Geometric Topology

Do NOT REMOVE

ON COMPLEXES THAT ARE LIPSCHITZ MANIFOLDS

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In this paper, we characterize those simplicial complexes that are locally Lipschitz isomorphic to euclidean space. The relevant lemmas concerning the conformal model of hyperbolic geometry as the open unit disc have wider implications and appear as appendices.

The objects of *Lipschitz topology*, cf. [9], are metric spaces X, Y, etc. and the maps $f : X \to Y$ (morphisms) are *(locally) Lipschitz* maps, i.e. each point in X admits a neighborhood U with a constant $K < \infty$ so that for each pair x,y in U, one has

$$d(f(x), f(y)) \leq K\, d(x,y).$$

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The restriction of f to U is said to be *uniformaly Lipschitz*[1], or more precisely *K-Lipschitz*.

Intuitively speaking, a Lipschitz map is one that obeys locally posted (or temporary) speed limits. Of course, on a compact space X, a Lipschitz map obeys a single speed limit, i.e. it is K-Lipschitz for some $K < \infty$ (depending on f).

THEOREM 1. *Suppose that a locally finite simplicial complex X with its barycentric metric is a Lipschitz n-manifold in the sense that each point admits an open neighborhood U and a Lipschitz isomorphism $h : U \to \mathbb{R}^n$. Then the link of every simplex of X is homotopy equivalent to a sphere.*

Remark a) : Our proof by Hausdorff measure general position uses only that, for each simplex σ of X, there exist an open $U \subset X$ with $U \cap \sigma \neq \emptyset$ and a homeomorphism $h : U \to \mathbb{R}^n$ such that the restriction $h|(U \cap \sigma)$ is Lipschitz.

Remark b) : R. D. Edwards and J. Cannon [4][1] have shown that, for a closed triangulated (homology) manifold H^k such that $H_*(H^k) = H_*(S^k)$, the double suspension $\Sigma^2 H^k$ is always homeomorphic to the sphere S^{k+2}. It is an amusing consequence of the above that when $\pi_1 H \neq 0$, the homeomorphism *cannot* be Lipschitz on as much as the suspension circle.

[1] *N.B. Many authors differ as to terminology (Sullivan* [17] *included), often preferring to say Lipschitz where this article says uniformly Lipschitz. With the present definition (only !), the Lipschitz property is local so that, for example, every C^1 differentiable map between Riemannian manifolds is Lipschitz, and every piecewise linear map between locally finite simplicial complexes is Lipschitz.*

Proof of theorem 1. An open k-simplex σ of X has an open neighborhood in X, e.g. its open star, that is naturally Lipschitz isomorphic to $\sigma \times c(L)$, where $c(L)$ is the open cone on the link L of σ. Thus local applications of Poincaré duality reveal that L has the integral homology of a $(n - k - 1)$-sphere and is itself an integral homology $(n - k - 1)$-manifold. Hence, when dim L is 0, 1 or 2, the link L is a triangulated sphere.

In view of the Hurewicz theorem, our task is to show that L is simply connected when dim $L \geq 3$, i.e. when σ has codimension ≥ 4. To prove this, consider any loop γ in L, and regard L as the base of a small standard sub-cone $x_0 \times c_\varepsilon(L)$, of radius ε, lying in some U as hypothesized. The cone on γ gives a map $f : B^2 \to U$ into $x_0 \times c_\varepsilon(L)$ contracting γ and meeting σ at $f(0) = x_0$ only.

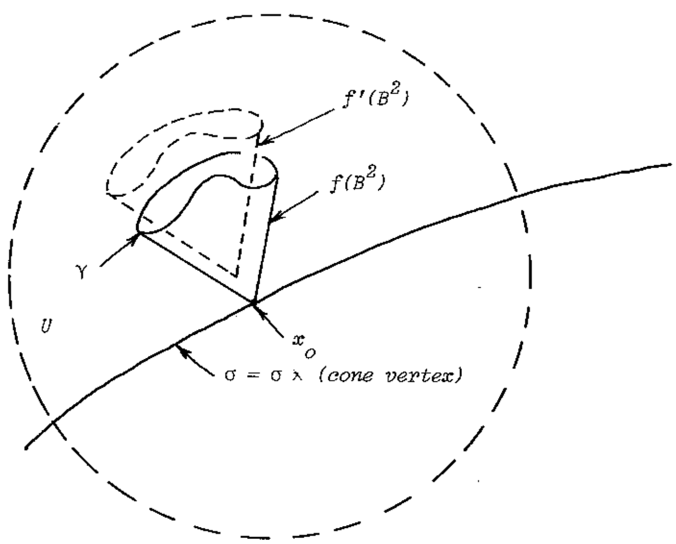

Fig. 1.

The general position lemma below shows that if we identify U with $\mathbb{R}^n$ by h and jiggle f a tiny bit, we get a map $f' : B^2 \to U$ that does not meet σ; thus γ is nul-homotopic in $(\sigma \times cL)-(\sigma \times \text{cone vertex}) = \sigma \times (0,\infty) \times L$ and therefore also in L itself. To apply this lemma, we must observe that $h|(U \cap \sigma)$ Lipschitz implies that the Hausdorff dimension of $h(U \cap \sigma)$ in $\mathbb{R}^n$ is $\leq \dim \sigma$.

GENERAL POSITION LEMMA. Let $A \subset \mathbb{R}^n$ be any subset of Hausdorff dimension $\leq a$. Then any continuous map $f : K \to \mathbb{R}^n$ of a finite simplicial k-complex K, with $k < n - a$, can be approximated by a map $f' : K \to (\mathbb{R}^n - A)$.

