foliation*" is by definition a p-dimensional foliation of a $2p+1$ manifold defined by an *exact* pure $(p+1)$-form $d\omega$ so that $\omega \wedge d\omega$ is a *nowhere zero volume* form. Geometrically, such a foliation carries a diffuse foliation cycle (defined by $d\omega$) which is homologous to zero (ω is the homology) and the self-linking number ($\omega \wedge d\omega$) is spread evenly over the entire manifold. When $p=1$ these self-linking foliations are just the contact flows.

COROLLARY 5. *Self-linking foliations are geometrically taut.*

Proof. Since $\omega \wedge d\omega$ is a volume form, ω is never zero on the leaves {kernel $d\omega$}. Clearly $d\omega$ is zero on any manifold tangent to the foliation of dimension $\leq 2p$ (in particular for $p+1$). Thus ω fulfills the condition of the Remark following the theorem. In fact, for any metric whose leaf volumes are $\{\omega \mid \text{leaf}\}$ and whose orthogonal projection agree with $\{P\omega\}$ the leaves are minimal surfaces.

Finally, non-taut examples can be constructed using the classical horospherical foliation as models. Define a "horospherical foliation" as one arising from another foliation of dimension one higher as the levels of distortion for a given transverse volume. More precisely, if ω defines one foliation then $d\omega = \eta \wedge \omega$ is pure and if it is nowhere zero defines a second foliation. This is the horospherical foliation associated to the first foliation defined by ω. In the tangent bundles of negatively curved manifolds the first foliation is made of all geodesics asymptotic at ∞ and the second is the foliation by horospheres.

COROLLARY 6. *The (generalized) horospherical foliations are never taut (geometrically or homologically).*



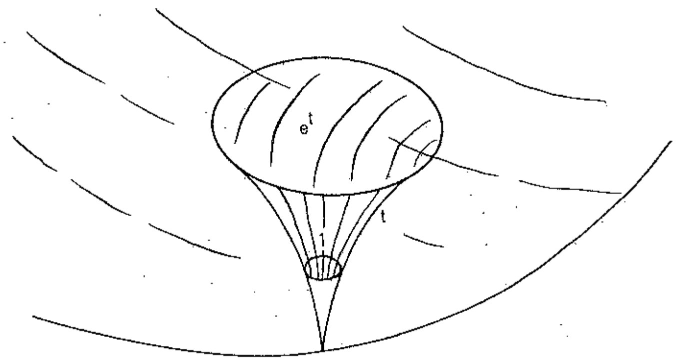
Proof. The diffuse foliation cycle defined by $d\omega$ actually bounds an (infinite) tangent homology defined by ω. This infinite homology may be approximated easily by a finite tangent homology $[S_1]$ or one may simply use Stokes again. For example let α be a form as in the Remark following the theorem. Then

$$0<\int_M \alpha\wedge d\omega=\int_M d\alpha\wedge\omega=0$$

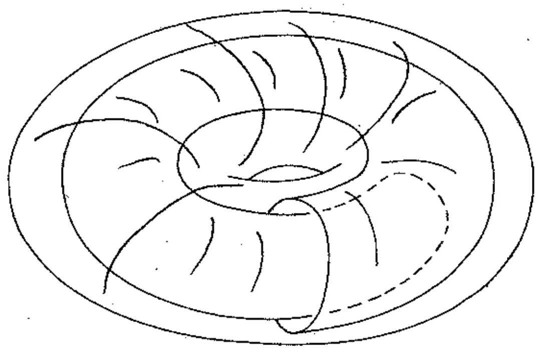
a contradiction.

EXAMPLES OF TANGENT HOMOLOGIES

In the classical horospherical tangent examples there is a picture* of the approximating homology defined by the region bounded by pieces of geodesics and horospheres of the indicated diameters.



Finite tangent homologies are easy to imagine. For example, Reeb components.



* Contained in Plante's thesis for the upper half plane. (cf. [P]).

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