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ON THE HOMOLOGY OF ATTRACTORS

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AN ATTRACTOR of a diffeomorphism f is a compact invariant set X which has an invariant neighborhood U satisfying $X = \bigcap_{n=0}^{\infty} f^n U$. We will study the real homology of hyperbolic expanding attractors (defined below in the appendix) using the branched manifolds of [8] and dynamical properties of [1] and [7].

One can assume that X is connected and in an appropriate sense oriented (expanding attractors are locally homeomorphic to Euclidean space cartesian product the Cantor set). We will replace f by a suitable power and then we have the

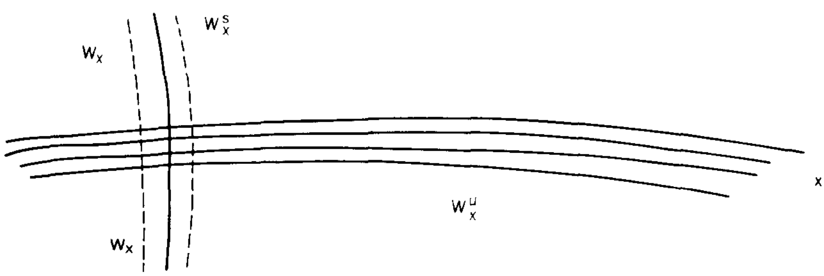
THEOREM. *The real Cěch homology of an oriented expanding attractor X in its top dimension is non-trivial and finite dimensional. In an appropriate basis the homology transformation induced by $f: X \to X$ is a matrix with positive entries. The log of the maximum eigenvalue of this transformation is the topological entropy of $f: X \to X$.*

Description of proof (see remark on smoothness assumption below).

We are assuming that X is a *hyperbolic* set for the diffeomorphism f, i.e. the tangent bundle along X splits into two df invariant subbundles, the stable bundle E_s which is contracted by df and the unstable bundle E_u which is expanded by df (relative to an appropriate metric). Under this hypothesis the attractor falls into finitely many connected components where some power of f is topologically transitive on each component [1]. We assume that E_s and E_u are oriented on one of the components X. If not, we could work in some covering of a neighborhood of X. Since X is connected the orientation must be preserved (or reversed) by f.

The stable bundle E_s is tangent to a foliation of some neighborhood of X by the stable manifolds

$$W_x^s(X) = \{y \in nghdX: d(f^n x, f_y{}^n) \to 0 \quad \text{as } n \to \infty\}.$$



The unstable bundle E_u is tangent to a "partial foliation" of X by the unstable manifolds

$$W_x^u(X) = \{y \in X: d(f^{-n}x, f^{-n}y) \to 0 \quad \text{as } n \to \infty\}.$$

We also assume that X is an *expanding attractor*, namely $\dim X = \dim W_x^u(X)$ for any $x \in X$. This implies each stable manifold intersects X in a Cantor set and that one can reasonably treat the quotient of some neighborhood of X by the stable foliation. The quotient space of suitable closed neighborhoods of X, by collapsing the components of the intersections with the stable leaves, are *compact branched manifolds*, triangulable spaces, with continuous tangent spaces, and a specific singularity structure [8].

The homological arguments for the theorem fall into two parts. For the first part, consider a standard closed differential form ω_x on some fixed neighborhood U of X. The form ω_x is supported in a small tubular neighborhood of the stable manifold through x and restricts to the unit volume form (with the correct sign) on each small normal disk to W_x^s.

PROPOSITION 1. *Any finite positive linear combination of the forms ω_x, $x \in X$ is not exact in any nghd of X.*

Proof of proposition 1. Take the case of one form ω_x. By Lemma 2 below ω_x restricted to the unstable manifold of a fixed point $W = W_p^s$ is commensurable with the unit Riemann volume of W. If ω_x were exact in a neighborhood of X this would contradict, using the proof of Lemma 3, the polynomial growth of W provided by Lemma 4. The same argument works for positive linear combinations.