Proof. By simplicial approximation, we can assume (after subdividing K) that f is linear and injective on simplices. Consider a closed t-simplex τ of K and the projection p parallel to $f(\tau)$ to a $(n - t)$-plane P normal to $f(\tau)$; since p is Lipschitz, $p(A) \subset P$ has Hausdorff dimension $\leq a < n-t$ and hence Lebesque measure zero in P. Thus, for almost every t-plane P parallel to $f(\tau)$, the intersection $P \cap A$ is empty. It follows that, for almost every translation T of $\mathbb{R}^n$, one has $Tf(K) \cap A = \emptyset$. We choose a small such T and set $f' = Tf$.

In [9], Luukainen and Vaisala derive stronger general position results from a theorem of Fubini type for Hausdorff measure [7; 2.10.25]. (We thank Vaisala for making the above proof perfectly elementary.)

There is a converse to Theorem 1.

THEOREM 2. (cf. [13]) Let X be a simplicial homotopy n-manifold (without boundary) i.e. a locally finite simplicial complex in which the link of every k-simplex is homotopy equivalent to S^{n-k-1}. Then X is a Lipschitz n-manifold provided that $n \neq 4$.

Remarks: (i) If the 3 and 4-dimensional Poincaré conjectures are true (piecewise linearly), then an easy induction shows something stronger : X^n is a piecewise-linear manifold (for all n). But, they well may be false, see [2]!

(ii) To lift the proviso that $n \neq 4$, the reader will find that it is sufficient (and necessary) to show that for any piecewise-linear 3-manifold L^3 homotopy equivalent to S^3, the product $L^3 \times S^1$ is Lipschitz isomorphic to $S^3 \times S^1$, cf. proof of Lemma 4 below.

The crux of Theorem 2 is

LEMMA 3. Let L be a triangulated closed 3-manifold that is homotopy equivalent to S^3. Then the simplicial join $L * S^1$ (with barycentric metric) is Lipschitz isomorphic to S^5.

From this point on we shall occasionally rely on the basic results of [17] stating that

(a) a topological manifold X^n, $n \geq 5$, $\partial X = \emptyset$, admits a metric making it a Lipschitz manifold and

(b) any two such Lipschitz structures are Lipschitz isomorphic by a map that (majorant) approximates the identity map.

We prove Lemma 3 by a hyperbolic version of the 'torus-to-suspension' trick in [13]. One can also prove it by the Lipschitz replication technique of [14, §3]; indeed this might lead to a proof of Theorem 2 not using [17]. We leave the reader to explore this alternative.

Proof of Lemma 3. Find a homeomorphism $L^3 \times T^2 \to S^3 \times T^2$ where T^2 is a smooth multiple torus with a metric of constant negative curvature -1. For example, letting $L_0^3 = L^3 -$ (open-3-ball), we can identify $L/L_0 = S^3$ topologically; then, the resulting quotient map $L^3 \times T^2 \to S^3 \times T^2$ can be approximated by a homeomorphism [5]; one uniformaly squeezes the sets $L_0^3 \times t$, for $t \in T^2$, by a radial engulfing isotopy of $L^3 \times T^2$ and applies the Bing shrinking criterion.

One can deform $L^3 \times T^2 \to S^3 \times T^2$ to a Lipschitz isomorphism [17], say h_0. The map of fundamental groups is still $id|\pi_1(T^2)$ up to conjugation.

The universal covering $h : L^3 \times \tilde{T}^2 \to S^3 \times \tilde{T}^2$ of h_0 above is a Lipschitz isomorphism that is Γ-equivariant, where $\Gamma = \pi_1(T^2)$. Identifying $\tilde{T}^2$ isometrically to $\overset{\circ}{B}{}^2$ with hyperbolic metric as in [17] and applying Appendix A, we find that h extends uniquely to a Lipschitz isomorphism $H : \Sigma^2 L^3 \to \Sigma^2 S^3$.

Here $\Sigma^2 L^3 = L^3 \times \overset{\circ}{B}{}^3 \cup (\partial B^3)$ has the suspension metric d_s of Appendix A. Now, d_s derives via the natural composed quotient map (of sets) $L \times [0,1] \times S^1 \to L \times B^2 \to \Sigma^2 L$ from the 'join' pseudo-metric on $L \times [0,1] \times S^1$:

$$(x,t,y),(x',t',y') \mapsto (1-\max\{t,t'\})d(x,x')+|t-t'|+\min\{t,t'\}d(y,y').$$

But this pseudo-metric induces a metric on the join $L * S^1$ that one can routinely calculate to be Lipschitz equivalent to the barycentric simplicial metric.

Similarly S^5 and $\Sigma^2 S^3$ are both naturally quotients of $S^3 \times I \times S^1$ by the same relation, the quotient map to S^5 being

$$(x,t,y) \mapsto (\cos\tfrac{\pi}{2}\, tx, \sin\tfrac{\pi}{2}\, ty) \in R^4 \times R^2$$

and one checks that a Lipschitz isomorphism $S^5 \cong \Sigma^2 S^3$ results.

Thus, in all, we have Lipschitz isomorphisms $L * S^1 \cong \Sigma^2 L^3 \overset{H}{\cong} \Sigma^2 S^3 \cong S^5$ as required.

We now can prove

LEMMA 4. Let L^4 be a compact simplicial homotopy 4-manifold that is homotopy equivalent to S^4. Then the simplicial join $L^4 * S^0$ is Lipschitz isomorphic to S^5.

Proof of Lemma 4. $L^4 \times R$ is homeomorphic to $S^4 \times R$, see [13; p. 83, Assertion] or [12, App. I]. Hence, by wrapping up [15], $L^4 \times S^1$ is homeomorphic to $S^4 \times S^1$. Also, Lemma 3 shows that $L^4 \times S^1$ is a Lipschitz manifold (since $L^3 * S^1$ contains cone $(L^3) \times S^1$ Lipschitz embedded). Thus [17] shows that $L^4 \times S^1$ is Lipschitz isomorphic to $S^4 \times S^1$.

