Topology Vol. 16, pp. 13–32. Pergamon Press, 1977. Printed in Great Britain

FOLIATIONS WITH ALL LEAVES COMPACT*

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(*Received* 20 *October* 1975)

§1.

THE PURPOSE of this paper is to present some information about the following Question:

If M is a compact manifold foliated by compact submanifolds (everything smooth), is there an upper bound on the volume of the leaves?†

In particular, if M is a compact manifold supporting a nonsingular flow in which each orbit is periodic, is there an upper bound on the lengths of the orbits? In his thesis, Reeb ([12], for an analytic example see [3, p. 68]) describes a smooth flow on a non-compact manifold such that all orbits are periodic and such that the lengths of the orbits are not locally bounded. After our research was completed, the third author found a smooth flow on a closed 5-manifold [15, 16], which showed that the answer, in general, was no. The former example shows that the question is global and cannot be answered by simply considering the structure of the foliation in a neighborhood of individual compact leaves. The latter example shows that some additional hypothesis on M is required. This example is worth keeping in mind while reading this paper due to its close connection with our main result.

The existence of an upper bound on the volume of the leaves has rather important consequences which provide a description of the local, as well as global, structure of the foliation. The boundedness of volume near any given leaf is equivalent to the finiteness of the holonomy group of that leaf, and also to the hausdroff separation property for the topology of the leaf space near the leaf ([4], [8], see also §4). Hence, in the presence of a bound on the volume, a structure theorem due to Ehresmann [4, Theorem 4.3] provides a nice picture of the local behavior of the foliation. In the absence of such a bound the geometrical possibilities are somewhat formidable.

Concerning the known cases of the Question, if the leaves have condimension 1 then the bound exists by a relatively elementary argument [12]. For periodic flows on compact 3-manifolds Epstein [3] has demonstrated the existence of a bound by a surprisingly delicate argument. Our two principal results were obtained in trying to understand Epstein's argument. Our first result is that in the presence of a certain homological assumption, the answer to the Question is yes.

THEOREM 1. *Suppose M is a compact smooth manifold which is smoothly foliated by compact leaves of dimension d. Suppose that the leaves are oriented in a continuous manner, and that the images of the fundamental classes of the leaves all lie in some open halfspace of the d-dimensional real homology of M. Then there is an upper bound on the volumes of the leaves of M. Consequently, all the holonomy groups of the foliation are finite.*

Another way of stating the homological condition is to say there exists a closed d-form ω on M, such that ω has a positive integral along each leaf. Here M may have boundary, in which case we assume that the boundary is a union of leaves. The with-boundary version of the Theorem follows from the without-boundary version by doubling M along its boundary.

Two particular situations to which the theorem applies are when (i) each leaf of the foliation

*The authors acknowledge the support of the Institut des Hautes Etudes Scientifiques and the National Science Foundation.

†Let the volumes be determined by a riemannian metric on the tangent bundle of the foliation, see §4.

has positive (respectively negative) euler characteristic, or (ii) the ambient manifold is kaehler and the foliation is complex analytic.

Details are in §7. We remark that case (ii) actually follows from the first part of the proof of the Theorem (the Moving Leaf Proposition, §5).

We point out, with regard to the volume question, that any lack of orientability of the leaves of M at the outset is no problem, for if the leaves are initially not continuously oriented, they can be made so by taking the appropriate double cover of M. Also, if one desires, M can be made oriented by taking a further double cover; this would make the foliation transversally oriented and hence would make all holonomy oriented. Note that the finiteness or nonfiniteness of holonomy is preserved by such covers. Similarly, so is the boundedness or unboundedness of the volume of the leaves.

Theorem 1 does not require so sweeping a homological hypothesis as stated above. The proof of the theorm deals only with a neighborhood of what Epstein calls the *bad set* of M, which is the union of leaves of M near which the volume function is not bounded. Of course the goal is to show that this bad set is empty; in general one can at least say that it is closed and nowhere dense (see §§4, 6). Our proof reveals that it must be empty if there is a closed d-form ω, defined on a neighborhood of the bad set, whose integral is positive along each leaf of the bad set. For this argument M need not be compact, as long as the bad set itself is compact.

Our second principal result is that the Epstein argument mentioned above can be adapted to hold in codimension two in general. This has also been done by Vogt.

THEOREM 2 (*extending* [3]). *Suppose M is a smooth compact manifold which is smoothly foliated by compact leaves of codimension two. Then there is an upper bound on the volumes of the leaves of M. Consequently, all the holonomy groups of the foliation are finite.*

As above, M may have boundary, in which case we assume that the boundary is a union of leaves.

This theorem can be regarded as a special case of Theorem 1, because in the proof we show in effect, that there is a closed $(m-2)$-form ω, defined on a neighborhood of the bad set, which has a positive integral along each leaf of the bad set. Nevertheless, the argument is of a sufficiently independent nature that we have isolated it in §8.

The motivation for the proofs of these theorems is this: We begin with the known fact (see §4) that the family of leaves whose holonomy is trivial (≡ the *generic* leaves) comprise an open dense subset of the manifold which is fibered by the leaves. Whenever the bad set is nonempty, there must exist a continuous family L_t, $0 \le t < \infty$, of generic leaves of M which approach the bad set and whose volumes are unbounded (see §5). Imagine for the moment that the bad set itself happens to be a nice smooth fiber bundle. Now the homological hypothesis of Theorem 1 says that the fiber (= leaf) of the bad set is not homologically trivial in the bad set. (For example, if the euler characteristic of the fiber is nonzero (cf. (i) above). This is true because an euler form on the total space of the bad set evaluates on each fiber to give its euler characteristic. Also for example, in the Epstein condimension two situation, the base of the fiber bundle must have dimension ≤ 1, and so the homological hypothesis holds. On the other hand, in the counter example mentioned above[15, 16], the bad set is S^3 foliated as the Hopf bundle.)

We wish now to get a contradiction. If the bundle had a smooth cross section, then this would be easy, for then we could extend the cross section over a neighborhood of the bad set and examine the intersection number of each leaf L_t of the continuous family with this fixed transverse manifold. On the one hand these numbers must all be equal, by homological considerations, and on the other hand they must be unbounded since the volumes of the moving leaves $\{L_t\}$ are unbounded. This is the contradiction.

In the absence of a cross section to the bad set, we use the hypothesized differential form and geometric currents. A geometric current is to be thought of as a homology class of M which is geometrically realized in a precise way. It consists of a certain collection of leaves of M (possibly uncountable), with the leaves weighted by transverse measures, so that this collection is an infinite homological sum of leaves. Such a geometric current gives rise, by integration along the leaves, to a de Rham current, i.e., a continuous homomorphism from the differential d-forms of M to the reals. Our construction of a geometric current on the bad set, and its application to

Theorem 1, can be summarized as follows: We begin with the continuous family of leaves L_t, $0 \leq t < \infty$, mentioned above. Regarding this family of leaves as a family of currents in the sense of de Rham we extract, after a certain normalization process to account for the unbounded volumes, a convergent subsequence. Arguing locally, we then show that there is a non trivial limiting geometric current. This we analyze geometrically to deduce homological relations among the fundamental classes of the leaves. For example, letting $\{L_i\}$ denote the sequence of leaves and letting $\{n_i\}$ denote a certain sequence of normalizing constants which approach infinity, we generate a possibly infinite collection $\{r_\alpha\}$ of positive real numbers and $\{L_\alpha\}$ of leaves of the bad set, determined by the distribution of the limiting geometric current, so that the following homology relation holds:

$$\lim_{i\to\infty} \frac{1}{n_i}[L_i] = \sum_\alpha r_\alpha [L_\alpha].$$

(If the bad set were a connected fiber bundle as imagined above, there would be only one term in the right hand side). From this homology relation, which holds in an arbitrarily small neighborhood of the bad set, we can use the homological hypothesis of Theorem 1 to deduce a contradiction, by evaluating the hypothesized d-form ω on each side of the equation. This shows that the bad set must be empty, establishing Theorem 1.

In the case of codimension two foliations we are able to mimic Epstein's construction, after passing to a 4-fold cover to make everything oriented, to obtain a transversally embedded oriented 2-manifold in M which is closed in some open neighborhood of the bad set, and which has nonempty intersection with each leaf of the bad set. Now one can argue in the manner described earlier, by considering the algebraic intersection numbers of this transverse 2-manifold with each leaf L_t in the continuous family of leaves of unbounded volume that has already been mentioned. These intersection numbers must on the one hand be constant and must on the other hand be unbounded, establishing the desired contradiction and proving Theorem 2.

In this paper we always work with smooth foliations of smooth manifolds, by which we mean C^r-smooth for some fixed r, $1 \leq r \leq \infty$. Most of the results and constructions of the paper do not require any differentiability hypotheses at all (e.g. the definition and construction of geometric currents in §§2,3 and the Moving Leaf Proposition in §5), although at times we may use differentiability as a convenience. In Theorem 1 only the rectifiability of the leaves is used. Thus to make the proof of Theorem 1 hold for topological manifolds, it remains to formulate a tool to take the place of differentiable forms, as something which can be continuously evaluated on continuously varying geometric currents. In Theorem 2 one uses the C^1 properties of holonomy. Thus to make the proof of Theorem 2 hold for topological manifolds, one only needs something to play the role of Weaver's Lemma [19] which concerns C^1 germs of diffeomorphisms of R^2.

§2. GEOMETRIC CURRENTS

The material in this section is a distillation of the early discussion in Ruelle–Sullivan [14], with a few trivial technical variations. Suppose that M^m is a smooth m-manifold without boundary. An l-dimensional current on M, as defined by de Rham [13], is any continuous homomorphism $\Lambda_c^l(M) \to \mathbf{R}$ from the compactly supported l-forms on M to the real numbers.* A special kind of current used in this paper is a *geometric current* (or *foliation current*), described as follows.

Suppose that M^m is smoothly foliated by l-dimensional leaves, where $m = k + l$. To fix notation, we suppose that $\{W_\sigma \times R^l\}$ is a locally finite collection of foliation charts whose interiors cover M, where each W_σ is compact. To be technically sound, one should regard each chart $W_\sigma \times R^l$ as lying inside an open foliation chart $U_\sigma \times R^l$, where $W_\sigma \subset U_\sigma$. We will tacitly assume this, without further mention of the $U_\sigma \times R^l$'s. We identify each W_σ with $W_\sigma \times 0 \subset W_\sigma \times R^l \subset M$, and we write $\operatorname{int}(W_\sigma \times R^l) = \mathring{W}_\sigma \times R^l$, where $\mathring{W}_\sigma \subset W_\sigma$ is a k-manifold which we assume is smoothly embedded in M. The W_σ's are called *transversals*. We assume that the leaves of M are continuously oriented by these coordinate charts, that is, for any two slices $x \times R^l \subset W_\sigma \times R^l$ and $y \times R^l \subset W_\tau \times R^l$ which overlap, the induced R^l-orientations agree.

Suppose that each compact transversal W_σ has defined on it a nonnegative measure μ_σ of

*The de Rham definition uses the C^∞ topology on forms. However, the currents arising here are even continuous in the C^0-topology on forms. From now on we replace "d" by "l" which is to be distinguished from "1".

finite mass. (We remark here that all measures employed in this paper are regular Borel measures of finite total mass, and we refer to them only as measures. Recall that, by the Riesz Representation Theorem, such measures on a compact space W are in 1-1 correspondence with bounded linear functionals on the Banach space of continuous maps from W to R^1). A collection of such measures $\{\mu_\sigma\}$ defined on the collection of transversals $\{W_\sigma\}$ is *translation invariant* if, roughly speaking, the measures are invariant under isotopies of the identity preserving the leaves. Precisely, the collection of measures $\{\mu_\sigma\}$ is translation invariant provided that for each Borel subset $F \subset W_\sigma$ and for each leaf-invariant homeomorphism (i.e. a homeomorphism of M which takes each leaf into itself) $h: M \to M$, if the restriction $h|: F \to M$ gives an embedding of F into another transversal W_τ (with possibly $\tau = \sigma$), then $\mu_\tau(h(F)) = \mu_\sigma(F)$.