We assume a fixed Riemannian metric on a fixed neighborhood U of X. Here we use $W^s_{y,\text{loc}}$ to denote a neighborhood in W_y^s chosen so as to contain the connected component of $W_y^s \cap U$ about y. On a Riemann manifold let $B(p,R) = \{x: d(p,x) \le R\}$.

LEMMA 2. *There is a number R such that each $B(p,R)$ in $W = W_x^u$ intersects each $W^s_{y,\text{loc}}$ for any unstable manifold W_x^u of X.*

Proof. This follows from the dynamical property that the closure of any unstable manifold is all of X [1a]. For suppose a sequence (p_i, R_i) exists with $R_i \to \infty$ and $B(p_i, R_i) \cap W^s_{y,\text{loc}} = \emptyset$. Let p be an accumulation point of the p_i. Since W_p^u is dense in X, W_p^u must come close to and hence intersect W_y^s, as they are uniformly tranverse. So $B(p,R)$ in W_p^u intersects W_y^s for some R. But then for p_i near p, $B(p_i, R')$ intersects W_y^s for R' near R, a contradiction.

LEMMA 3. *If W is a complete Riemannian manifold whose volume form is exact by a bounded form, then for any $p \in W$ the function volume $B(p,r)$ grows as fast as an exponential in r.*

Proof of Lemma 3. Suppose $\omega = d\eta$ where ω is the unit volume form and η is a bounded form. Let V_r = volume $B(p,r)$ and A_r = area of $\partial B(p,r)$. Then for some constant c we have $V_r = \int_{B(p,r)} \omega = \int_{\partial B(p,r)} \eta \le cA_r$. The lemma is proved by integrating the differential inequality $(dV_r/dr) = A_r \ge (1/c)V_r$. See [3] for more details. This proof does involve checking to see that the sets $B(p,r)$ are not too pathological; see for example Plante[3].

The next lemma applies to the unstable manifolds which fill up X.

LEMMA 4. *Let W be a complete Riemannian manifold which admits a uniformly expanding self-diffeomorphism.*† *Then for some p, the growth of volume $B(p,r)$ is dominated by a polynomial. Also W is diffeomorphic to Euclidean space.*

Proof of Lemma 4. Let p be the unique fixed point of the contracting map f^{-1}. If D is a small ball about p of radius ρ and volume ν, then $D_n = f^n D$ is an increasing union of balls which exhaust W. Thus W is diffeomorphic to R^n. [2].

Then clearly $B(p, a^n\rho) \subseteq D_n$ and vol $D^n \le b^{dn}$ where $d = \dim W$. The lemma follows.

This completes the first part of the homological argument of the theorem, namely Proposition 1. For the second part, we use the branched manifold theory of [8], which we summarize here, for completeness.

For an appropriate closed neighborhood U, by collapsing the components of $W_x^s \cap U$ to points, one obtains a quotient space and quotient map $q: U \to B$. B is a *branched manifold*, of class C^1 (as we have assumed that the stable foliation is C^1) and fits into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[g]{} & B \end{array} \qquad \text{or} \qquad \begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ B & \longrightarrow & B \end{array}$$

Branched manifolds have good tangent spaces, which relate nicely to smooth maps. In particular the map $g: B \to B$ is an *immersion*, in that its differential dg is 1–1 on each tangent space T_xB, $x \in B$. In addition, g inherits certain properties from f, which we list as

AXIOM 1. *The non-wandering set $\Omega(g) = B$.*

AXIOM 2. (*Flattening*) *each point of B has a neighborhood V such that $g^i(v)$ is a d-disk for some i.*

AXIOM 3^+. *g is an expanding map.*

†For all points the eigenvalues of $df \circ df^*$ lie in an interval $[a,b]$ with $a > 1$ and $b < \infty$.

A basic result of [8] is that $(x, f(x))$ is recoverable from $g: B \to B$, as

$$X = \varprojlim (B \xleftarrow[g]{} B \xleftarrow[g]{} \cdots) \qquad f|X = \varprojlim g.$$

In detail, there is a homeomorphism $h: X \to \varprojlim (B, g)$ defined by $h(x) = (qx, qf^{-1}x, qf^{-2}x, \ldots)$ and

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \varprojlim B & \xrightarrow[\varprojlim]{} & \varprojlim B \end{array}$$

commutes.