There results, on passing to universal coverings, a Lipschitz isomorphism $\Sigma^1 L^4 \cong \Sigma^1 S^4$ for the suspension metrics. (This is a trivial case of Appendix A.)

Thus $L^4 * S^0 \cong \Sigma^1 L^4 \cong \Sigma^1 S^4 \cong S^5$.

Proof of Theorem 2. For $n = 5$, note that every point lies in the open star of some vertex v, which is the open cone on the link L^4 of v, a Lipschitz 5-manifold by Lemma 4.

For $n = 6$, the link L^5 of any vertex v is, by the case $n = 5$ just proved, a Lipschitz 5-manifold. By the topological Poincaré theorem, L^5 is homeomorphic to S^5, so by [17] L^5 is Lipschitz isomorphic to S^5. Thus $v * L^5$ is Lipschitz isomorphic to D^6, which proves X^6 is Lipschitz 6-manifold.

The proof continues in this fashion by induction on n.

This completes the proof of Theorem 2.

Parallel triangulation theories of T. Matumoto and of D. Galewski and R. Stern (see [10][8]) have shown that a necessary and sufficient condition that every topological manifold (without boundary) of a fixed dimension $n \geq 5$ be triangulated by a simplicial homotopy n-manifold is that

(*) *There exist a closed smooth homotopy 3-sphere L^3 with Rohlin invariant 1 in $\mathbb{Z}/2\mathbb{Z}$ such that the connected sum $L^3 \# L^3$ bounds a smooth contractible 4-manifold.*

Combining this with Theorems 1 and 2 and [17], we immediately get

LIPSCHITZ TRIANGULATION THEOREM 5. Fix $n \geq 5$. Every Lipschitz n-manifold (without boundary) of dimension n is Lipschitz isomorphic to some locally finite simplicial complex if and only if (*) is true.

Concluding remark: Theorems 1, 2 and 5 admit fairly obvious generalisations for manifolds with boundary. See [13, p. 84] and [3] for some assistance.

PROBLEMS

1) Is Theorem 1 true replacing Lipschitz embeddings by quasiconformal embeddings ([11]) ?

One can observe, looking back to Theorem 1, that if σ is an open k-simplex, $k < n - 3$, and $h|(U \cap \sigma)$ has a non-singular differential Dh_x at at least one point x, then Link(σ) is simply connected. Indeed a small $(n - k - 1)$- sphere centered at $h(x)$ in any $(n-k)$-plane cutting transversally across the k-plane Image(Dh_x) at $h(x)$ misses $h(U \cap \sigma)$ and maps with degree ± 1 onto Link(σ) proving it to be simply connected.

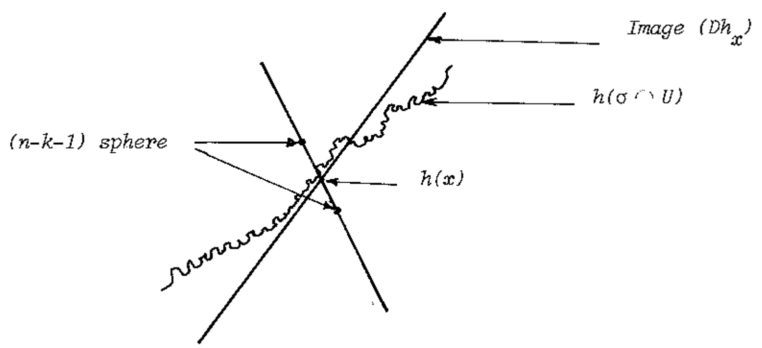

Fig. 2.

However a generic quasi-conformal homeomorphism of the complex plane seems to map the x-axis to a curve that has no tangent line at all!

2) Analytic spaces have (locally defined) Lipschitz structures. For example, the Lipschitz structures of the cusps $x^2 = y^{2k+1}$ in the plane R^2, $k = 1, 2, \ldots$ are all non-isomorphic. Up to (local) Lipschitz isomorphism are there only countably many such objects? Is there Lipschitz equisingularity along the strata of a suitable Whitney stratification of analytic spaces?

3) Define the Pontrjagin classes locally in terms of the Lipschitz structure on topological manifolds.

4) Is Rohlin's theorem on signature of 4-manifolds true for Lipschitz 4-manifolds? Perhaps an analytic proof?

5) Can the basic Lipschitz isomorphism extension theorem of [17] (cf. App. A, below) be proved by elementary means? By A. V. Cernavskii's methods? Or can one construct almost parallizable hyperbolic manifolds directly?

APPENDIX ℵ. LIPSCHITZ CONTINUATION

Recall that the open unit ball $\mathring{B}^n \subset R^n$ is a model for hyperbolic n-space when we replace the euclidean Riemannian metric dS_E on $\mathring{B}^n$ by the hyperbolic Riemannian metric $dS_H = dS_E/(1-r^2)$ where $r : R^n \to [0,\infty)$ is euclidean distance from the origin in R^n. We recall that the hyperbolic isometries of $\mathring{B}^n$ are the restrictions to $\mathring{B}^n$ of the Möbius (sphere preserving) transformations of $R^n \cup \infty$ that respect $\mathring{B}^n$ [11, §1].

THEOREM OF LIPSCHITZ CONTINUATION TO ∂B^n. Let $h : \mathring{B}^n \to \mathring{B}^n$ be a self-map that is K-Lipschitz, $K < \infty$, for the hyperbolic metric, and of hyperbolic distance $\leq \delta$ from the identity map (i.e. for all x in $\mathring{B}^n$ the hyperbolic distance $d(x,h(x))$ is $\leq \delta$).

