Genetics of homotopy theory and the Adams conjecture

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Dedicated to Norman Steenrod

This is the first in a series of papers devoted to the invariants and classification of compact manifolds.

One might hope for a classification theory of manifolds having the following form: First, there is an understandable algebraic description of the homotopy theory of manifolds. Second, the salient geometric properties of manifolds beyond homotopy theory can be isolated and understood. Third, the algebraic structure of these geometric invariants can be determined. And fourth, this structure can be successfully intertwined with the algebraic description of the homotopy theory of manifolds to complete the theory.

We shall try to outline and motivate this paper in terms of this program.

First of all, the Adams conjecture concerns the homotopy theory of sphere bundles associated to vector bundles. This is important in the second and third parts of the program for a theory of smooth manifolds and follows from the fact that the possible tangent vector bundles of closed manifolds in a homotopy type must have sphere bundles of the same fiber homotopy type. In fact the work of Browder shows that this sphere bundle condition is almost a sufficient condition for a bundle to be a tangent bundle.

The affirmed Adams conjecture then allows us to find the possible tangent bundles of all manifolds in the homotopy type X by calculating the K-group of the homotopy type $K(X)$ and the Adams' operations $\{\psi^k\}$ in this group. A more detailed description of Adams' statement is given below. This explains our interest in the Adams conjecture beyond that inspired by the natural beauty of the statement.

Now it was well-known from the outset that the Adams conjecture concerned only questions of torsion and divisibility of a certain subgroup of $K(X)$ of maximal rank. So the question was trivial after tensoring with the field of rationals.

Thus it is natural (and indeed imperative) to discuss the question in a context where only p-primary considerations are left, where the influence of the rationals is minimal. This accounts for the profinite homotopy theory of Section 3 of this paper which is, I must say, at first flush somewhat foreboding.

In this profinite setting the Adams' statement assumes its most elegant form (see below). Also the homotopy equivalences of the unstable grassmannians required for its proof appear very naturally from the fact that the profinite homotopy type of an algebraic variety can be constructed using only the abstract algebraic structure of the variety (étale homotopy theory).

This latter point is in fact the main motive for the profinite homotopy theory of Section 3. In order to derive benefit from the rich Galois symmetry in the homotopy theory of varieties we had to domesticate the abstract beasts of [5] to make them usable in ordinary algebraic topology.

The Adams conjectures become immediate consequences of general homotopy theoretical discussions.

The localization in homotopy theory discussed in Section 2, which is easier to conscience than the completion of Section 3, plays the following role:

First, localization arises in a natural way in each of the four steps of the outline for a classification theory of manifolds. Many workers have found in discussions about manifolds persistent disparity between the prime 2 and odd primes (see for example the statements about the universal spaces G/PL and G/TOP in Section 2). Analysis of these spaces essentially solves the third step in the program for a manifold theory of the piecewise linear and topological categories. A detailed solution of the Adams conjecture leads to an analysis of G/O, the relevant space for this step in smooth manifold theory.

Second, localization allows us to construct the rational homotopy type which complements the information in the profinite homotopy type. This complementation makes very clear the relationship between the ordinary (or transcendental) homotopy theory of complex algebraic varieties and their étale (or algebraic) homotopy theory. Also in beautiful counterpoint to the relationship between profinite homotopy theory and abstract algebraic geometry is a new relationship between a rational de Rham theory on polyhedra and the rational homotopy type (see [17]). This new relationship allows a determination of the "real homotopy" of a smooth manifold by the de Rham algebra of C^∞-forms.

Thus analytical and arithmetical structure[1] play a role in real and rational homotopy theory akin to that of abstract algebraic structure in profinite homotopy theory.

[1] I.e., C^∞ manifolds and triangulated spaces.

Now we shall outline our structural analysis of homotopy theory and discuss the Adams conjectures in some detail.

First of all a remark on terminology: We are studying the structure of homotopy types to deepen our understanding of more complicated or richer mathematical objects such as manifolds or algebraic varieties.

The relationship between these two types of objects is I think rather strikingly analogous to the relationship in biology between the *genetic* structure of living substances and the visible structure of completed organisms or individuals.

The specifications of the genetic structure of an organism and of the homotopy structure of a manifold have similar texture; they are both discrete, combinatorial, rigid, interlocking and sequential.[1]

On the other hand, a completed individual has shape, form, geometrical substance. Its visible expression admits continuous variation. All these attributes are possessed by a manifold in Euclidean space.