The geometric structure of B can be summarized as follows:

Each point $x \in B$ has neighborhood V where

(a) $V = D \cup \ldots \cup D_k$, each D_i a closed, smooth d-dimensional disk.
(b) x is in the interior (as a disk) of each D_i.
(c) $D_i \cap D_j$ is a closed d-cell.

Note that part (c) implies that D_i and D_j are mutually tangent along $\partial(D_i \cap D_j)$, which is part of the "branch set." Also the neighborhood V mentioned in Axiom 2 can be taken as in (a, b, c), and one can assume that g^i maps each D_i (of part a) diffeomorphically onto the same disk, say $D \subset B$.

Let $B \xrightarrow{g} B$ be the expanding endomorphism of the branched manifold constructed in [8]. Then B is triangulable, there is the commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow[g]{} & B \end{array}$$

and we can make the identifications $X = \varprojlim B$ and $f = \varprojlim g$. The tangent spaces of B are continuously oriented. Let t be a sufficiently small triangulation of B. Then from Proposition 1 we deduce that any positive d-cochain of B ($d = \dim B = \dim X$) is not a coboundary. Now let V denote the real vector space of d-chains. Define a chain map on V, $\hat{g}$, by

$$\hat{g}(\sigma) = \sum_{\tau} \frac{\text{volume}\,(g(\sigma) \cap \tau)}{\text{volume}\,\tau} \cdot \tau$$

where σ and τ are the d simplices provided with convenient volumes. It is geometrically clear that $\hat{g}$ preserves the subspaces of cycles Z. In the simplex basis $\hat{g}$ is a positive matrix and by the remark above about positive cochains the subspace of cycles intersects the positive quadrant of V.

Now $\hat{g}|Z$ can be identified with the homology map induced by g on H_dB. and we can pass to f and H_dX by taking inverse limits. Thus there is an invariant subspace Z_0 of Z such that $\hat{g}|Z_0$ can be identified with the homology map induced by f in $H_d(X)$.

Proof. Let $Z_0 = \bigcap_{n \geq 0} \hat{g}^n(Z)$. As Z is finite dimensional this is $\hat{g}^{n_0}(Z)$ for some n_0, and $\varprojlim \hat{g}|Z = \hat{g}|Z_0$.

If λ is the maximum eigenvalue of $\hat{g}$ on V with positive eigenvector v, then $(\hat{g})^n/\lambda^n$ approaches projection onto the linear subspace generated by v. Thus v is a cycle since $\hat{g}Z \subset Z$.

Also $(\hat{g})^n$ squeezes the positive quadrant of V closer and closer to the ray generated by v as n

increases. It follows that a positive simplicial cone in Z containing v is kept invariant by $(\hat{g})^n$. This means there is a basis in which $H_dX \xrightarrow{f^n} H_dX$ has positive entries for large enough n. (We can take this basis over Q if we wish here.)

To see that λ is the topological entropy of $X \xrightarrow{f} X$, let $A^{(n)}$ = area chain map of g^n relative to this Markov measure which is uniformly expanded by g. The log of expansion constant ν is the exponential growth rate (egr) of each column sum and thus also the egr of the sum of the matrix elements of $A^{(n)}$. But this latter quantity is also the log of the maximum eigenvalue for $A^{(n)}$. Thus $\lambda = \nu$ and we are done.

Remark on smoothness. There are two kinds of smoothness assumptions. The first is the smoothness of the diffeomorphism f and the second is the smoothness of the stable foliation of the neighborhood of the attractor. If f is C^k then each stable or unstable manifold is C^k. The first part of our argument works if the unstable manifolds are C^3 or probably even C^1.

The second part of the argument requires the stable foliation to be C^1 so that the branched manifold can be formed. In general the stable foliation only has Hölder continuous tangent planes even if f is $C\infty$.

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