This data—the foliation with continuously oriented leaves together with the translation invariant measures $\{\mu_\sigma\}$—comprise a *geometric current* on M, which we denote by $C\{\mu_\sigma\}$. The definition can be made independent of the charts $\{W_\sigma \times R^l\}$ (as in [14]), but that is unnecessary here, and at any rate it is implicit in later discussions.

A subset X of M is *saturated* or *invariant*, if it is a union of leaves of the foliation. Given a geometric current $C\{\mu_\sigma\}$ and a saturated subset X, then $C\{\mu_\sigma\}$ is *supported on* X if, for each σ, the restricted measure $\mu_\sigma | W_\sigma - X$ is trivial.

Given a geometric current $C\{\mu_\sigma\}$, one can define a de Rham current $C\{\mu_\sigma\}: \Lambda_c^l(M) \to \mathbf{R}$ by using a smooth partition of unity, $\{p_\sigma\}$ say, with compact supports $\{cl\, p_\sigma^{-1}(0, 1]\}$ which are subordinate to the *open* cover $\{\mathring{W}_\sigma \times R^l\}$. Given any compactly supported l-form ω, the partition of unity can be used to decompose ω into a finite sum of l-forms, $\omega = \Sigma_\sigma p_\sigma \cdot \omega$, where each $p_\sigma \cdot \omega$ has compact support in $\mathring{W}_\sigma \times R^l$, and then one can define

$$\langle C\{\mu_\sigma\}, \omega\rangle = \Sigma_\sigma \int_{x \in W_\sigma} \langle x \times R^l, p_\sigma \cdot \omega > d\mu_\sigma.$$

It is clear that this defines a continuous homomorphism. An essential fact, which is a consequence of the translation invariance of the measures $\{\mu_\sigma\}$, is that this definition is independent of the partition of unity chosen. That is, if $\{p'_\sigma\}$ is any other smooth partition of unity, each p'_σ having compact support in $\mathring{W}_\sigma \times R^l$, then the above sum of integrals remains unchanged using $\{p'_\sigma\}$ in place of $\{p_\sigma\}$. This is a consequence of the familiar process of carefully choosing a much finer cover and partition of unity, and then evaluating with respect to it, cf. [14]. Because of its importance, we repeat the argument here.

Let $\{Y_\lambda \times R^l\}$ be any locally finite cover of M by open foliation charts which refine the cover $\{\mathring{W}_\sigma \times R^l\}$ and satisfy the following two special properties:

(i) for each $Y_\lambda \times R^l$ and each $W_\sigma \times R^l$, if $Y_\lambda \times R^l \cap (\text{support } p_\sigma \cup \text{support } p'_\sigma) \neq \emptyset$, then $Y_\lambda \times R^l \subset \mathring{W}_\sigma \times R^l$, and

(ii) for each $Y_\lambda \times R^l$ and each $W_\sigma \times R^l$, if $Y_\lambda \times R^l \subset W_\sigma \times R^l$, then the natural map $Y_\lambda = Y_\lambda \times 0 \subset Y_\lambda \times R^l \subset W_\sigma \times R^l \to W_\sigma$ is an embedding of Y_λ into W_σ (rather than just a submersion) which is given by the restriction of some leaf-invariant homeomorphism of M.

Thus each transversal Y_λ has induced on it in unambiguous fashion a measure ν_λ. Fix a smooth partition of unity $\{q_\lambda\}$, with compact supports subordinate to $\{Y_\lambda \times R^l\}$. Then for any l-form $\omega \in \Lambda_c^l(M)$, the evaluation of ω using this new cover and partition of unity agrees with the evaluation using the original cover and either associated partition of unity $\{p_\sigma\}$ or $\{p'_\sigma\}$. The appropriate calculations for the partition of unity $\{p_\sigma\}$ are as follows:

The evaluation of ω using the cover $\{Y_\lambda \times R^l\}$ and partition of unity $\{q_\lambda\}$

$$= \Sigma_\lambda \int_{y \in Y_\lambda} \langle y \times R^l, q_\lambda \cdot \omega \rangle d\nu_\lambda$$

$$= \Sigma_\lambda \Sigma_\sigma \int_{y \in Y_\lambda} \langle y \times R^l, q_\lambda \cdot p_\sigma \cdot \omega \rangle d\nu_\lambda \quad (\text{since } \Sigma_\sigma p_\sigma \equiv 1)$$

$$= \Sigma_\sigma \Sigma_{\{\lambda | Y_\lambda \times R^l \subset \mathring{W}_\sigma \times R^l\}} \int_{y \in Y_\lambda} \langle y \times R^l, q_\lambda \cdot p_\sigma \cdot \omega \rangle d\nu_\lambda$$

(for if $Y_\lambda \times R^l \not\subset \mathring{W}_\sigma \times R^l$, then $q_\lambda \cdot p_\sigma \equiv 0$)

$$= \Sigma_\sigma \Sigma_{\{\lambda | Y_\lambda \times R^l \subset \mathring{W}_\sigma \times R^l\}} \int_{x \in \mathring{W}_\sigma} \langle x \times R^l, q_\lambda \cdot p_\sigma \cdot \omega \rangle d\mu_\sigma$$

(using property (ii))

$$= \Sigma_\sigma \int_{x \in \mathring{W}_\sigma} \langle x \times R^l, p_\sigma \cdot \omega \rangle d\mu_\sigma$$

(for $\Sigma\{q_\lambda | \lambda$ such that $Y_\lambda \times R^l \subset \mathring{W}_\sigma \times R^l\} = 1$ on the support of p_σ).

Thus the value $\langle C\{\mu_\sigma\}, \omega\rangle$ is independent of the original partition of unity $\{p_\sigma\}$.

The basic consequence of the invariance of measures above is that the geometrically defined current $C\{\mu_\sigma\}: \Lambda_c^l(M) \to \mathbf{R}$ is closed, that is, if $\omega = d\kappa$, then $\langle C\{\mu_\sigma\}, \omega\rangle = 0$. This follows by writing $\kappa = \Sigma\kappa_\sigma$ as a finite sum with each κ_σ having compact support in $\mathring{W}_\sigma \times R^l$, and then observing that for each σ, $\langle C\{\mu_\sigma\}, d\kappa_\sigma\rangle = 0$.

We remark in passing that geometric currents can be defined more generally on foliated subsets of non-foliated manifolds. For example, an l-dimensional compact oriented submanifold may be regarded as an l-dimensional geometric current. In particular we find it useful to regard individual compact oriented leaves of a foliation as defining geometric currents in this manner. For further discussion along these lines, see Ruelle–Sullivan[14].

§3. A SPECIAL GEOMETRIC CURRENT

In this section we describe a certain basic construction of a geometric current, which we will subsequently use in the proof of Theorem 1. Suppose that M is a smoothly foliated manifold as in §2, with continuously oriented leaves, and having the additional property that all leaves are compact. Suppose that X is a compact saturated subset of M, and that $\{L_i\}$ is a sequence of leaves in M which converge to X in the sense that, given any neighborhood of X in M, the sequence eventually lies in this neighborhood. (A suitable X and sequence $\{L_i\}$ for our application will be carefully chosen in §§5,6). We show how to choose a subsequence $\{L_j\}$ of $\{L_i\}$ and sequence of positive integers $\{n_j\}$ such that $\{(1/n_j)L_j\}$, regarded as a sequence of geometric currents in the sense of §2, converges to some nonzero limit geometric current $C\{\mu_\sigma\}$ which is supported on X.

Let $\{W_\sigma \times R^l\}$ be a locally finite collection of foliation charts for M as described in §2. Each compact leaf L_i intersects each compact transversal W_σ in at most a finite number of points, say $n_{i,\sigma}$ of them. For each i, let $n_i = \max_\sigma\{n_{i,\sigma}\}$. Let $\mu_{\sigma,i}$ be the nonnegative measure on W_σ which assigns the mass $1/n_i$ (not $1/n_{i,\sigma}$) to each point of $W_\sigma \cap L_i$, and is zero on any subset of $W_\sigma - L_i$. Then for each σ and i, $\mu_{\sigma,i}(W_\sigma) \leq 1$. By passing to a subsequence of (L_i, n_i), we can assume that there is a fixed transversal W_ξ such that $n_i = n_{i,\xi}$ for each i. Now for each σ, $\{\mu_{\sigma,i}\}$ is a sequence of uniformly bounded, nonnegative measures on W_σ. Hence some subsequence of these measures converges to a bounded nonnegative measure, call it μ_σ, and this μ_σ is concentrated on $W_\sigma \cap X$. Furthermore, when $\sigma = \xi$, we have that $\mu_\xi(W_\xi \cap X) = 1$. Hence, by taking a subsequence of (L_i, n_i), say (L_j, n_j), we may arrange that for each σ, $\lim_{j\to\infty} \mu_{\sigma,j} = \mu_\sigma$ as well as $\mu_\xi(W_\xi \cap X) = 1$.

To be able to assert that these charts $\{W_\sigma \times R^l\}$ and $\{\mu_\sigma\}$ define a geometric current $C\{\mu_\sigma\}$ supported on X, we must show that the measures are translation invariant.

Lemma A. *Suppose $W_\sigma \times R^l$ and $W_\tau \times R^l$ are two of the prescribed foliation charts for M and suppose that F is a Borel subset of W_σ. Furthermore suppose that there is an embedding of F into W_τ given by the restriction $h|F$ of some leaf-invariant homeomorphism $h: M \to M$. Then $\mu_\sigma(F) = \mu_\tau(h(F))$. Consequently, the measures $\{\mu_\sigma\}$ are translation invariant.*

Proof. In brief, the Lemma is true because, for each j, it is true for the collection $\{\mu_{\sigma,j}\}$. Specifically, we have that for each j, $h(F \cap L_j) = h(F) \cap L_j$ because h is leaf-invariant. Thus $\mu_{\sigma,j}(F) = \mu_{\tau,j}(h(F))$. Taking limits gives the desired conclusion (the reader can supply the necessary measure theoretic details here).

Having established the existence of the limiting geometric current $C\{\mu_\sigma\}$, we wish to show that as a de Rham current, $C\{\mu_\sigma\}$ equals $\lim_{j\to\infty}\langle(1/n_j)L_j, -\rangle$: $\Lambda_c^l(M) \to \mathbf{R}$. Here convergence and equality are interpreted pointwise, that is, on each l-form individually.

To show this equality for a particular compactly supported l-form ω, it suffices to decompose ω as a finite sum $\omega = \Sigma\omega_\sigma$ via a partition of unity subordinate to the open cover $\{\mathring{W}_\sigma \times R^l\}$ in the manner of §2, and then to apply the following lemma to each term ω_σ in the sum. Let ω_τ denote a specific ω_σ.