So we shall think of our fracturing process for homotopy types as a genetic analysis. We shall refer to the array of irreducible pieces of information in the homotopy type as the *genotype.* This is similar to the use of the term *genus* in the theory of quadratic forms.

Thus the genotype of a space is determined by the rational type and profinite type. The latter often splits completely into a collection of p-adic types, one for each prime.

We cannot combine these ingredients arbitrarily to form a space. A certain compatibility must persist among the pieces.

This *coherence* in the *genotype* is the second piece of structure in our analysis of homotopy theory. Now the detailed description:

Let X denote an "ordinary homotopy type". We shall think of X as arising from a compact manifold, an algebraic variety, or one of the universal spaces associated to geometric problems concerning these spaces. The homology groups of these spaces are almost always finitely generated Abelian groups, and one often has some analogous control over the fundamental group and the higher homotopy groups.

We shall associate to X a natural array of more basic homotopy types which are not such "ordinary homotopy types" but which satisfy analogous finiteness conditions over the rings

$$Z_l = \left\{\frac{a}{b} \in Q: (b, l) = 1\right\}$$

[1] In fact many of our constructions on spaces proceed by a replication process using the Postnikov tower as a template.

or

$$\hat{Z}_l = \varprojlim_{n \in \{l\}} Z/nZ$$

where $\{l\}$ is the multiplicative subset of Z generated by a collection l of primes. The two important cases for us correspond to the ground rings Q and $\hat{Z} = \varprojlim_n Z/n$.

These more elemental spaces associated to X are constructed by the processes of localization and completion. These processes might be thought of as analogous to laboratory techniques which break a more complicated substance into simpler components.

The first process, localization at l, is a kind of direct limit procedure which strips away the p-torsion and p-divisibility structure in the algebraic topology of the space for the primes p *not* in the collection l. The remaining information in X is carried by the associated space X_l to which X maps $X \to X_l$.

This localization procedure is well-understood if X has a principal Postnikov decomposition; this happens precisely when each π_1-module $\pi_n X$, $n \geqq 1$ has a finite filtration

$$e \subset \Gamma_k \subset \Gamma_{k-1} \subset \cdots \subset \Gamma_1 = \pi_n X$$

with trivial action on the successive quotients.[1] In this "nilpotent case" the homology groups of X_l are the localizations of those of X:

$$H_* X_l \cong H_* X \otimes Z_l \ , \qquad * > 0 \ .$$

Similarly, the homotopy groups of X_l are the localizations of those of X. More precisely, each $\pi_n X_l$ admits a finite filtration so that the action of $\pi_1 X_l$ is trivial on the successive quotients. The map $X \to X_l$ induces a filtration preserving map on π_n which is localization on the successive (Abelian) quotients.

If we try to strip away all of the prime information (take $l = \varnothing$) we obtain X_Q, the rational type of X. In X_Q the "algebraic" topology of X has become uniquely divisible.

Now we turn to the process of finite[2] completion. This is an inverse limit procedure which allows us to strip away much of the rational information in the homotopy type of X.

An associated homotopy type $\hat{X}$ is constructed from "inverse systems"

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[1] D. Kan and E. Dror who observed this point about Postnikov systems refer to such spaces as "nilpotent spaces"—a natural generalization of nilpotent groups.

[2] In earlier versions of this work, e.g., [18], this was called "*profinite* completion". Similarly, in the algebraic context (groups, modules, etc.) we use "finite" or "l-finite" for "profinite or "l-profinite", before the word "completion".

indexed by the category $\{f\}_X = \{X \xrightarrow{f} F\}$ of all maps of X into spaces with finite homotopy groups.

$\hat{X}$ is defined in the homotopy category by the formula

$$[\ ,\hat{X}] = \varprojlim_{\{f\}_X} [\ , F]$$

using a little theory about "compact Brownian functors".

In the most general discussion of finite completion one needs both the space $\hat{X}$ and the "inverse systems" over $\{f\}_X$. However, if X has "good" homotopy groups then all the information after finite completion is carried by the homotopy theory of $\hat{X}$, which is simply related to that of X.

For example, if $\pi_1 X$ contains a solvable subgroup which has finite index and finite type[1], and if the higher homotopy groups of X are finitely generated, then[2]

$$\pi_n \hat{X} \cong \text{finite completion of } \pi_n X , \qquad n \geqq 1$$

and

$$H^i(\hat{X}, M) \cong H^i(X, M)$$

for all finite π_1-modules M.

We also have l-completions for a set of primes l. For example if l consists of one prime p we have the p-adic completion $\hat{X}_p$ constructed from the category of maps of X into spaces whose homotopy groups are finite p-groups.