Then, for the euclidean metric, h is uniformly Lipschitz; in fact, it is L-Lipschitz where $L < \infty$ is a constant depending *only on K and δ. Hence, h extends uniquely to a Lipschitz self-map $h : B^n \to B^n$.*

Remark: Since the euclidean radius of a ball in $\mathring{B}^n$ of fixed hyperbolic radius r and variable center $x \in \mathring{B}^n$ tends to zero as x tends to ∂B^n, it is clear that *the extension $h : B^n \to B^n$ is the identity on $S^{n-1} = \partial B^n$.*

Remark: It is amusing that h need not be a bijective map, although it must be surjective, and that even, if it is bijective, its inverse need not be Lipschitz.

Proof of Theorem. Let D and D_δ in $\mathring{B}^n$ be the compact balls centered at the origin, of hyperbolic radius 1 and $1 + \delta$ respectively. As a map from hyperbolic to euclidean metrics, the identity map $f = id|D_\delta$ is a Lipschitz isomorphism, and as D_δ is compact, both f and f^{-1} are uniformly Lipschitz. Hence $h|D : D \to D_\delta \subset \mathring{B}^n$ is L_0-Lipschitz for euclidean metric where L_0 is a constant depending on K and δ only.

Next, we prove

ASSERTION: If x and x' in $\mathring{B}^n$ lie in <u>any</u> ball D' of hyperbolic radius 1,

$$(*) \quad |h(x) - h(x')| < L|x - x'| ,$$

where L is a constant depending on K and δ only, i.e. $h|D'$ is L-Lipschitz for the euclidean metric.

From this, note that the theorem follows; indeed $(*)$ holds for any x, x' in $\mathring{B}^n$ as one can see by cutting up the *euclidean* segment x to x' into finitely many segments of *hyperbolic* length ≤ 2.

To prove the assertion, let α be a hyperbolic isometry that carries D onto D'. The conjugate $h' = \alpha^{-1} h\, \alpha$ verifies the same conditions as h, hence is L_0-Lipschitz on D (for the euclidean metric).

Observe that if α were a euclidean similarity, we would have that h' is L_0-Lipschitz on $D \Leftrightarrow h$ is L_0-Lipschitz on $D' = \alpha D$.

In reality, α is only infinitesimally such a similarity, but we prove an *almost similarity lemma* (read it's statement below) that lets us deduce the assertion as follows.

Write y, y' for $\alpha^{-1}(x)$, $\alpha^{-1}(x')$, a pair in D, and write z, z' for $h'(y)$, $h'(y')$, a pair in D_δ.

$$\begin{tikzcd} x \arrow[d, "h"'] & y \arrow[l, "\alpha"'] \arrow[d, "h'"] \\ h(x) & z \arrow[l, "\alpha"'] \end{tikzcd} \qquad \text{in } D_\delta$$

When $h(x) \neq h(x')$, we have

$$\frac{|h(x)-h(x')|}{|x-x'|} = \frac{|\alpha(z)-\alpha(z')|}{|z-z'|} \cdot \frac{|h'(y)-h'(y')|}{|y-y'|} \cdot \frac{|y-y'|}{|\alpha(y)-\alpha(y')|}$$

$$\leq L_0 a$$

where a is the approximate similarity constant for the disc D_δ. Thus h is L-Lipschitz on D' where $L = L_0 a$.

We have made essential use of the

APPROXIMATE SIMILARITY LEMMA. Given a compactum $A \subset \mathring{B}^n$, there exists a constant $a < \infty$ such that, for every quadruple x, x', y, y' of distinct points in A and every hyperbolic isometry γ of $\mathring{B}^n$, one has

$$\frac{|\gamma(y') - \gamma(y)|}{|\gamma(x') - \gamma(x)|} \leq a \frac{|y' - y|}{|x' - x|} .$$

Proof of lemma. Without losing generality, we assume that A is a ball centered at the origin $0 \in \mathring{B}^n$. Each hyperbolic isometry γ is naturally a product $\gamma = \tau\theta$ where τ is the hyperbolic translation along the line 0 to $\gamma(0)$ carrying 0 to $\gamma(0)$, while θ is the rotation $\tau^{-1}\gamma$ fixing 0. Since, θ is an euclidean isometry and respects A, it suffices to prove the lemma for translations as described. Further, by symmetry, it suffices to deal with the translations along a single ray from 0 toward $x_0 \in \partial B^n = S^{n-1}$.

There is a Mobius (sphere preserving) diffeomorphism $f : (B^n - \{-x_0\}) \to\to \mathbb{R}^{n-1} \times [0,\infty)$ with $f(x_0) = (0,0)$, $f(0) = (0,1)$, and sending dS_H to Poincarés Riemannian metric $dSp = dS_E/y$, where y is the projection to $[0,\infty)$. In the Poincaré model (see Figure 4 below), the hyperbolic translations $\overline{\gamma}$ from $f(0)$ towards $f(x_0)$ are euclidean similarities, multiplication by γ for $0 < \gamma \leq 1$. Set $\overline{\gamma} = f\gamma f^{-1}$ and $\overline{x} = f(x)$, etc..., and let K be a Lipschitz constant for both f and f^{-1} on the compacta $f^{-1}\hat{A}$ and $\hat{A}$ respectively, where $\hat{A}$ is the convex hull of $f(A)$ and $f(x_0)$. Then

$$\frac{|\overline{\gamma}(\overline{y}') - \overline{\gamma}(\overline{y})|}{|\overline{\gamma}(\overline{x}') - \overline{\gamma}(\overline{x})|} = \frac{|\overline{y}' - \overline{y}|}{|\overline{x}' - \overline{x}|} \quad , \text{ by similarity,}$$

while $$|\bar{y}' - \bar{y}| \leq k|y' - y| ,$$
$$|\bar{x}' - \bar{x}| \geq \frac{1}{k}|x' - x|$$

and $$|\bar{\gamma}(\bar{y}') - \bar{\gamma}(\bar{y})| = |f\gamma(y') - f\gamma(y)| \geq \frac{1}{k}|\gamma(y') - \gamma(y)|,$$
$$|\bar{\gamma}(\bar{x}') - \bar{\gamma}(\bar{x})| = |f\gamma(x') - f\gamma(x)| \leq k|\gamma(x') - \gamma(x)|.$$