LEMMA B. *Suppose ω_τ is an l-form with compact support in $\mathring{W}_\tau \times R^l$. Then*

$$\langle C\{\mu_\sigma\}, \omega_\tau\rangle = \lim_{j\to\infty}\left\langle \frac{1}{n_j} L_j, \omega_\tau\right\rangle.$$

Proof. In §2 we showed that $\langle C\{\mu_\sigma\}, \omega_\tau\rangle$ can be evaluated using any partition of unity $\{p_\sigma\}$ subordinate to the cover $\{\mathring{W}_\sigma \times R^l\}$. In particular, we can choose one such that $p_\tau = 1$ on the support of ω_τ. Hence

$$\begin{aligned}\langle C\{\mu_\sigma\}, \omega_\tau\rangle &= \int_{x\in \mathring{W}_\tau} \langle x\times R^l, \omega_\tau\rangle d\mu_\tau \\ &= \lim_{j\to\infty}\int_{x\in \mathring{W}_\tau} \langle x\times R^l, \omega_\tau\rangle d\mu_{\tau,j} \\ &= \lim_{j\to\infty}\frac{1}{n_j}\sum_{x\in \mathring{W}_\tau\cap L_j} \langle x\times R^l, \omega_\tau\rangle \\ &= \lim_{j\to\infty}\frac{1}{n_j}\langle L_j, \omega_\tau\rangle.\end{aligned}$$

To complete this section, we show that $C\{\mu_\sigma\}$ is a *nonzero* geometric current. That is to say, we show

LEMMA C. *For some τ, $\mu_\tau(\mathring{W}_\tau) > 0$.*

Proof. By construction $\mu_\xi(W_\xi) = 1$ for the specific index ξ chosen earlier. However it is possible that $\mu_\xi(\mathring{W}_\xi) = 0$, i.e., the measure μ_ξ is concentrated in the "boundary" $W_\xi - \mathring{W}_\xi$. If this is the case, write $W_\xi - \mathring{W}_\xi = \cup\{F_q\}$ as a finite disjoint union of Borel sets such that, for each q, there is a chart $W_{\sigma(q)}\times R^l$ and a leaf-invariant homeomorphism $h_q: M\to M$ such that $h_q(F_q)\subset \mathring{W}_{\sigma(q)}$. For some q_0, $\mu_\xi(F_{q0}) > 0$, so that by Lemma A, we have $\mu_\tau(\mathring{W}_\tau) > 0$ for $\tau = \sigma(q_0)$.

In summary we have proved in this section the following.

PROPOSITION. *Suppose M is a smooth manifold foliated by compact l-dimensional leaves, and suppose there is a compact saturated subset X of M and a sequence of leaves $\{L_i\}$ in M converging to X. Then there is a subsequence of leaves $\{L_j\}$ and an associated sequence of positive integers $\{n_j\}$ such that the sequence of geometric currents $\{(1/n_j)L_j\}$ converges in the sense of de Rham to a nonzero geometric current $C\{\mu_\sigma\}: \Lambda_c^{\,l}(M)\to \mathbf{R}$ which is supported on X.*

Remark. This construction of a nontrivial limiting geometric current on a compact subset X of a smooth manifold M is possible even if M is not foliated, as long as X itself is smoothly foliated and the approaching compact submanifolds $\{L_i\}$ have tangent l-planes which converge to the tangent l-planes of the leaves of X (see e.g. [14], [17]). Also, these L_i's need not be compact; they can instead be assumed to satisfy a certain growth-of-volume condition, e.g. subexponential growth. These conditions are pursued in [5] and [11], for example.

§4. HOLONOMY AND THE VOLUME FUNCTION

In this section we recall the volume-of-the-leaves function, and the relation between its continuity properties and the holonomy of the foliation.

We suppose in this section that M is a smooth manifold without boundary, foliated by smooth compact submanifolds of condimension k. (Actually, with suitable reinterpretation, the material in this section is valid for any locally compact metric space which is topologically foliated by compact leaves. See [4]). First we introduce some notation which will be used frequently throughout the paper. The leaf of M through a point $x\in M$ is denoted L_x. Given a compact leaf L_x, let W be an open k-disc bundle neighborhood of L_x, with bundle retraction $p: W\to L_x$, whose disc fibers $\{p^{-1}(y)|y\in L_x\}$ are all transverse to the foliation. Let D_y denote $p^{-1}(y)$.

Let α be a loop in L_x, based at x. The basic property of a foliation is that some open neighborhood U of x in D_x can be translated around α, through the transverse discs $D_{\alpha(t)}$ to be reembedded into D_x, fixing x. The group of germs at x of such open embeddings is by definition the holonomy group of L_x, which we denote $\mathcal{H}_x$.

Suppose X is an invariant (= saturated) subset of M, that is, X is a union of leaves of M. If $x \in X$, then each holonomy embedding germ takes a neighborhood of x in $D_x \cap X$ into $D_x \cap X$, thus inducing an open embedding germ of $D_x \cap X$ at x. The group of such restricted germs, i.e. the holonomy group at x of the foliation restricted to X, is denoted $\mathcal{H}_x|X$.

If X is a locally compact invariant subset of M, we say that the restricted foliation on X is *hausdorff* if each leaf in X has arbitrarily small invariant open neighborhoods in X. That is to say, the quotient space of X by its leaves is hausdorff. In this case, the quotient space is locally compact metric and the quotient map $\pi: X \to X/\text{leaves}$ is a proper map (see e.g. [4, Theorem 4.1]). Hence, given any $x \in X$, there are arbitrarily small neighborhoods $\{V\}$ of x in $D_x \cap X$ which are saturated with respect to leaf interesctions with D_x, that is, for any $y \in V$ we have $L_y \cap D_x \subset V$. Such a small transversal V in X is invariant under the holonomy translations defined by the fundamental group of L_x, and consequently the holonomy epimorphism factors into two epimorphisms

$$\pi_1(L_x) \to H_V \to \mathcal{H}_x|X$$

where H_V denotes the image of $\pi_1(L_x)$ in the group of homeomorphisms of V. Thus each holonomy element can be represented by an automorphism of V, and $\mathcal{H}_x|X = \text{direct limit } \{H_V\}$, where the connecting homomorphism $H_{V_0} \to H_{V_1}$ can be assumed epic. It is advantageous to understand when this limit is stable, i.e. when the connecting epimorphisms are isomorphisms (see the example at the end of this section). We shall see that it is in certain natural cases (Proposition 4.1 below).

Let M be given a fixed riemannian metric. This induces on each leaf a riemannian metric hence a natural measure. (This is defined by assigning orientations locally and using the locally defined positive volume form determined by the orientation and the riemannian metric. Note that this measure is independent of the choices). Thus each oriented leaf L has a certain volume, denoted vol L, and we define a function $\text{vol}: M \to (0, \infty)$ given by $x \to \text{vol } L_x$. The following proposition gives a convenient (if lengthy) listing of all the properties of this function that we use. The proposition reflects work of Ehresmann, Reeb, Haefliger and Epstein and the second author. Full details are provided in [4, 8]. We offer below a sketch of the proofs.

PROPOSITION 4.1. (*Properties of the volume function*). *Suppose M is a smooth manifold-without-boundary, smoothly foliated by continuously oriented compact submanifolds, and suppose that X is any locally compact invariant subset of M. Then the restricted volume function* $\text{vol}|X: X \to (0, \infty)$ *has the following properties, at any* $x \in X$:

I. Discrete lower-semicontinuity (*see Figure* 1 *below*). *vol|X is lower-semicontinuous at x in the following especially discrete manner: for any integer* $n > 0$ *and any* $\epsilon > 0$, *it is true that for any y in a sufficiently small neighborhood of x in X: either* (i) $\text{vol } L_y > n \text{ vol } L_x$, *or* (ii) *there exists an integer j*, $1 \le j \le n$, *such that* $|\text{vol } L_y - j \text{ vol } L_x| < \epsilon$.

Consequently, the subset of X consisting of all points of continuity of vol $|X$, *is an open dense subset of X.*

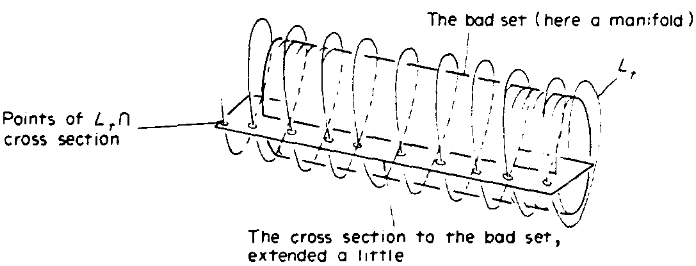


FIG. 1.

II. Continuity. *The following conditions are equivalent:*

(i) *the restricted volume function* vol $|X$ *is continuous at x.*

(ii) *the restricted holonomy group* $\mathcal{H}_x|X$ *is trivial.*

(iii) (*letting* $p: W \to L_x$ *and* $D_x = p^{-1}(x)$ *be the bundle retraction and transverse disc described above*). *There exists a transversal neighborhood V of x in* $D_x \cap X$ *such that each holonomy translation along any loop in* L_x *carries V identically onto itself. That is to say, the limit of automorphisms* $\varinjlim\{H_V\}$ *described above is stable and trivial.*

III. Boundedness. *The following conditions are equivalent*:

(i) *the restricted volume function* vol$|X$ *is bounded on some neighborhood of x in X.*

(ii) *the restricted holonomy group* $\mathcal{H}_x|X$ *is finite.*

(iii) (*letting* $p: W \to L_x$ *and* $D_x = p^{-1}(x)$ *be as described above*). *There exists a transversal neighborhood V of x in* $D_x \cap X$ *such that each holonomy translation along any loop in* L_x *carries V onto itself, and the group of automorphisms* H_V *of V so produced is finite and isomorphic to the holonomy group* $\mathcal{H}_x|X$. *That is to say, the sequence of automorphism groups* $\{H_V\}$ *described above is stable and finite, and* $\varinjlim\{H_V\} \approx \mathcal{H}_x|X$.

Remark 1. In case the conditions of part III hold, then the structure of X near L_x is particularly nice (these comments apply for part II, also). The union of all leaves which intersect a sufficiently small open transversal V from (iii) provides an open invariant neighborhood, U say, of L_x in $X \cap W$ such that the restriction $p|: U \to L_x$ is a fiber bundle projection, with fiber V and group H_V.

Remark 2. Epstein[4] and Millett[8] have observed that in part III, if X is an open subset of M, then the equivalent conditions are all true merely under the assumption that the quotient space X/leaves is hausdorff. This fact is based on a clever generalization of a theorem of Montgomery[9] concerning pointwise periodic homeomorphisms of manifolds. This paper does not make use of this fact.

The following figure, conveyed to us by Epstein, captures graphically the discrete lower-semicontinuous behavior of the restricted volume function vol$|X$ at a typical point x in X (see part I of the Proposition above). The horizontal axis represents the distance from x of an arbitrary point $y \in X$, the vertical axis represents the volume vol L_y of the leaf through y, and the crosshatched region represents the allowable values for vol L_y.

$$r = \text{the distance } \|y - x\|, \text{ for } y \in X.$$

Allowable values for vol L_y, for $y \in X$ lying near a fixed $x \in X$.

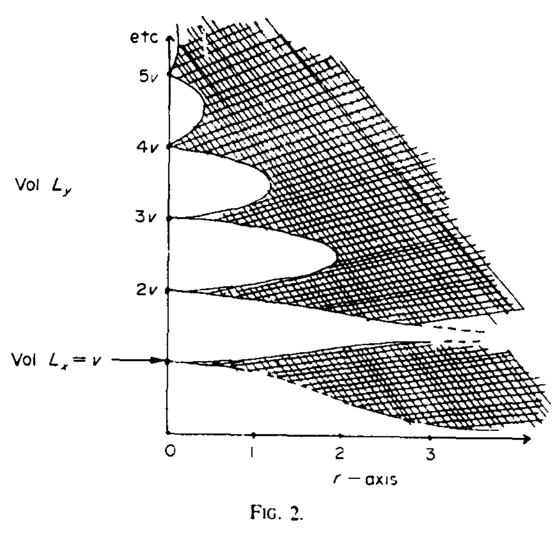


FIG. 2.

