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A REMARK ON THE LEFSCHETZ FIXED POINT FORMULA FOR DIFFERENTIABLE MAPS

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If 0 is an isolated fixed point for the continuous map $f: U \to R^m$, where U is an open subset of R^m, then the index of f at 0, $\sigma_f(0)$, is the local degree of the mapping Id-f restricted to an appropriately small open set about 0. If 0 is an isolated fixed point of f^n, then $\sigma_{f^n}(0)$ is defined for all $n > 0$, where f^n means f composed with itself n times restricted to a small neighborhood of 0. We will use a little elementary calculus to show:

Proposition. *Suppose that $f: U \to R^m$ is C^1 and that 0 is an isolated fixed point of f^n for all n. Then $\sigma_{f^n}(0)$ is bounded as a function of n.*

The proposition is not true for continuous functions as the mapping of the complex plane $f(z) = 2z^2/\|z\|$ shows. In fact, for this f, $\sigma_{f^n}(0) = 2^n$. Our interest in the proposition arose from the Lefschetz fixed point formula as applied to a smooth endomorphism f of a compact differentiable manifold M. The Lefschetz formula says that the Lefschetz numbers

$$L(f^n) \equiv \sum (-1)^i \quad \text{trace} \quad f_{*i}^{\,n}: H_i M \to H_i M$$

can be computed locally by these fixed point indices,

$$L(f^n) = \sum_{P \in Fix f^n} \sigma_{f^n}(P),$$

provided that the fixed points of f^n are isolated.

Corollary. *If $f: M \to M$ is C^1, and the Lefschetz numbers $L(f^n)$ are not bounded then the set of periodic points of f is infinite.*

In particular, any C^1 degree two map of the two sphere, S^2, has an infinite number of periodic points and hence an infinite non-wandering set [see 1].† The corollary suggests the possibility of getting sharper estimates on the asymptotic growth rate of $N_n(f)$, the number of fixed points of f^n.

Problem. If $f: M \to M$ is smooth, is

$$\limsup \frac{1}{n} \log |L(f^n)| \leqslant \limsup \frac{1}{n} \log N_n(f)?$$

† Note that the one-point compactification of $f(z) = 2z^2/\|z\|$ is a *continuous* degree two map of S^2 with only two periodic points.

As remarked in [1] this inequality is rather obviously true for the set of C^r endomorphisms f of M which have the property that all periodic points of f are transversal. Then, of course, $|L(f^n)| \leqslant N_n(f)$.

We now proceed with the proof of the proposition. In all that follows below f is C^1 and 0 is an isolated fixed point of f^n for all n. The idea is to try to approximate $I - f^n$ by $(I + f + f^2 + \cdots + f^{n-1})(I - f)$ so that if $I + f + f^2 + \cdots + f^{n-1}$ is a local diffeomorphism then degree $(I - f^n) = \pm$degree $(I - f)$. To make this precise and to do the estimates we work with the derivatives of f^n at 0 which we denote by Df^n.

LEMMA 1. *If* $\sum_{j=0}^{n-1} Df^j$ *is non-singular then* $\sigma_f(0) = \pm\, \sigma_{f^n}(0)$.

Before we prove Lemma 1 we will show how it proves the proposition. $\sum_{j=0}^{n-1} Df^j$ is singular precisely when $n = mk$, $k > 1$, and Df has a primitive kth root of unity as an eigenvalue. For each integer n, let λ be the least common multiple of these orders k. Then we may apply the proposition to see that $\sigma_{f^n}(0) = \pm\, \sigma_{f^\lambda}(0)$. (If $(k_1, k_2, \ldots)$ are the orders of roots of unity in the spectrum of Df, then $(k_1/\text{g.c.d.}(k_1, \lambda), \ldots)$ are the orders for Df^λ. But now n/λ is not a multiple of any of these orders greater than 1.)

Since we only need finitely l.c.m.'s λ to take care of all the integers n, this argument proves the proposition.

A standard fact that we shall use in proving Lemma 1 is:

LEMMA 2. *If* $h, k: U \to R^n$ *are continuous, have* 0 *as an isolated* 0 *and* $\|h(x) - k(x)\| < \|h(x)\|$ *then degree* $(h) =$ *degree* (k).

Proof of Lemma 1. Let $f = Df + \theta_1$ and $f^n = Df^n + \theta_n$.

Then
$$\begin{aligned} I - f^n &= I - Df^n - \theta_n \\ &= (I + Df + \cdots + Df^{n-1})(I - Df) - \theta_n \\ &= (I + Df + \cdots + Df^{n-1})(I - f) + (I + Df + \cdots + Df^{n-1})\theta_1 - \theta_n. \end{aligned}$$

We will show by induction that given (n, ε) there is a neighborhood $U_{n,\varepsilon}$ of 0 such that
$$\left\| \left(\sum_{j=0}^{n-1} Df^j\theta_1 - \theta_n \right)(x) \right\| < \varepsilon \|(I - f)(x)\| \text{ for all } x \in U_{n,\varepsilon}.$$
So that if $\sum_{j=0}^{n-1} Df^j$ is non-singular then by Lemma 2,
$$\text{degree } (I - f^n) = \text{degree } \left(\textstyle\sum_{j=0}^{n-1} Df^j\right)(I - f) = \pm \text{ degree } (I - f).$$

To estimate $\sum_{j=0}^{n-1} Df^j\theta_1 - \theta_n$ first observe that $\theta_n = \sum_{j=0}^{n-1} Df^{n-j-1}\theta_1 f^j$ as can easily be seen by induction. So
$$\begin{aligned} \sum_{j=0}^{n-1} Df^j\theta_1 - \theta_n &= \sum_{j=0}^{n-1} Df^{n-1-j}\theta_1 - \sum_{j=0}^{n-1} Df^{n-j-1}\theta_1 f^j \\ &= \sum_{j=1}^{n-1} Df^{n-1-j}(\theta_1 - \theta_1 f^j). \end{aligned}$$
By the mean value theorem
$$\left\| \left(\sum_{j=0}^{n-1} Df^j\theta_1 - \theta_n \right)(x) \right\| \leqslant \sum_{j=1}^{n-1} \|Df^{n-1-j}\|\, \|D\theta_1\|_{U_{n,\varepsilon}} \|(I - f^j)(x)\|$$

where $\|D\theta_1\|_{U_{n,\varepsilon}} = \sup_{x \in U_{n,\varepsilon}} \|D_x \theta_1\|$. Since $D_0 \theta_1 = 0$ it clearly suffices to prove inductively that given $j < n$ there is a neighborhood V_j of 0 and a $0 \leq k_j < \infty$ such that

$$\|(I - f^j)(x)\| \leqslant k_j \|(I - f)(x)\| \quad \text{for all} \quad x \in V_j.$$

Since

$$I - f^j = \left(\sum_{i=0}^{j-1} Df^i\right)(I - f) + \sum_{i=0}^{j-1} Df^i \theta_1 - \theta_j,$$

we can inductively choose $U_{j,\varepsilon}$ so that

$$\|(I - f^j)(x)\| \leqslant \sum_{i=0}^{j-1} \|Df^i\| \, \|(I - f)(x)\| + \varepsilon \|(I - f)(x)\|,$$

and we are done.

REFERENCE

1. M. SHUB: Dynamical systems, filtrations and entropy, *Bull. Am. math. Soc.* **80** (1974), 27–41.

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