Genericity Theorems in Topological Dynamics.

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1. Introduction.

Some recent theorems in differentiable dynamical systems are of a C^0 nature, referring to C^0 Ω-explosions and C^0 density for example, see [11, 12, 14, 15]. As far as we know however, no one has explained what these theorems imply about the generic homeomorphism of a compact manifold M or the generic C^0 vector field on M. We record here the result of several conversations on this matter.

First the C^0 topology makes Homeo(M) a Baire space. The usual C^0 metric

$$d(f,g) = \sup_{x\in M} d(f(x), g(x))$$

gives the same topology on Homeo(M) as does the metric

$$d_H(f,g) = \max(d(f,g), d(f^{-1},g^{-1})) .$$

Under d_H, Homeo(M) is complete and hence, as a topological space, it has the Baire property: every countable intersection of open dense sets is dense.

A set G is generic (relative to a Baire space $B \supset G$) if G contains a countable intersection of open dense sets. A generic property is one enjoyed by a generic set of elements of B.

Theorem 1. The following properties of $g \in$ Homeo(M) are generic

(a) g has no C^0 Ω-explosion,

(b) g has no C^0 Ω-implosion,

(c) g is a continuity point of the map Ω : Homeo(M) $\to$ K(M) where K(M) is the space of compact subsets of M under the Hausdorff topology,

(d) g has a fine sequence of filtrations, fields on M . A remarkable but easily proved result of Orlicz [8] (see also Choquet's book [3]) says that the generic $X \in X^0(M)$ generates a continuous flow. It then makes sense to ask whether Theorem 1 remains true for such an X-flow ϕ . (It does - see Theorem 1' below.) One might also ask about the Entropy Conjecture for flows (Theorem 2) but unfortunately its natural generalization is trivial: the time t map of any flow, ϕ_t, induces the identity on $H_*(M)$ because $\phi_t \simeq 1$. On the other hand there might be an interesting Flow Entropy Conjecture if ϕ_t were forced to act on some sort of "transverse homology groups".

Returning to Theorem 1, we shall restate only the part having to do with filtrations. A _global_ _Lyapunov_ _function_ for the continuous flow ϕ is a real valued continuous function on M which strictly decreases on ϕ-trajectories off Ω and is constant along trajectories of Ω . (Ω is the non-wandering set of ϕ .)

Theorem 1'. Generically $X \in X^0(M)$ generates a flow having a C^∞ global Lyapunov function.

Proof. Takens' proof of (a) extends to flows. Also (a) continues to be equivalent to (d): a fine sequence of filtrations [7]. Such a fine sequence produces a continuous global Lyapunov function. This can be made C^∞ by the smoothing theory of Wilson [16].

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