Proof of Proposition. Given $x \in X$, the basic data are the fixed open tubular neighborhood W of the leaf L_x and the bundle retraction $p: W \to L_x$ described above, containing the transverse open k-disc $D_x = p^{-1}(x)$. Even though W is not necessarily invariant, it is true that for each $y \in W$ the restriction $p|$: $W \cap L_y \to L_x$ is a codimension 0 submersion (i.e., a local open embedding).

To prove part I, and also to establish Figure 1, one observes that if y is sufficiently close to x, then the image $p(W \cap L_y)$ must cover all of L_x. In fact, fixing any integer $n > 0$, then if y is sufficiently close to x, it must be the case that either (i) $p|W \cap L_y$ is greater than n-to-1 everywhere, or (ii) $p|W \cap L_y$ is a j-to-1 covering projection, for some j, $1 \le j \le n$. This establishes the first part of I. The consequent denseness assertion is simply the fact that the points of continuity of a semicontinuous function on a locally compact space comprise a dense subset (which in general is G_δ, but here it is open).

Part II is established by keeping in mind again the fixed smooth submersion-retraction p: $W \to L_x$ mentioned above, and by arguing that each of conditions (i), (ii) and (iii) is equivalent to the following: if $y \in X$ is sufficiently close to x, then the leaf L_y intersects D_x in precisely a single point. The only mildly subtle point in these equivalences is the fact that (ii) implies this property; this uses the fact that the holonomy group is finitely generated.

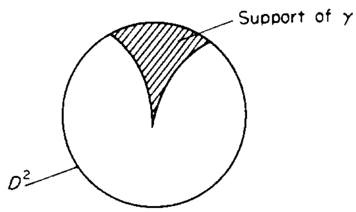
In part III, we always work inside the above-mentioned tubular neighborhood W of the leaf L_x in M. It seems most convenient to show separately the equivalence of (iii) to each of (i) and (ii). The implication (ii)$\Rightarrow$(iii) is a general fact about finite groups of germs. Let $\{f_i: V_i \to D_x \cap X\}$ be a finite collection of open embeddings of open neighborhoods $\{V_i\}$ of x in $D_x \cap X$, with distinct germs $[f_i]$, such that these germs comprise the holonomy group $\mathscr{H}_x|X_0$. We assume that one of the f_i's, say f_0, is the identity on its domain V_0. For each pair i, j of indices, let $k = k(i, j)$ denote the (unique) index such that $[f_j] \circ [f_i] = [f_k]$ (as germs). By shrinking the domains $\{V_i\}$ if necessary, we can assume that for each pair i, j, $f_jf_i|(V_i \cap f_i^{-1}(V_j) \cap V_k) = f_k|$ same. Define $V = \bigcap_{i,j}(V_i \cap f_i^{-1}(V_j))$ and $h_i = f_i|V$. Then $H_V \equiv \{h_i\}$ is a group of automorphisms of V, isomorphic to the holonomy group $\mathscr{H}_x|X$. The implication (iii)$\Rightarrow$(ii) is clear.

The implication (iii)$\Rightarrow$(i) is straightforward. Let U be the union of all leaves intersecting an open transversal V in X provided by (iii). If V is sufficiently small, then $U \subset W$. If (i) fails, there must be a leaf L in U which intersects V in an arbitrarily large finite collection of points, say $\{y_0, y_1, \dots, y_p\}$. For each $j \ge 1$, there is a path in L from y_0 to y_j, hence there is a holonomy automorphism h_j: $V \to V$ such that $h_j(y_0) = y_j$ (h_j is induced by following the projected image of this path in L_x). This shows that H_V must be arbitrarily large, hence infinite, contradicting (iii).

For the final implication (i)$\Rightarrow$(iii), one argues that if there is a bound on vol L_y for y in X near x, then given any neighborhood U_0 of L_x in $W \cap X$, there is a smaller neighborhood U_1 of L_x in X such that the union U of all leaves in X which intersect U_1, lies in U_0. Otherwise there would be leaves in X which intersect both $X - U_0$ and any arbitrarily small U_1, which would force these leaves to have large volume. Now U is an open invariant neighborhood of L_x in U_0, and its intersection with D_x, call it V, gives a subset of $D_x \cap X$ which is taken homeomorphically onto itself by every holonomy translation of L_x. Let the group of automorphisms so produced be denoted H_V. It remains to show that H_V is finite.

By the assumption on bounded volume, we can assume that V is so small that there is an integer n such that each leaf L_y, $y \in V$, interesects V in n or fewer points. In particular, then, the orbit of any point of V under H_V has n or fewer points. We now use an argument of Epstein [4,§7.2], which shows that given any finitely generated group H_V of automorphisms of any set V, if the orbit of each point in V has $\le n$ points (n fixed), then H_V is finite. This is proved as follows. For each orbit $Y \subset V$, arbitrarily number its points $Y = \{y_1, \dots, y_{n(Y)}\}$, where $n(Y) \le n$. For each orbit Y, this provides a homomorphism $H_V \to S(n(Y)) \subset S(n)$ to the symmetry group $S(n)$, by restricting to Y. Since H_V is finitely generated, there are only finitely many homomorphisms $H_V \to S(n)$. Therefore we can group the orbits $\{Y\}$ into finitely many disjoint collections $\{Y\}_i$, according to the homomorphism each orbit determines. Now each $h \in H_V$ is completely determined by its images $\{h_i\}$ in $S(n)$, one image h_i for each collection $\{Y\}_i$. That is, h is determined by a finite number of choices from the finite group $S(n)$. Hence H_V is finite. This completes the proof of the Proposition.

To complete this section, we offer an example to illuminate the earlier discussion on the stability of the limit, $\mathscr{H} =$ direct limit $\{H_V\}$. This is an example of a codimension 2 foliation of a manifold, $L \times D^2$, which has a compact leaf $L = L \times 0$ with arbitrarily small invariant neighborhoods, such that the above limit is not stable when $\mathscr{H}$ is the holonomy group of L. First, we define three diffeomorphisms of the 2-disc D^2, α, β and γ, each of which is radial-distance preserving (i.e. $\|f(x)\| = \|x\|$, for each $x \in D^2$). For convenience, we regard S^1 as $R^1 \cup \infty$, and write $D^2 = \cup\{tS^1 | 0 \le t \le 1\}$. We leave it to the reader to make the following discussion smooth. Define a homeomorphism α_1: $S^1 \to S^1$ by $\alpha_1(\infty) = \infty$ and $\alpha_1(x) = x/2$, for $x \in R^1$. Let α: $D^2 \to D^2$ be gotten by coning α_1 to the origin, that is, $\alpha(x) = \|x\| \cdot \alpha_1(x/\|x\|)$, for $x \in D^2 - 0$, and $\alpha(0) = 0$. Let β_1: $S^1 \to S^1$ be a homeomorphism such that $\beta_1 = \alpha_1$ on $[-1, 1] \subset R^1 \subset S^1$, and $\beta_1 =$ identity on a neighborhood of $\infty \in S^1$, i.e., $\beta_1 | R^1$ has compact support. Let β: $D^2 \to D^2$ be gotten by coning β_1 to the origin. Finally, let γ_1: $S^1 \to S^1$ be a homeorphism such that for some interval $[-c, c]$, $\gamma_1 | S^1 - [-c, c] =$ identity, and in addition $|\gamma_1(x)| < |x|$ for $x \in (-c, c)$, $x \neq 0$. That is, γ_t: $S^1 \to S^1$ by $\gamma_t(x) = t\gamma_1(x/t)$, for $0 < t \le 1$, (here multiplication is in the R^1 structure on S^1) and define γ: $D^2 \to D^2$ by $\gamma | tS^1 = \gamma_t$, for $0 < t \le 1$, and $\gamma(0) = 0$. Hence, rather than defining γ simply to be the cone on γ_1, it is defined by a scaled coning process, so that the support of γ looks like a cone in D^2 which has been pinched at the origin.



Let H be the group of homeomorphisms of D^2 generated by α, β and γ. Then the group of germs at 0 is $\mathcal{H} = \text{direct} \lim_{t \to 0} H | V_t$, where V_t is the disc of radius t. This limit is not stable, for there is a sequence of homeomorphisms in H, which have identity germ, but which move points which lie arbitrarily close to 0. A sequence of such homeomorphisms is $(\alpha^{-n}\gamma\alpha^n)^{-1}\beta^{-n}\gamma\beta^n$, $n \to \infty$.

To build a foliation with G as the holonomy group of some leaf L, let L be a closed manifold for which there is an epimorphism $\pi_1(L) \to H$, e.g., L a 3-holed torus. On $L \times D^2$, construct a codimension 2 foliation, transverse to the D^2's, by means of this homomorphism. Then the leaf $L = L \times 0$ has holonomy group G.

§5. THE MOVING LEAF PROPOSITION

In this section M continues to be a smooth manifold-without-boundary which is foliated by continuously oriented compact submanifolds. For convenience we assume in addition that M is oriented, and hence that all of the holonomy is oriented.

Let X_1 denote the set of all points $x \in M$ at which the volume function is not bounded, i.e., $x \not\in X_1$ iff there is no neighborhood of x in M on which the volume function is bounded. Then X_1 is a closed invariant subset of M, which Epstein has called the *bad set*. The basic question is whether $X_1 = \emptyset$. We know at least that X_1 has no interior in M, since it can be expressed as a countable union of closed subsets, $X_1 = \cup_{n=1}^{\infty} (X_1 \cap \text{vol}^{-1}(0, n])$, each of which by the definition of X_1, has no interior in M. If X_1 is compact and nonempty, the following proposition provides the data for the construction of a geometric current supported on X_1, as described in §3.

MOVING LEAF PROPOSITION. If X_1 is compact and nonempty, then there exists a sequence of generic leaves $\{L_i\}$ ($\equiv$ leaves with trivial holonomy) which lie in $M - X_1$ such that

(i) the L_i's converge to X_1 and are homologous to each other near X_1. Precisely stated: Given any neighborhood U of X_1, there is an integer i such that for all $j, k > i$, the leaves L_j and L_k lie in U and are homologus in U, and

(ii) $\lim_{i \to \infty} \text{vol}(L_i) = \infty$.

If X_1 is noncompact (i.e. only closed in M), the Proposition fails. This is the key place where the proofs of Theorems 1 and 2 break down, in the case that X_1 is not compact.

We note that the proof below shows that the L_i's may in fact be chosen as the integer-point images $L_i = h_i(L)$ of a homotopy of leaves h_t: $L \to M - X_1$, $0 \le t < \infty$ (which can even be an embedded family of leaves), where for each t, $h_t(L)$ is a generic leaf of M and $h_t(L)$ approaches X_1 as $t \to \infty$.

Remarks. (Neither of which is used in this paper, but they are of independent interest.)

1. This Proposition remains true in the absence of any orientation assumptions whatever. This is most easily proved by reducing the unoriented cases to the oriented case by taking covers, c.f. §§7,8.

2. Although we have assumed in this paper that M and its foliation are smooth, the Proposition in fact holds for topological foliations of topological manifolds (where the volume functions is defined in the general manner of [4, §3]). The proof below adapts quite readily.

Proof of Proposition. Our goal is to construct an infinite path of generic leaves in some component of $M - X_1$, such that the path of leaves approaches X_1 and the volumes of the leaves in the path are unbounded.

The generic leaves of $M - X_1$ comprise an open dense subset G say, such that $(M - X_1) - G$ is a countable union of smooth submanifolds of codimension ≥ 2. This is a consequence of the local boundedness of the volume function on $M - X_1$, the orientation hypothesis and the structure theorem of Ehresmann (see [4, Theorem 4.3]). In more detail:

Suppose L is a leaf in $M - X_1$. By Proposition 4.1 III, the holonomy at L is finite. By assumption, the holonomy is orientation preserving, so by the structure theorem [*ibid*], the holonomy group can be identified with a finite subgroup of $SO(k)$ acting on a disc D^k transverse to the foliation at some point of L. Now, any nontrivial orientation preserving linear transformation of finite period has fixed point set of codimension ≥ 2. Hence the set of leaves near L having nontrivial holonomy is a finite union of submanifolds having codimension ≥ 2 in M. Thus the set of generic leaves in $M - X_1$ has the properties stated above.