Thus, the similarily equation gives

$$\frac{k^{-1}|\gamma(y') - \gamma(y)|}{k|\gamma(x') - \gamma(x)|} \qquad \frac{k|y' - y|}{k^{-1}|x' - x|} \quad ;$$

so, the lemma is proved with $a = k^4$.

Quasi-conformal remarks, cf. [11]. The composition of a K-quasi-conformal homeomorphism with a conformal homeomorphism is a K-quasi-conformal homeomorphism. This makes immediate the proof of the above theorem with quasi-conformal in place of Lipschitz[1]. The quasi-conformal version of the continuation theorem of the next appendix is almost as immediate.

As for the last appendix B, beware that if $f : I \times X \to I \times Y$ is an open quasi-conformal embedding respecting projection to I, *i.e. a quasi-conformal isotopy*, then f is necessarily a Lipschitz isomorphism onto its image, i.e. a Lipschitz isotopy.

APPENDIX A. – LIPSCHITZ CONTINUATION TO SUSPENSION

We prove (for Theorem 2) a result whose statement and proof are direct generalizations of the theorem for Lipschitz-continuation from $\mathring{B}^n$ to B^n as proved in Appendix $\aleph$. It also lets us deform Lipschitz isomorphisms of polyhedra and discuss Lipschitz isotopies (Appendix B).

[1] *At boundary points, the extension is 1-quasi-conformal; elsewhere it is K-quasi conformal.*

Given any metric space V of diameter ≤ 2, there is a *suspension pseudo-metric*[1] d_s on $V \times B^n$ given by

$$(v,x),\ (v',x') \mapsto (1-\max\{|x|,|x'|\})d(v,v') + d_c(x,x')$$

where d_c is the cone metric on B^n given in polar co-ordinates $[0,1] \times \partial B^n \to B^n$, $(\lambda,\hat{x}) \mapsto \lambda\hat{x}$ by

$$d_c : (\lambda\hat{x}, \lambda'\hat{x}') \mapsto |\lambda-\lambda'| + \min\{\lambda,\lambda'\}\ |\hat{x},\hat{x}'|.$$

Remark: One is tempted to simplify d_s, replacing d_c by the (Lipschitz equivalent) euclidean metric. However, as J. Väisälä pointed out to us, the triangle inequality would then not quite hold for the simplified distance function d_σ. By identifying points of $V \times B^n$ whose d_s distance is zero, we obtain a genuine metric d_s on the suspension $\Sigma^n V = V \times B^n/\sim$ where $\sim$ identifies to points the sets $V \times x$, $x \in \partial B^n$; these points $\Sigma^n V - V \times \mathring{B}^n$, naturally and isometrically identified to ∂B^n, form the *suspension sphere*.

Observe that the restriction of d_s to each disc $\{v\} \times B^n$, $v \in V$, is the cone metric of B^n.

Observe that, if $f : V \to W$ is a K-Lipschitz map of metric spaces of diameter ≤ 2, the induced map $\Sigma^n f : \Sigma^n V \to \Sigma^n W$ K-Lipschitz.

Concerning metric spaces of any diameter $\leq \infty$, recall that the modified metric d_λ , $\lambda > 0$,

$$(x,x') \mapsto \min\ \{\lambda, d(x,x')\}$$

is (locally) Lipschitz isomorphic to the old one and has diameter $\leq \lambda$. The metrics d_λ are all mutually uniformly Lipschitz isomorphic. Again, the scaled metrics λd, $\lambda > 0$, are all uniformly Lipschitz isomorphic to d.

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[1] *The triangle inequality would fail for diameter $V > 2$.*

We shall consider also the metric on $V \times \mathring{B}^n$ that is the product of the metric on V with the *hyperbolic* metric on $\mathring{B}^n$:

$$(v,x),(v',x') \mapsto d(v,v') + d(x,x').$$

We shall call this, for short, the *product-hyperbolic metric*.

THEOREM OF LIPSCHITZ CONTINUATION TO A SUSPENSION SPHERE. Consider a map $h : V \times \mathring{B}^n \to W \times \mathring{B}^n$ where V, W are metric spaces of diameter ≤ 2. Suppose that, for the product-hyperbolic metric on source and target, h is K-Lipschitz, $K < \infty$, and that $p_2h : V \times \mathring{B}^n \to \mathring{B}^n$ is of bounded hyperbolic distance $\leq \delta < \infty$ from the projection $p_2 : V \times \mathring{B}^n \to \mathring{B}^n$.

Then, h is L-Lipschitz for the suspension metrics where L is a constant depending on K and δ only. Hence, a L-Lipschitz map $h : \Sigma^n V \to \Sigma^n W$ is induced.

Remark: The extension $h : \Sigma^n V \to \Sigma^n W$ is necessarily the identity on the suspension sphere. This follows from the parallel remark in Appendix $\aleph$ applied to

$$\mathring{B}^n \xrightarrow{v\times} V \times \mathring{B}^n \xrightarrow{h} W \times \mathring{B}^n \xrightarrow{p_2} \mathring{B}^n.$$

The argument proving the theorem is strictly parallel to that in Appendix $\aleph$ and we are content to give an outline discussing the new technical points. V is metric of diameter ≤ 2.