From now on we argue assuming that M is compact. We leave it to the interested reader to make the simple changes necessary when only X_1 is assumed to be compact.

Let $\{N_i\}$ be the (countable) collection of components of $M - X_1$. Recall that for each i, frontier $N_i \subset X_1$, hence $N_i \cap X_1$ is a compact invariant subset of M. The bulk of our proof is to show that for some i, vol $|N_i$ is unbounded. We should point out that if dim $X_1 \leq$ dim $M - 2$, then $M - X_1$ has only one component (assuming M is connected), and so this fact is a straightforward consequence of Proposition 4.1 I (cf. the next few sentences).

Let $x \in X_1$ be a point of continuity of the restricted function vol $|X_1$ (Proposition 4.1 I). To establish the above unboundedness fact, we suppose conversely that for each i, vol $|N_i$ is bounded (with possibly varying bounds), and show that as a consequence the full holonomy group $\mathcal{H}_x$ at x is trivial. This will imply that the volume function is continuous at x in M (Proposition 4.1 II), contradicting that $x \in X_1$. Our argument is adapted from a proof of Montgomery[9], which in turn rests on a theorem of Newman[10].

To begin, let D^k be a transverse open k-disc at x in M, so small that vol $|(X_1 \cap \text{cl}D^k)$ is continuous and therefore also bounded. Let $[f]$ be an element of the holonomy group $\mathcal{H}_x$, which we can take to be represented by an embedding $f\colon V \to D^k$, fixing x, of some open connected neighborhood V of x in D^k. Since vol $|V \cap X_1$ is continuous, we have that f = indentity on $V \cap X_1$ (Proposition 4.1 II). We will show that f = identity on V.

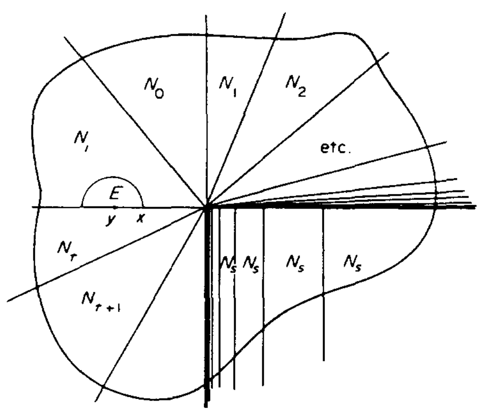


Fig. 3.

It suffices to show that for each i, f = identity on $V \cap (N_i \cup X_1)$. (We emphasize that it does *not* suffice to show that for each i, f = identity on a neighborhood of x in $V \cap (N_i \cup X_1)$. To do this, we will use the fact, which is a consequence of the boundedness of vol $|V \cap (N_i \cup X_1)$, that for each point y of $V \cap (N_i \cup X_1)$ which is fixed by f, there is a neighborhood E of y in $V \cap (N_i \cup X_1)$ and an integer $p \geq 1$ such that $f(E) = E$ and $f^p|E$ = identity (Proposition 4.1 III; with "x" = y, "X" = $V \cap (N_i \cap X_1)$, and "V" = E). Having fixed i, define a new map f_i: $V \to D^k$ by letting $f_i|V \cap (N_i \cup X_1) = f|V \cap (N_i \cup X_1)$ and letting $f_i|V - N_i$ = identity. This is well-defined because the intersection of the two closed domains of definition lies in $V \cap X_1$, where f = identity. We claim that f_i = identity on V (hence f = identity on $V \cap (N_i \cup X_1)$). For let A_i be the fixed point set of f_i in V, and let $B_i = \text{cl}(\text{int}\, A_i) \subset A_i$. Then $B_i \neq \emptyset$, since $V - (N_i \cup X_1)$ is nonempty, by our hypotheses that vol $|N_i \cup X_1$ is bounded at x but vol $|V$ is not. If $B_i \neq V$, then there exists a point y in the frontier of B_i in V. By the above remarks, f_i has periodic germ at y, i.e. there exists an open neighborhood E of y in V such that $f_i^p|E$ = identity, for some $p \geq 1$. Then $f_i|E$ = identity, as a conseqeuence of Newman's theorem (which is nicely explained in [1]). Hence $E \| B_i$, contradicting that $y \in$ frontier of B_i in V. Consequently, f = identity on $V \cap (N_i \cup X_1)$ and so the first part of the argument is complete.

At this point we have established the existence of a component of $M - X_1$, N_1 say, such that vol $|N_1$ is not bounded. Hence there exists a point $y \in \text{fr} N_1 \subset X_1$ such that vol $|\text{cl}\, N_1$ is not bounded at y. Let $\{y_i\}$ be a sequence of points in N_1 converging to y such that vol $L_i \to \infty$, where L_i is the leaf through y_i. Clearly we can assume the L_i's are generic leaves, by our earlier remarks. We wish to choose an infinite path in the open subset of generic leaves G_1 of N_1, say α: $[0, \infty) \to G_1$, which pierces a subsequence of the y_i's, and which converges to X_1. Recall that $N_1 - G_1$ is a countable union of codimension ≥ 2 submanifolds of N_1, so that any path α: $[0, \infty) \to N_1$ (with image in N_1) having the other properties, can be perturbed by an arbitrarily small homotopy to in addition lie in G_1. We should point out that it may not be possible to choose any path α in N_1 which converges to y itself, as y may not be pathwise accessible from N_1. Nevertheless, to get the path α in N_1 as asserted, involves only a simple argument with the ends of N_1, which goes as follows. Recall that for any compact subset C of N_1, the complement $N_1 - C$ has at most finitely many components which have noncompact closures in N_1. So some subsequence of the y_i's must lie in one of these unbounded, path-connected components. Doing this refinement process for ever larger compact subsets C of N_1, selecting subsequences of subsequences of $\{y_i\}$, and taking a final diagonal subsequence, one can construct a path α in N_1, as claimed.

Let L_t denote the leaf through the point $\alpha(t)$. The family of leaves L_t, $0 \leq t < \infty$, converges to X_1 in the sense stated in the Proposition, since X_1 has arbitrarily small invariant neighborhoods in M. This is a consequence of the fact that $M - X_1$ contains arbitrarily large invariant subsets which are closed in M, by Proposition 4.1 III. This completes the proof of the Moving Leaf Proposition.

§6. THE NATURAL FILTRATION OF THE BAD SET

In this section we assume M is a smooth manifold-without-boundary (not necessarily compact), foliated by compact submanifolds which are oriented in a continuous manner. We wish to show that any geometric current $C\{\mu_\sigma\}$ (as defined in §2) which is supported on the bad set X_1, can be expressed in a more convenient manner via a measure ν defined on the quotient space $X_1/\mathcal{F}$ of X_1 by its leaves.* In short, we aim for an expression of the following form, which is to hold for any compactly supported l-form ω defined on M:

$$\langle C\{\mu_\sigma\}, \omega\rangle = \int_{L \in X_1/\mathcal{F}} \langle L, \omega\rangle d\nu$$

(L should be thought of both as a point and a leaf; recall that the left hand side is defined in §2).

Unfortunately the quotient space $X_1/\mathcal{F}$ may not be hausdorff, in which case the existence of the measure ν is not so obvious. In brief, the reason we are able to find ν and to write the above equation, is that the quotient $X_1/\mathcal{F}$ can be expressed as a countable disjoint union of subsets which are locally compact metrizable.

*All of the discussion in this section applies as well to any locally compact invariant subset X_1 of M, not just the bad set.

First we give a simple technical lemma, which treats the case when $X_1/\mathcal{F}$ is hausdorff (hence locally compact and metrizable; see e.g. [4, Theorem 4.1]).

LEMMA. *Suppose X is any locally compact invariant subset of M such that the restriction of the volume function to X is continuous, and suppose $C\{\mu_\sigma\}$ is a geometric current supported on X. Then there is a nonnegative measure ν induced on the (locally compact metrizable) quotient space $X/\mathcal{F}$ with the property that the de Rham current C_ν: $\Lambda_c^l(M) \to \mathbf{R}$ defined by*

$$\langle C_\nu, \omega\rangle = \int_{L\in X/\mathcal{F}} \langle L, \omega\rangle d\nu$$

agrees with the current defined by $C\{\mu_0\}$.

Proof. In brief, the point is that X is a fiber bundle with fibers = leaves and base = $X/\mathcal{F}$, and there is a measure ν naturally induced on the base. In detail, we argue that for any open transversal $\mathring{W}_\sigma \subset M$, the restriction to $\mathring{W}_\sigma \cap X$ of the quotient map $\pi: X \to X/\mathcal{F}$ is locally an open embedding, since the holonomy $\mathcal{H}_x|X$ is trivial at each $x \in X$. Define a measure ν on $X/\mathcal{F}$ by requiring this restriction to be locally an isomorphism of the measure $\mu_\sigma|(\mathring{W}_\sigma \cap X)$ with $\nu|(\mathring{W}_\sigma \cap X)/\mathcal{F}$. This is well defined because of the translation invariance of the measures $\{\mu_\sigma\}$. To show that the de Rham current C_ν defined using ν agrees with the current defined by $C\{\mu_\sigma\}$, it suffices to check agreement on the l-form ω which is compactly supported in a single open chart $\mathring{W}_\sigma \times R^l$ in M. Consider the following sequence of equalities.

$$\begin{aligned}\langle C_\nu, \omega\rangle &= \int_{L\in X/\mathcal{F}} \langle L, \omega\rangle d\nu \\ &= \int_{L\in(\mathring{W}_\sigma\cap X)/\mathcal{F}} \langle L \cap (\mathring{W}_\sigma \times R^l), \omega > d\nu \\ &= \int_{x\in \mathring{W}_\sigma} \langle x \times R^l, \omega\rangle d\mu_\sigma \\ &= \langle C\{\mu_\sigma\}, \omega\rangle.\end{aligned}$$

The first equality is the definition of C_ν; the second is simply the fact that ω is supported on $\mathring{W}_\sigma \times R^l$. The third equality is the key; it is a change of variables using the fact that the measure μ_σ on $\mathring{W}_\sigma$ is the pullback of the measure $\nu|(\mathring{W}_\sigma \cap X)/\mathcal{F}$ by the quotient map $\pi|$: $\mathring{W}_\sigma \cap X \to (\mathring{W}_\sigma \cap X)/\mathcal{F}$. The last equality is the definition of $C\{\mu_\sigma\}$.

We wish to apply this Lemma to the bad set X_1, but unfortunately the volume function may not be continous on X_1. To circumvent this problem we employ Epstein's filtration $\{X_\alpha\}$ of X_1 into a decreasing collection of closed subsets on whose successive differences the volume function is continuous [3, §6].* The filtration is indexed by the ordinals ≥ 1. It is defined as follows. Suppose that for each ordinal α less than some given ordinal β a collection of closed subsets $\{X_\alpha | \alpha < \beta\}$ of X_1 has been defined. Then X_β is defined according to one of the following two cases: if β is a limit ordinal let $X_\beta = \bigcap_{\alpha<\beta} X_\alpha$, and if β is a successor ordinal, say $\beta = \alpha + 1$, let $X_\beta = \{x \in X_\alpha |$ vol $|X_\alpha$ is not continuous at $x\}$. (Note: this filtration commences in a different place than Epstein's, at X_1 instead of M. Hence the above sets $\{X_\alpha\}$ may not coincide with Epstein's $\{B_\alpha\}$).