APPROXIMATE SIMILARITY LEMMA. Given a compactum $A \subset \mathring{B}^n$, there exists a constant $a < \infty$ such that, for every quadruple x,x',y,y' of distinct points of $V \times A$ and every hyperbolic isometry α of $\mathring{B}^n$, one has, writing $\gamma = (\mathrm{id}|V) \times \alpha$:

$$\frac{d_s(\gamma(y),\gamma(y'))}{d_s(\gamma(x),\gamma(x'))} \leq a\frac{d_s(y,y')}{d_s(x,x')},$$

for the suspension metric d_s on $V \times \mathring{B}^n$.

Proof of Lemma. In imitating the argument in Appendix $\aleph$, one uses the enhanced comparison with Poincaré's model:

$$(\mathrm{id}|V) \times f : V \times (B^n - \{-x_0\}) \to V \times \{R^{n-1} \times [0,\infty)\}$$

and checks that on each set V x (compactum), it is a uniformly Lipschitz isomorphism, when, on the left, we use the suspension pseudo-metric d_s, and on the right, we use the 'wedge' pseudo-metric

$$(v,x,t),(v',x',t') \mapsto \min\{t,t'\}\ d(v,v') + |x-x'| + |t-t'|$$

where p is projection to $[0,\infty)$. To check this, one needs only to know that f (from Appendix $\aleph$) is a Lipschitz isomorphism.

The pseudo-metric just mentioned is designed to make $(v,x,t) \mapsto (v,\lambda x,\lambda t)$ a similarity.

Proof of Theorem. Form the metric space X from the disjoint union of the two metric spaces V, W of diameter ≤ 2 by decreeing the distance between v in V and w in W to be 2. Then, X has diameter ≤ 2.

By applying to X this approximate similarity lemma and imitating faithfully the argument of Appendix $\aleph$, one shows that there exists a constant $L < \infty$ depending on K and δ only, such that, for each ball $D' \subset \mathring{B}^n$ of hyperbolic radius 1, the map h is L-Lipschitz on $V \times D'$ for the suspension metrics.

Finally, one deduces that $h : V \times \mathring{B}^n \to W \times \mathring{B}^n$ is likewise L-Lipschitz by using the fact that, for any pair of points (v,x), (v',x') of $V \times B^n$ with $|x| \leq |x'|$, the d_s distance is exactly the sum of these from (v,x) to (v,x') and from (v,x') to (v',x'). Note this is false for $|x| > |x'|$.

Now, for the pair (v,x'), (v',x'), the map h has just been proved L-Lipschitz, while, for the pair (v,x), (v',x), it is prov~~ed K-Lipschitz using the euclidean segment x to x' as in Appendix~~ $\aleph$.

APPENDIX B. - LIPSCHITZ ISOTOPY EXTENSIONS

Here, exploiting Appendix A, we describe extensions and improvements for the Lipschitz isotopy extension constructed in [17, Theorem 1]. One is that the theory of [16] for deforming open topological embeddings of complexes can be carried out for Lipschitz embeddings. Another is that the isotopy extensions can preserve a Lipschitz condition with respect to parameters.

A *Lipschitz isotopy* of open embeddings of X into Y is an open embedding $F : I \times X \to I \times Y$ that respects projection to $I = [0,1]$ and is a Lipschitz isomorphism onto its image. One sometimes allows I to be a disc or a general metric space.

Note that, with $X = Y = R$, the map $F : (t,x) \mapsto (t,x + \sqrt{t})$ is not a Lipschitz isomorphism, although, for each t, the self map $x \mapsto x + \sqrt{t}$ of R is one.

We shall show, for example, that if we write the Lipschitz isotopy $F(t,x) = (t,F_t(x))$ and suppose $I = [0,1]$, while X, Y are Lipschitz manifolds, or locally finite simplicial complexes, and F_0 is inclusion $X \hookrightarrow Y$, then, for any compactum $C \subset X$, we can find a bijective Lipschitz isotopy $G : I \times Y \to I \times Y$ such that $G = F$ near $I \times C$ and $G_0 = \mathrm{id}|Y$. This is the Lipschitz version of R. Thom's smooth isotopy extension theorem.

DEFORMATION THEOREM FOR LIPSCHITZ EMBEDDINGS FOR COMPLEXES. Let X be a locally finite simplicial complex. We assert the property, cf. [16, §0]:

$\boxed{\mathfrak{L}(X)}$ *For each open set $U \subset X$ and each compactum $B \subset U$, the following holds:* $\boxed{\mathfrak{L}(X\ ;\ B\ ;\ U)}$ *If $h : U \to X$ is an open Lipschitz embedding sufficiently near to the identity $i : U \to X$, there is a rule assigning to h a Lipschitz isomorphism $h' : X \to X$ equal to h on B and the identity outside of U. For h near i, the rule $h \mapsto h'$ can be continuous (for the compact open topology) ; it sends i to* $\mathrm{id}|X$.

The rule $h \mapsto h'$ can be such that, if h respects a subcomplex $Y < X$ and its complement, then h' will too; and if h' fixes $Y < X$, then h' will too.

Remark. An equivalent property $\boxed{\mathfrak{L}^*(X\ ;\ B\ ;\ U)}$ is obtained by assuming $B \subset U$ to be closed (not compact) in X, with closure $(X - B)$ compact, and then relaxing the condition that $h' =$ identity outside U. Still other equivalent statements are sometimes helpful - see [16, §§ 0,2,4].

Remark. Unfortunately this result does *not* apply to analytic spaces with their natural Lipschitz structure, since cusps prevent their triangulations from being Lipschitz (compare Problem 2 in main text).

Proof of Deformation Theorem. The proof, when X is a Lipschitz manifold, is given in [17] (except of course for the statement concerning subcomplexes). We now indicate how to generalize this argument by combining the proof of the topological analog of given in [16], with the technical theorem of this appendix.

Here is the recipe. After reading [17], read [16, §2] to get the plan of proof by induction on *depth* (the depth of an open subset V of a complex is $d = h - h'$ where h is the maximum simplex dimension met in V and h' is the minimum). Then solve the relevant handle problems making two important changes in the argument of [16, §3]. These handle problems amount to proving $\mathfrak{L}(R^m \times cL\ ;\ B\ ;\ U)$ for any $B \subset R^m \times v$ assuming inductively $\mathfrak{L}(R^m \times (cL - v))$, where L is any finite complex.

Recall that

$$cL = L \times [0,\infty)/\{L \times 0 = v = \text{vertex}\}$$

~~is the open cone and that c_1L the radius 1 cone is the quotient of~~ $L \times [0,1]$. The metric can come from the pseudo metric on

$L \times [0,\infty)$ given by

$$(x,t),\ (x',t') \mapsto \min\{t,t'\}d(x,x') + |t-t'| .$$

The barycentric metric for the standard triangulation of c_1L is Lipschitz isomorphic to this one.

CHANGE 1. *Instead of wrapping up to derive a torus problem, use Kirby's punctured torus immersion idea in its hyperbolic form.*

This requires a capping off process as in [17]; beware that $\mathfrak{L}(R^m \times (cL-v))$ is required to make it a problem of type $\mathfrak{L}^*(cK\ ;\ K \times [2,\infty)\ ;\ K \times (1,\infty))$ on an open cone cK on a complex K, which can be solved by wrapping up as in [17]. (The complex K will be a link in $R^m \times cL$ of $0 \times v$, i.e. $K = \Sigma^m L$.)

CHANGE 2. *Adjust the 'horn device' at the unfurling stage of the proof (construction of g_4, g_5 in [16, §3]) so that it functions for Lipschitz.*

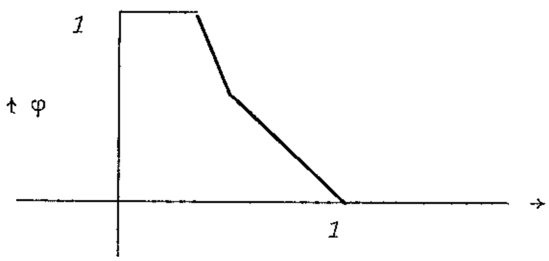
One is faced with a Lipschitz isomorphism $g_3 : \mathring{B}^m \times cL \to \mathring{B}^m \times cL$ that is the universal covering of a small Lipschitz isomorphism $g_2 : T^m \times cL \to T^m \times cL$, where T^m is a closed hyperbolic m-manifold. One has arranged that g_2 is the identity outside $T^m \times c_1L$ where $c_1L = L \times [0,1]/\{L \times 0 = v\}$ and that correspondingly g_3 = identity outside $\mathring{B}^m \times c_1L$. The horn modification of g_3

$$g_4 : \mathring{B}^m \times cL \to \mathring{B}^m \times cL,$$

is $g_4 = \Theta g_3 \Theta^{-1}$ where Θ is the Lipschitz automorphism of $\mathring{B}^m \times cL$ defined by

$$\Theta : B^m \times [0,\infty) \times L \ni (x,t,y) \mapsto (x,\varphi(|x|)t,y) \in B^m \times [0,\infty) \times L.$$

Here $\varphi : [0,1] \to [0,1]$ is Lipschitz, decreasing and satisfies $\varphi(t) = 1$ for t near 0 and $\varphi(t) = (1-t)$ for t near 1.



It is easily seen that g_4 extends to a homeomorphism

$$g_5 : R^m \times cL \to R^m \times cL$$

that is the identity outside $\Theta(\mathring{B}^m \times c_1L)$.

ASSERTION. g_5 *is a Lipschitz isomorphism.*

Proof of Assertion. The technical theorem of this appendix shows that

$$g_3 \mid : \mathring{B}^m \times c_1L \to \mathring{B}^m \times c_1L$$

extends to a Lipschitz isomorphism

$$\bar{g}_3 : \Sigma^m(c_1L) \to \Sigma^m(c_1L).$$

But there is a canonical Lipschitz isomorphism

$$\Theta' : \Sigma^m(c_1L) \to \Theta(\mathring{B}^m \times c_1L) \cup \partial B^m \times v$$

induced by the map Θ. This shows that g_5 gives a Lipschitz automorphism on the compact target of Θ'; since g_5 is the identity elsewhere, the proof is complete.

The verification that the standard map Θ' is indeed a Lipschitz isomorphism is left to the reader.

The isomorphism g_4 essentially solves the handle problem. This completes our outline of the proof of $\mathcal{L}(X)$.

We now turn to Lipschitz isotopy extensions, and show that the deformation rule $h \mapsto h'$, *as already constructed* in [17] and above, to satisfy $\mathfrak{L}(X\ ;\ B\ ;\ U)$, enjoys a further:

PARAMETER PROPERTY. If $h_t : U \to X$, $t \in I$, *gives a Lipschitz isotopy*[1] *so near to the inclusion* $U \to X$ *that the rule* $h_t \mapsto h'_t$: $X \to X$ *is defined for all* $t \in I$, *then (automatically)* $h'_t : X \to X$, $t \in I$, *likewise gives a Lipschitz isotopy.*

This property is to be checked by looking back over the solutions of the finitely many handle problems that yield the deformation rule $h \mapsto h'$. In each handle problem, one has to check that both *capping off* and *unfurling* preserve Lipschitz isotopies, using the