For each $\beta > \alpha$, X_β is a closed nowhere dense subset of X_α (Proposition 4.1 I). Eventually $X_\beta = \emptyset$, as all the points of X_1 are exhausted. Let δ be the least ordinal for which $X_\delta = \emptyset$. Then we note that δ is a successor ordinal, for otherwise $\emptyset = X_\delta = \bigcap\{X_\alpha | \alpha < \delta\}$, which violates the finite intersection property for compact sets. Furthermore, δ is a countable ordinal. To see this, let υ be some fixed countable basis for the topology of M. For each $\alpha < \delta$ there is a member $V_\alpha \in \upsilon$ such that $V_\alpha \cap X_\alpha \neq \emptyset$ and $V_\alpha \cap X_{\alpha+1} = \emptyset$. This assignment $\alpha \to V_\alpha$ is one-to-one.

Using this countable filtration $\{X_\alpha\}$ of the bad set X_1 we can obtain the desired alternative description of the geometric current $C\{\mu_\sigma\}$ supported on X_1.

PROPOSITION. *Given a geometric current $C\{\mu_\sigma\}$ supported on X_1, there is a collection of measures $\{\nu_\alpha\}$, where each ν_α is defined on the locally compact metrizable quotient*

*All subsequent discussion could also be carried out using the coarse Epstein filtration, determined by the local boundedness of the volume function instead of its continuity. This so is because the above Lemma also holds under the weaker assumption that the volume function is locally bounded on X, instead of continuous, by Proposition 4.1.III.

$\{(X_\alpha - X_{\alpha+1})/\mathcal{F}\}$, *such that the two associated de Rham currents* $C\{\mu_\sigma\}$ *and* $\Sigma_\alpha C_{\nu/\alpha}$: $\Lambda_c^l(M)\to\mathbf{R}$ *agree, where (as above)*

$$\langle C_{\nu/\alpha}, \omega\rangle = \int_{L\in(X_\alpha - X_{\alpha+1})/\mathcal{F}} \langle L, \omega\rangle d\nu_\alpha.$$

Proof. For each σ and α define $\mu_{\sigma,\alpha} = \mu_\sigma | W_\sigma \cap (X_\alpha - X_{\alpha+1})$, and note that $\mu_\sigma = \Sigma_\alpha \mu_{\sigma,\alpha}$. For each α, the collection of measures $\{\mu_{\sigma,\alpha}\}$ is translation invariant, and so this collection defines a geometric current $C\{\mu_{\sigma,\alpha}\}$. Clearly as geometric currents, $C\{\mu_\sigma\} = \Sigma_\alpha C\{\mu_{\sigma,\alpha}\}$. For any given α, we know that vol $|X_\alpha - X_{\alpha+1}$ is continuous, and so the preceding lemma shows that the de Rham current defined by $C\{\mu_{\sigma,\alpha}\}$ is equal to C_{ν_α}, where ν_α is the measure induced on $(X_\alpha - X_{\alpha+1})/\mathcal{F}$. Summing over α, the Proposition follows.

Remark. It is illuminating to note another interpretation of the geometric current $C\{\mu_\sigma\}$ obtained by considering it as an l-dimensional real homology class. Each difference $X_\alpha - X_{\alpha+1}$ is a fiber bundle over its quotient $(X_\alpha - X_{\alpha+1})/\mathcal{F}$, since the holonomy is trivial. Assuming for the moment that $X_\alpha - X_{\alpha+1}$ is connected, let L_α be a leaf in $X_\alpha - X_{\alpha+1}$, and recall that the homology class of L_α (in real Čech homology $\check{H}_l(X_\alpha - X_{\alpha+1}; \mathbf{R})$) is independent of this choice. We have then that as homology classes in $\check{H}_l(X_1; \mathbf{R})$,

$$[C\{\mu_\sigma\}] = \Sigma_\alpha t_\alpha [L_\alpha],$$

(where $t_\alpha = \nu_\alpha((X_\alpha - X_{\alpha+1})/\mathcal{F})$), since the evaluations of each side on closed l-forms agree. If $X_\alpha - X_{\alpha+1}$ is not connected but has countably many components, this same type of analysis holds; however if $X_\alpha - X_{\alpha+1}$ has uncountably many components, this analysis must be done using a Čech type limiting process.

We note that the above sort of homological equality also holds even if we use the coarse Epstein filtration $\{X_\lambda\}$ on X_1 given by using the local boundedness of the volume function (c.f. earlier footnote). That is, if we filter X_1 by throwing away at each stage the points where the volume function is bounded (instead of continuous), obtaining $\{X_\lambda\}$, then the geometric current $C\{\mu_\sigma\}$ can be expressed as a homology class in $\check{H}_l(X_1, \mathbf{R})$ by

$$[C\{\mu_\sigma\}] = \Sigma_\lambda s_\lambda [L_\lambda].$$

This is because (again assuming each $X_\lambda - X_{\lambda+1}$ is connected) the leaves of $X_\lambda - X_{\lambda+1}$ are all positive multiples of each other in homology. If $X_\lambda - X_{\lambda+1}$ is not connected, then the same comments as above, concerning the connectedness of $X_\alpha - X_{\alpha+1}$, also apply here.

§7. PROOF OF THEOREM 1

Suppose the bad set is compact and nonempty, and suppose there exists a closed l-form, defined on some neighborhood of the bad set, whose integral on each leaf on the bad set is always positive. We shall apply the results established in the previous sections to get a contradiction.

As already noted in §1, we can assume that M itself is oriented (and hence the foliation is transversally oriented), by taking a double cover if necessary.

Let X_1 be the bad set and let ω be the l-form. Choose a sequence $\{L_i\}$ of generic leaves of M approaching the bad set, with volumes going to infinity, as described in the Moving Leaf Proposition (§5). Let $C\{\mu_\sigma\} = \lim_{j\to\infty} 1/n_j L_j$ be the geometric current constructed in §3, where the normalizing constants $\{n_j\}$ go to infinity because the volumes of the L_j's go to infinity. Then, using the fact that the L_j's are homologous near the bad set and ω is closed, we have that

$$\begin{aligned}\langle C\{\mu_\sigma\}, \omega\rangle &= \left\langle \lim_{j\to\infty} \frac{1}{n_j} L_j, \omega\right\rangle &&\text{(definition of } C\{\mu_\sigma\}) \\ &= \lim_{j\to\infty} \frac{1}{n_j}\langle L_j, \omega\rangle &&\text{(see Proposition, §3)} \\ &= \lim_{j\to\infty} \frac{\text{constant}}{n_j} = 0.\end{aligned}$$

On the other hand we can use the Proposition in §6 to write $\langle C\{\mu_\sigma\}, \omega\rangle = \Sigma_\alpha \langle C_{\nu_\alpha}, \omega\rangle$. To establish a contradiction, it suffices to show that for some α, $\langle C_{\nu_\alpha}, \omega\rangle > 0$. According to Lemma C in §3, $C\{\mu_\sigma\}$ is a nontrivial geometric current (i.e. $\mu_\sigma(\mathring{W}_\sigma) > 0$ for some σ), so there is an α such that $\nu_\alpha((X_\alpha - X_{\alpha+1})/\mathcal{F}) > 0$. But then clearly $\langle C_{\nu_\alpha}, \omega\rangle > 0$, since the value $\langle C_{\nu_\alpha}, \omega\rangle$ is obtained by integrating a positive continuous function $\langle L, \omega\rangle$ over the set $(X_\alpha - X_{\alpha+1})/\mathcal{F}$, which has positive measure. This establishes the contradiction, proving Theorem 1.

The first consequence of Theorem 1 (see §1) follows immediately from the existence of a closed l-form on M whose integral on each leaf is the euler characteristic of the leaf. This form is simply the euler form of the bundle of tangents to the leaves of the foliation. It restricts to the euler form of the tangent bundle to each individual leaf. For the second consequence of Theorem 1, concerning foliations of kaehler manifolds, we note that the $(l/2)$-th power ω^l of the kaehler 2-form ω restricts to the volume form on the leaves, which are complex submanifolds. Thus for each leaf L we have $\langle L, \omega^l\rangle$ is positive, so that Theorem 1 applies. Note that, since ω^l is pointwise positive on leaves, a direct contradiction arises from the moving leaf proposition.

§8. CODIMENSION TWO FOLIATIONS

For foliations of compact 3-manifolds by circles Epstein[3] has demonstrated the existence of a bound on the length (= volume) of the circles. In this section we give a distillation of the ingredients in Epstein's proof which allows us to extend this theorem to codimension two foliations of compact manifolds of any dimension. This has also been done by Vogt.

THEOREM 2. *Suppose M is a smooth compact manifold which is smoothly foliated by compact leaves of codimension 2. Then there is an upper bound on the volume of the leaves of the foliation. Consequently, all holonomy groups of the foliation are finite.*

We note that there are no orientation assumptions here at all.

In brief, the point of the proof is that in the codimension 2 case, if everything is oriented and if the bad set is nonempty, then the homological assumption of Theorem 1 does in fact hold in a neighborhood of the bad set. Hence the bad set must be empty. This homology fact is established by a barehands construction of a 2-dimensional oriented surface in M, transverse to the foliation and closed in a neighborhood of the bad set, which intersects each leaf of the bad set. With the normal orientation this surface represents a nontrivial relative integral 2-cycle in M modulo the complement of the bad set. The real cohomological (Alexander) dual of this cycle is represented by the $(m-2)$-form ω that we hypothesized in Theorem 1.

The material in this section is independent of any of the preceding material on currents, but it does make use of the other material in §§4,5 and part of §6.

As before we assume without loss of generality, by doubling M if necessary, that M has no boundary. Also we assume, by passing to a 4-fold cover if necessary, that everything is oriented (the leaves, the normal bundle and M itself). Continuing our previous notation, let X_1 be the bad set of M, i.e., the compact set of leaves of M near which the volume function is not bounded. We wish to show that $X_1 = \emptyset$. Suppose instead that $X_1 \neq \emptyset$. Let $\{X_\alpha\}$ be the Epstein decreasing filtration of X_1 by compact saturated subsets, which is described in §6. The following proposition (for $\alpha = 1$) provides the key ingredient for showing that the bad set is empty; it will be proved by induction on decreasing α.

TRANSVERSAL PROPOSITION. *For each index $\alpha \geq 1$ there exists a smoothly embedded open 2-manifold $T_\alpha \subset M$ such that*

(i) *T_α is transverse to the foliation,*

(ii) *T_α is a closed subset of some open neighborhood U_α of X_α in M (i.e., T_α is properly embedded in U_α), and*

(iii) *T_α has nonempty intersection with each leaf in X_α.*

Suppose T_α is such a transverse surface in M. Then T_α is oriented in a natural manner. Each leaf L lying in U_α intersects T_α transversally in a finite number of points, and each intersection point is algebraically positive. So this number of geometric intersections is equal to the algebraic intersection number $L \cdot T_\alpha$.

Proof of Theorem 2 from the Transversal Proposition. Assuming that $X_1 \neq \emptyset$, let T_1 be the transverse surface associated with X_1, as promised by the Transversal Proposition. By the

Moving Leaf Proposition in §5, there is a sequence of leaves $\{L_i\}$ in U_1, all representing the same integral homology class in U_1, such that the leaves approach X_1 and their volumes increase without bound. From the homological considerations discussed above, the geometric intersection numbers $\#(L_i \cap T_1)$ must all be the same. On the other hand, these numbers must become arbitrarily large as $i \to \infty$, since the volumes of the L_i's go to infinity and therefore by Proposition 8.1 (III), they must pass more and more frequently through the transverse surface T_1. This contradiction establishes the Theorem.

We note that our geometric current argument in earlier sections was expressly designed to take the place of this transversal intersection number argument, in the absence of a transverse manifold T_1 lying near the bad set X_1.

Remark. The proof of the Transversal Proposition below actually shows the following: If M is a manifold (not necessarily compact) foliated by submanifolds of codimension 2 and if X is a closed nowhere dense invariant subset of M which is a union of compact leaves, then X is a Seifert fiber bundle, with leaves as fibers and base of dimension ≤ 1. Recall that a Seifert fibering of X is simply a foliation such that $\mathcal{H}_x|X$ is a finite group for each $x \in X$ (and hence a trivial group for all but a locally finite collection of leaves).

For use in the proof of the Transversal Proposition, we introduce a function, called the *intersection number function*, which will play a role analogous to that of the volume function in earlier sections.

Suppose $T \subset M$ is any smoothly embedded transverse open 2-manifold (not necessarily closed in M, but closed in some *open* subset of M) e.g. as in the Transversal Proposition. If L is a compact leaf in M, then the intersection $L \cap T$ is a discrete hence countable subset of T (but it may not be discrete, i.e. finite in M, if $L \cap (\text{cl } T - T) \neq \emptyset$). Let $\text{num}_T L$ denote the (finite or countably infinite) number of intersection points. Thus we have a function $\text{num}_T\colon M \to \mathbf{Z}_+ \cup \infty =$ nonnegative integers $\cup\, \infty$ given by $x \to \#(L_x \cap T)$. (We could if desired force this number to be finite, by working only with transverse surfaces $\{T\}$ which are *extendible* in the sense that they are relatively compact open subsets of larger transverse surfaces. However this modification would save us no effort).

The intersection number function behaves very much like the volume function, as the Proposition below shows. It would be most convenient if we could express the relation between these functions by saying that in some neighborhood of any given $x \in M$, the quotient function $\text{vol}/\text{num}_T\colon M \to (0, \infty)$ is continuous (even though the individual functions may not be continuous). However, this may not be true if $L_x \cap (\text{cl } T - T) \neq \emptyset$, or even if $L_y \cap (\text{cl } T - T) \not\to \emptyset$ for points $\{y\}$ in M arbitrarily close to x. This is the (only) subtle point to keep in mind when reading the following Proposition, whose statements parallel the statements of Proposition 4.1.

PROPOSITION 8.1. (*Properties of the intersection number function*). *Suppose M is a smooth manifold-without-boundary, smoothly foliated by compact submanifolds of codimension 2, with everything oriented. Suppose that $T \subset M$ is any smoothly embedded transverse open 2-manifold (not necessarily closed in M), and suppose that X is any locally compact invariant subset of M. Let $x \in X$ be such that $L_x \cap (\text{cl } T - T) = \emptyset$. Then the restricted intersection number function* $\text{num}_T|X$: $X \to$ *nonnegative integers* $\cup\,\infty$ *has the following properties at* x:

I. $\text{num}\, T|X$ *is lower-semicontinuous at x, that is, for any $y \in X$ which is sufficiently close to x,* $\text{num}_T L_y \geq \text{num}_T L_x$.

II. $\text{num}_T|x$ *is continuous at* $x \Leftrightarrow \text{vol}|X$ *is continuous at* x.

III. $\text{num}_T|_X$ *is bounded at* $x \Leftrightarrow \text{vol}|X$ *is bounded at* x.

Recall that the volume properties given in statements II and III have equivalent interpretations in terms of holonomy, c.f. Proposition 4.1. The proof of the above Proposition is straightforward, using the ideas of §4.

Proof of the Transversal Proposition. In the Epstein filtration $\{X_\alpha\}$ of the bad set X_1, let γ be the countable ordinal such that $X_\gamma \neq \emptyset$ but $X_{\gamma+1} = \emptyset$ (see §6). We prove the Proposition using downward induction on the indices $\{\alpha\}$ (starting with $\gamma + 1$), by assuming the Proposition is true for some $\beta \leq \gamma + 1$ (and hence for each larger ordinal), and showing that it is true for some $\alpha < \beta$.

Consider first the trivial case where β is a limit ordinal. Suppose that T_β is a transverse open 2-manifold given by the Transversal Proposition for some neighborhood U_β of X_β. The union of all leaves which intersect T_β is an open neighborhood of X_β, and so assuming without loss that U_β lies in this neighborhood, we have that for each point $x \in U_\beta$, the leaf L_x intersects T_β. By definition $X_\beta = \cap\{X_\alpha | \alpha < \beta\}$ is an intersection of compact sets, so there is an ordinal $\alpha < \beta$ such that $X_\alpha \subset X_\beta$. The Proposition is established for α by taking $T_\alpha = T_\beta$ and $U_\alpha = U_\beta$.

The remaining discussion is devoted to the case where β is a successor ordinal, say $\beta = \alpha + 1$.

There are three natural steps in the construction of T_α from $T_{\alpha+1}$. They are:

Step I. The separation step, which is patterned after [3, §10], using Weaver's Lemma in the same manner.

Step II. The multiplying-up step, which is equally as easy as the analogous part of Epstein's proof, and

Step III. The extension step, which plays a role similar to that of Epstein's Pasting Lemma.

The details of these steps follow. For economy we will often write num_* in place of num_{T^*}, where $*$ denotes an ordinal.

Step I. Suppose that $T_{\alpha+1}$ is a transverse 2-manifold in M satisfying the conclusion of the Proposition for some neighborhood $V_{\alpha+1}$ of $X_{\alpha+1}$. In this step our goal is to show the existence of a small compact invariant neighborhood G of $X_{\alpha+1}$ in $U_{\alpha+1} \cap X_\alpha$ such that G can be written as a finite disjoint union $G = \bigcup_{i=1}^{r} G_i$ of compact invariant subsets, with the property that on the frontier in X_α of each G_i, call it F_i, the intersection number function $\text{num}_{\alpha+1}$ is constant.

As remarked in §6, $X_{\alpha+1}$ has arbitrarily small invariant neighborhoods in X_α, since the volume function is continuous on $X_\alpha - X_{\alpha+1}$ (by definition). On any such sufficiently small open invariant neighborhood N of $X_{\alpha+1}$ lying in $U_{\alpha+1} \cap X_\alpha$ the function $\text{num}_{\alpha+1}$ is nonzero and finite valued. Furthermore it is continuous on $N - X_{\alpha+1}$. Proceeding just as in Epstein, we choose a compact invariant neighborhood G of $X_{\alpha+1}$ in N and consider the restricted function $\text{num}_{\alpha+1}|N$ near $\text{fr}_\alpha G \equiv$ the frontier of G in X_α, which is a compact invariant subset of $N - X_{\alpha+1}$. Since $\text{num}_{\alpha+1}|N$ is continuous at $\text{fr}_\alpha G$, one can write $\text{fr}_\alpha G = F_1 \cup \ldots \cup F_r$ as a finite union of disjoint compact invariant subsets such that

(i) for each i, $1 \leq i \leq r$, there is a invariant neighborhood N_i of F_i in N such that $\text{num}_{\alpha+1}|N_i =$ some constant, n_i say, and

(ii) the n_i's are all distinct.

Following Epstein and Weaver we prove

Lemma. *The F_i's can be separated in G, that is, G can be expressed as a union $G = G_1 \cup \ldots \cup G_r$ of disjoint compact invariant subsets such that for each i, $F_i \subset G_i$.*

Proof. (derived from [19] and [3, §10]). The bulk of the argument below is to show that there is no component of G which intersects more than one distinct F_i. Assuming for the moment that this is true, the proof is completed by the following standard point-set argument. Let F_j, F_k be two distinct F_i's. The preceding assumption says that any two points $x \in F_j$ and $y \in F_k$ lie in different components of G. Hence there is a separation of G, say $G = U_x \cup U_y$, into disjoint open sets such that $x \in U_x$ and $y \in U_y$. This is because in a compact metric space, each component is an intersection of open-closed sets [7, §42.II.3]. Now, one can argue as in the proof of the fact that a compact hausdorff space is normal, that there is a separation of G, say $G = U_j \cup U_k$, into disjoint open sets such that $F_j \subset U_j$ and $F_k \subset U_k$. That is, F_j and F_k can be separated in G, and so from this the Lemma is easily deduced.

To complete the proof of the Lemma, it remains to show that no component of G intersects more than one F_i. This will be accomplished by appealing to a theorem of Sierpinski, which says that no compact connected metric space is a nontrivial countable union of disjoint closed nonempty subsets [7, §42.III.6]. For each integer $n \geq 1$, define $A_n = \{x \in G | \text{num}_{\alpha+1}(x) \leq n\}$, which is a closed invariant subset of G by the lower semicontinuity of $\text{num}_{\alpha+1}$. Let C_n be the union of the non-isolated leaves of A_n. C_n is a closed invariant subset of A_n with the property that every point of C_n is a limit of other points of A_n, and also $A_n - C_n$ is a countable union of leaves. Clearly $A_n \subset A_{n+1}$ and $C_n \subset C_{n+1}$.

Claim. *C_n is an open-closed subset of C_{n+1}.*

Proof. We already know that C_n is closed in C_{n+1}. To show openness, suppose the contrary. Then there exists an $x \in C_n \cap T_{\alpha+1}$ such that x is a limit point of $(C_{n+1} - C_n) \cap T_{\alpha+1}$, and hence x is a limit point of $(A_{n+1} - A_n) \cap T_{\alpha+1}$. That is, x is a limit point of a sequence $\{x_j\}$ of distinct points in $A_{n+1} \cap T_{\alpha+1}$ such that for each j, $\mathrm{num}_{\alpha+1}(x_j) = n+1$. Also, x is the limit of a sequence of distinct points $\{y_j\}$ in $A_n \cap T_{\alpha+1}$. Without loss there exists m, $0 < m \le n$, such that for each j, $\mathrm{num}_{\alpha+1}(y_j) = m$ (but note that possibly $\mathrm{num}_{\alpha+1}(x) < m$; if we had that $\mathrm{num}_{\alpha+1}(x) = m$, the following argument would be a bit simpler). Consider the closed set $Z_0 \equiv \{x_j\} \cup \{y_j\} \cup \{x\}$ and the union Z of all leaves passing through points of Z_0, which is a closed set by the boundedness of the function $\mathrm{num}_{\alpha+1}$ on Z_0. This amounts to saying that the leaves L_{x_j} and L_{y_j} converge as sets to the leaf L_x. We wish to find a holonomy germ h defined on a neighborhood V of x in $T_{\alpha+1}$ such that h has a certain fixed nontrivial period, say $p > 1$, on a subsequence of points from $\{x_j\}$, and at the same time h is fixed on the corresponding subsequence from $\{y_j\}$. Such an h will violate Weaver's Lemma. To find h, we appeal to the observation of Epstein that the restricted holonomy group $\mathscr{H}_x|Z$ is finite (Proposition 4.1 III). Let D be a small open disc about x in $T_{\alpha+1}$, whose closure intersects L_x only in x. For j sufficiently large, the intersection $L_{x_j} \cap D$ consists of $(n+1)/\mathrm{num}_{\alpha+1}(x)$ $(\equiv n_X)$ points, and the intersection $L_{y_j} \cap D$ consists of $m/\mathrm{num}_{\alpha+1}(x)$ $(\equiv n_Y < n_X)$ points. For each j sufficiently large, we can find a holonomy map h_j such that $h_j(x_j) \neq x_j$ but $h_j(y_j) = y_j$. (Argue thus. Let $L_{x_j} \cap D = \{x_{j,k} \mid 1 \le k \le n_X\}$ be the orbit of $x_j = x_{j,1}$, and let $L_{y_j} \cap D = \{y_{j,l} \mid 1 \le l \le n_Y\}$ be the orbit of $y_j = y_{j,1}$. For each k, let g_k be a holonomy map such that $g_k(x_{j,1}) = x_{j,k}$. Then there must be two distinct indices k_0, k_1 such that $g_{k_0}(y_{j,1}) = g_{k_1}(y_{j,1})$. Let $h_j = g_{k_1}^{-1}(g_{k_0})$. By restricting to a subsequence, we can assume that there is some fixed $p > 1$ such that each h_j has period p on x_j. Finally, by restricting to a further subsequence if necessary, we can assume the h_j's coincide, i.e., they are all the same element, h_0 say, of $\mathscr{H}_x|Z$, because $\mathscr{H}_x|Z$ is finite. The desired h is then any holonomy element of $\mathscr{H}_x$ whose restriction to Z is h_0. (The interested reader may find it more satisfying to combine all of the above arguments directly with those of Proposition 4.1 III, to come up with a more economical proof). This completes the Claim.