THEOREM OF LIPSCHITZ PARAMETERS. Let $F : I \times J \times \mathring{B}^n \to I \times J \times \mathring{B}^n$ *be a self-map that respects projection to the metric space* I, *while* J *is a metric space of diameter* ≤ 2. *Suppose* F *is* K-*Lipschitz,* $K < \infty$, *for the product metric on* $I \times J \times \mathring{B}^n$, *where* $\mathring{B}^n$ *carries the hyperbolic metric, and suppose also that* $p_3 \circ F : I \times J \times \mathring{B}^n \to \mathring{B}^n$ *is of bounded hyperbolic distance* $\leq \delta$ *from the projection* p_3 *to* $\mathring{B}^n$. *Let* $\Sigma^n J = J \times \mathring{B}^n \cup (\partial B^n)$ *carry the suspension metric and give* $I \times \Sigma^n J$ *the product metric. Then* F *extends to a map* $G : I \times \Sigma^n J \to I \times \Sigma^n J$ *that is* L-*Lipschitz where* L *is a constant depending on* K *and* δ *only.*

Remark (cf. Appendix A): Necessarily, G is the identity on $I \times (\partial B^n)$ and respects projection to I.

Proof of theorem. This variant of Appendix A is fortunately an easy corollary thereof.

[1] *That is,* $H : (t,x) \mapsto (t, h_t(x))$ *mapping* $I \times U \to I \times X$ *is a Lipschitz isomorphism onto its image; here* I *is any metric space.*

To lighten the notation, we prove only the extreme case where J is a point so that $I \times J \times \mathring{B}^n = I \times \mathring{B}^n$ and $I \times \Sigma^n J = I \times B^n$. Let p_1 and p be the projections from $I \times \mathring{B}^n$ to I and $\mathring{B}^n$ respectively.

Since, it is a local question whether a Lipschitz extension of F exists, we can assume that diameter I is ≤ 2. Appendix A then asserts that F extends uniquely to an L-Lipschitz map

$$H : I \times B^n \to I \times B^n$$

for the *suspension distance function*

$$(t,x),(t',x') \mapsto \{1 - \max(|x|,|x'|)\}d(t,t') + |x - x'|$$

and that $H|(I \times \partial B^n) = \text{id}$. It sufficies to check that H is $2L$-Lipschitz on $I \times B^n$ for the *product metric*

$$d_\pi((t,x),(t',x')) = d(t,t') + |x - x'| \ .$$

We can assume $1 < L < \infty$, so that

$$d_s(H(t,x),H(t',x') \leq Ld_s((x,t),(x',t')).$$

Hence

$$|pH(t,x) - pH(t',x')| \leq L(d(t,t') + |x-x'|) = Ld_\pi((t,x),(t',x')).$$

But, since H respects projection to I,

$$d(p_1H(t,x),p_1H(t',x')) = d(t,t') \leq Ld_\pi((t,x),(t',x')) \ .$$

Thus, adding,

$$d_\pi(H(t,x),H(t',x')) \leq 2Ld_\pi((t,x),(t',x')).$$

REFERENCES

1. Cannon, J., Shrinking cell-like decompositions, Annals of Math. (1978), to appear.
2. Cappell, S., and Shaneson, J., Construction of some new four-dimensional manifolds, Bull. Amer. Math. Soc. 82 (1976), pp. 69-70.
3. Cohen, M., Homeomorphisms between homotopy manifolds and their resolutions, Invent. Math. 10 (1970), pp. 239-250.
4. Edwards, R. D., Approximating certain cell-like maps by homeomorphisms, Ann. of Math., to appear.
5. Edwards, R. D., and Glaser, L. C., A method for shrinking decompositions of certain manifolds, Trans. Amer. Math. Soc. 165 (1972), pp. 45-56.
6. Edwards, R. D., and Kirby, R. C., Deformations of spaces of imbeddings, Ann. of Math. 93 (1971), 63-88.
7. Federer, H., Geomet c measure Theory, Grundlehren Series no. 153, Springer, 1969.
8. Galewski, D., and Stern, R., Ann. of Math, to appear and Proc. AMS Stanford Topology Conf. 1976, to appear.
9. Luukkainen, J., and Väisälä, J., Elements of Lipschitz topology, Ann. Acad. Sci. Fenn. 3 (1977), pp. 85-122
10. Matumoto, T., Variétés simpliciales d'homologie et variétés topologiques métrisables, thèse, Univ. Paris-Sud, 91405 Orsay, 1976. Proceedings of AMS Stanford Topology Conference 1976, to appear.
11. Mostow, G., Rigidity of hyperbolid space forms, Publ. Math. I.H.E.S. 34 (1968), pp. 53-104.
12. Siebenmann, L. C., Approximating cellular maps by homeomorphisms, Notices Amer. Math. Soc. 17 (1970, p. 532 and Topology 11 (1973), pp. 271-294.
13. Siebenmann, L. C., Are non-triangulable manifolds triangulable?, in Topology of manifolds, edited by J. C. Cantrell and C. H. Edwards, (Athens Georgia Conf. 1969), Markham, Chicago, 1970, pp. 77-84.
14. Siebenmann L. C., Topological Manifolds, Proc. I.C.M. Nice (1970), Vol. 2, pp. 143-163, Gauthier-Villars, 1971.
15. Liebenmann, L. C., Pseudo-annuli and invertible cobordisms, Archiv der Math. 19 (1968), pp. 28-35.
16. Siebenmann, L. C., Deformation of homeomorphisms on stratified sets, Comment. Math. Helv. 47 (1972), pp. 123-163. Note errata in 6.25, p. 158: on line 10, N becomes B; on lines 23, 24, 25, $\frac{!}{2}$ becomes $\frac{t}{2}$.
17. Sullivan, D., Hyperbolic geometry and homeomorphisms (these proceedings).