To complete the proof of the Lemma, we argue as in Weaver, using the Claim to express G as a countable union of disjoint closed subsets, $G = \bigcup_{n=1}^{\infty}(C_{n+1} - C_n) \cup$ leaves in $(G - \bigcup_{n=1}^{\infty} C_n)$. By (i) above, no two distinct F_j and F_k intersect the same member of this countable collection, since $\mathrm{num}_{\alpha+1}(F_j) \neq \mathrm{num}_{\alpha+1}(F_k)$. Thus, no component of G can intersect both F_j and F_k, as a consequence of the theorem of Sierpinski mentioned above. This completes the proof of the Lemma.

Step II. In this step we alter the transverse surface $T_{\alpha+1}$ to arrange that the n_i's of Step I become equal. We do this by choosing some open neighborhood of $G \cap T_{\alpha+1}$ in $T_{\alpha+1}$ and then taking multiple copies of the various components of this neighborhood. Stating our goal precisely (continuing the notation of the previous step), we will show that there exists a smooth transverse surface T_G in M satisfying the conditions of the Transversal Proposition (with G in place of X_α), such that num_G is constant on some neighborhood of the frontier $\mathrm{fr}_\alpha G$ in X_α.

Let F_i and G_i, $1 \le i \le r$, be the compact saturated subsets of G described in Step I. For each i, let $U_{\alpha+1,i}$ be a neighborhood of G_i in $U_{\alpha+1}$ such that the $U_{\alpha+1,i}$'s are disjoint, and define $T_{\alpha+1,i} = T_{\alpha+1} \cap U_{\alpha+1,i}$. Letting n be the product of the n_i's, we wish to multiply up each $T_{\alpha+1,i}$ by the integer n/n_i to produce a new transverse surface $\tilde{T}_{\alpha+1,i}$ whose intersection number function has the constant value n near F_i in X_α. The union of the $\tilde{T}_{\alpha+1,i}$'s is the desired T_G.

The multiplying up is accomplished by using the fact that the normal bundle of each $T_{\alpha+1,i}$ in M is trivial (see the next paragraph). We can construct the normal bundle so that the fibers lie in the leaves of M. Regarding $T_{\alpha+1,i}$ as the zero section of the normal bundle, let $\tilde{T}_{\alpha+1,i}$ be the union of n/n_i nearby disjoint cross sections. We can ensure that $\tilde{T}_{\alpha+1,i}$ is a closed subset of $U_{\alpha+1,i}$ by choosing the sections to taper down sufficiently rapidly to $T_{\alpha+1,i}$ as they approach the "boundary" $\mathrm{cl}\, T_{\alpha+1,i} - T_{\alpha+1,i}$. $\tilde{T}_{\alpha+1,i}$ is transverse to the foliation, by construction. Consequently the union $T_G = \bigcup_{i=1}^{r} \tilde{T}_{\alpha+1,i}$ satisfies the conditions of the Transversal Proposition, for G in place of X_α, as claimed.

The triviality of the normal bundle of $T_{\alpha+1,i}$ follows because the bundle is oriented and its base $T_{\alpha+1,i}$, being an open 2-manifold, has the homotopy type of a 1-complex (possibly infinite). Here we point out a variation of our argument which obviates this general fact. Since G_i has no interior in M, and consequently dim $G_i \cap T_{\alpha+1,i} \leq 1$, we can triangulate $T_{\alpha+1,i}$ so that the barycenters of the 2-simplexes do not lie in G_i. Letting $T'_{\alpha+1,i}$ denote $T_{\alpha+1,i}$ minus the barycenters, clearly $T'_{\alpha+1,i}$ has the homotopy type of a 1-complex, namely, the 1-skeleton of $T_{\alpha+1,i}$. Now $T'_{\alpha+1,i}$ can be used in place of $T_{\alpha+1,i}$ in the previous discussion.

Step III. This is the extension step, and it begins with the compact invariant neighborhood G of $X_{\alpha+1}$ in X_α and its associated transverse surface T_G constructed in Steps I and II. We wish to extend T_G over the remainder of X_α, i.e. over $H_\alpha \equiv \mathrm{cl}(X_\alpha - G)$, to get a transverse suface T_α for X_α. In brief, the reasons why we can do this are (1) H_α is a fiber bundle with 1-dimensional base (since vol $|H_\alpha$ is continuous and H_α has no interior in M), and (2) near $H_\alpha \cap G$, the surface T_G provides n distinct cross sections of this bundle. So in effect, in order to extend the surface T_G, we must extend the n distinct cross sections. To do this simpleminded argument rigorously, we will do the extension process a bit at a time, making use of the following lemma.

Basic Extension Lemma. *Suppose G is a compact invariant neighborhood of $X_{\alpha+1}$ in X_α, with associated transverse surface T_G satisfying the properties of the Transversal Proposition (with G in place of X_α), and having the additional feature that its intersection number function* num_G *is constant on some neighborhood in X_α of* $\mathrm{fr}_\alpha G$*(= frontier of G in X_α). Suppose H is a compact invariant subset of $X_\alpha - X_{\alpha+1}$ which lies sufficiently close to a single leaf so that H has natural product structure, in the precise sense described below. Then one can extend a neighborhood of $G \cap T_G$ in T_G to a transverse surface $T_{G\cup H}$ for $G \cup H$ in M, so that $T_{G\cup H}$ satisfies the conditions of the Transversal Proposition (with $G \cup H$ in place of X_α), and in addition the intersection number* $\mathrm{num}_{G\cup H}$ *is constant on some neighborhood of* $\mathrm{fr}_\alpha(G \cup H)$ *in X_α (in fact constant on some neighborhood of* $(\mathrm{fr}_\alpha G) \cup H$ *in X_α).*

The precise sense in which the invariant set H is to have trivial product structure is this (cf. §4): H must lie in some open tubular neighborhood W of some leaf L_0 of H, which has smooth bundle retraction $p: W \to L_0$ such that the open 2-disc fibers $\{p^{-1}(x) | x \in L_0\}$ are transverse to the foliation, and such that each leaf L in H is projected diffeomorphically onto L_0 by p. Let $* \in L_0$ be a basepoint and let $D_* = p^{-1}(*)$. Then there is a natural product structure on H given by $d \times p: H \to (H \cap D_*) \times L_0$, where $d: H \to H \cap D_*$ is defined by $d(y) = L_y \cap D_*$ for $y \in H$. In the proof we will refer to such an H as being *submersion-trivial*.

Given the Basic Extension Lemma, one can get the full extension T_α of T_G over X_α by first choosing a finite collection $\{H_j | 1 \leq j \leq s\}$ of compact invariant submersion-trivial subsets of $X_\alpha - X_{\alpha+1}$ such that $X_\alpha = G \cup (\cup_{j=1}^{s} H_j)$. Then one can apply the Basic Extension Lemma successively to the pairs G and H_1, then $G \cup H_1$ and H_2, then $G \cup H_1 \cup H_2$ and H_3, etc...., until one has constructed the transverse surface T_α for X_α. So it remains to prove the Lemma.

Proof of the Basic Extension Lemma. As As we have already remarked, the key to the extension process is the fact that the locally compact metric quotient space $(X_\alpha - X_{\alpha+1})$/leaves has dimension ≤ 1. For these dimension theory facts we refer the reader to [6].

By enlarging G an arbitrarily small amount, we can assume that in addition to the already stated properties, we have that the dimension of the quotient $(\mathrm{fr}_\alpha G)$/leaves is ≤ 0, i.e., each leaf in $\mathrm{fr}_\alpha G$ has arbitrarily small open-closed (hence invariant) neighborhoods in $\mathrm{fr}_\alpha G$. Also, for convenience we replace H by $\mathrm{cl}(H - G)$, so that we can assume $G \cap H \subset \mathrm{fr}_\alpha G$ and the quotient $(G \cap H)$/leaves is 0-dimensional.

Let $p: W \to L_0$ be the disc bundle retraction of a neighborhood W of H in M onto a leaf L_0 of H, as hypothesized above. Letting $z_1, \ldots, z_n$ be n distinct points in L_0, let $D_i = p^{-1}(z_i)$ denote the 2-disc fiber through z_i. For each leaf L in $G \cap H$, let W_L be an open tubular neighborhood which is sufficiently small to have the following two properties (where n is the constant $\mathrm{num}_G(\mathrm{fr}_\alpha G)$):

(i) $T_G \cap W_L$ has exactly n components (each a 2-disc), denoted $T_1, \ldots, T_n$, each intersecting L in a single point, and

(ii) given any compact invariant subset S of $G \cap H$ lying in W_L, and given any neighborhood U of S in M, there is a leaf-invariant diffeomorphism $h: M \to M$, fixed off of U, such that h takes the germs of $D_1, \ldots, D_n$ at S onto the germs of $T_1, \ldots, T_n$ at S (not necessarily in order preserving fashion; this merely says that for each D_i there is a neighborhood V_i of $S \cap D_i$ in D_i such that $h(V_i) \subset T_k$, for some $k = k(i)$, and hence $h(V_i)$ is a neighborhood of $S \cap T_k$ in T_k).

This second property is established by using a simple holonomy translation process; we leave details to the reader.

Applying the 0-dimensionality of the quotient $(G \cap H)$/leaves, there is a separation $G \cap H = S_1 \cup \ldots \cup S_q$ of $G \cap H$ into a finite number of disjoint compact invariant subsets, such that each S_j lies in a tubular neighborhood $W_j(= W_{L_j})$ of some leaf L_j in $G \cap H$, having the two properties described in the preceding paragraph. Thus, we can choose disjoint neighborhoods $U_1, \ldots, U_q$ of $S_1, \ldots, S_q$ in M, together with leaf-invariant diffeomorphisms $h_1, \ldots, h_q$ of M, each supported in the corresponding U_j, and each taking germs of $D_1, \ldots, D_n$ at S_j onto germs of $T_1, \ldots, T_n$ at S_j. Let $V_{i,j}$, for $1 \leq i \leq n$ and $1 \leq j \leq q$, denote a neighborhood of $S_j \cap D_i$ in D_i such that $h_j(V_{i,j})$ lies in one of the T_k's. Define h: $M \to M$ to be the composition of the h_j's, which may be taken in any order as their supports are disjoint. Let $V = \cup_{i,j} V_{i,j} \subset D = \cup_{i=1}^{n} D_i$. Choose open neighborhoods N_G of G and N_H of H in M, so small that $h(D \cap N_H) \cap (T_G \cap N_G) = h(V \cap N_H) \cap N_G$ and also $h(V \cap N_H) \cap N_G$ lies in a compact subset of $h(V)$. Finally, define $T_{G \cup H} = (T_G \cap N_G) \cup h(D \cap N_H)$. This is the desired transverse surface for the conclusion of the Basic Extension Lemma.

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