There is a natural map $\hat{X} \xrightarrow{h} \prod_p \hat{X}_p$ where h is an isomorphism in the intersection of the two good cases mentioned above, i.e., if X is nilpotent with finitely generated homotopy groups.

We refer to the array of spaces $\{X_Q, \hat{X}_2, \hat{X}_3, \cdots\}$ as the "genotype of X".

The homotopy type X can be reconstructed from a coherence between rational type X_Q and the finite type $\hat{X} = \prod_p \hat{X}_p$. We describe the coherence.

First, construct a formal completion of X_Q by taking the increasing union of the finite completions of its finite subcomplexes X_Q^α,

$$\text{formal completion } X_Q \equiv \varinjlim_\alpha (X_Q^\alpha)^\wedge .$$

Then construct the rational type of $\hat{X}$. This may be described in terms of the rational types of the p-adic factors of $\hat{X}$ by the formula

$$\text{rational type of } \hat{X} \cong \varinjlim_S \prod_{p \in S} (\hat{X}_p)_Q \times \prod_{p \in S} \hat{X}_l ,$$

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[1] All its subgroups are finitely generated.

[2] If G is a group we refer to the inverse limit of its finite quotients as its finite completion.

where S ranges over finite sets of primes.[1]

Because X_Q and $\hat{X}$ came from the same ordinary homotopy type X there is a canonical homotopy class of equivalences

$$\zeta_X\colon \text{formal completion of } X_Q \sim \text{rational type of } \hat{X}$$

preserving the $\hat{Z} \otimes Q$ structure in the "homotopy groups". We refer to ζ_X as the coherence of the homotopy type X.

If we add ζ_X to the genotype we obtain the "coherent genotype" of X:

$$\{X_Q, \hat{X}_2, \hat{X}_3, \cdots; \zeta_X\} .$$

Now we describe the reconstruction of the homotopy type X from its coherent genotype $\{X_Q, \hat{X}; \zeta_X\}$. The coherence allows us to construct a common space containing X_Q and $\hat{X}$. Then X is simply the space of all paths in this common space whose endpoints lie respectively in X_Q and $\hat{X}$.

The complete theorem about these constructions goes as follows:

Form the homotopy category $\mathfrak{N}$ generated by nilpotent spaces of finite type and visible[2] homotopy classes of maps between them. Also form the category of coherent genotypes $\mathcal{G}_c$. The objects of $\mathcal{G}_c$ are arrays of nilpotent spaces $\{X_Q, X_2, \cdots\}$ whose "homotopy groups" are finitely generated over the appropriate ground rings $\{Q, \hat{Z}_2, \hat{Z}_3, \cdots\}$, together with a coherence

$$\zeta\colon \text{formal completion of } X_Q \sim \text{rational type of } \pi_p X_p ,$$

a class of homotopy equivalences respecting the $\hat{Z} \otimes Q$ module structure on "homotopy". The morphisms of $\mathcal{G}_c$ are arrays of homotopy classes of maps $\{f_Q, f_2, f_3, \cdots\}$ respecting the coherences. Thus we have the

THEOREM. *Localization and completion define a functor* $\mathfrak{N} \xrightarrow{a} \mathcal{G}_c$. *The path construction defines a functor* $\mathcal{G}_c \xrightarrow{i} \mathfrak{N}$. *These are mutually inverse equivalences of categories.*

As suggested above we will show in later discussion and papers how the two objects X_Q and $\hat{X}$ are respectively related to analytic and algebraic structure in X.

Now we turn to the final section of the homotopy analysis, where we apply this discussion to the theory of fibrations where the fiber has the homotopy type of a sphere. Thus we study local and complete spherical fibration theory. By fiberwise localization and completion we construct a canonical diagram of theories

[1] Hereafter, "$\simeq$" means "has the same homotopy type", or "is homotopic to".

[2] Two maps are visibly homotopic if they are homotopic on every finite subcomplex of the domain.

$$\begin{array}{ccc} & & \{S_l^{n-1}\text{-fibrations}\} \\ & \nearrow & \downarrow \\ \{S^{n-1}\text{-fibrations}\} & & \\ & \searrow & \\ & & \{\hat{S}_l^{n-1}\text{-fibrations}\}\ . \end{array}$$

We show that the corresponding diagram of classifying spaces has for π_1 the diagram of units[1]

$$\begin{array}{ccc} & & Z_l^* \\ & \nearrow & \downarrow \\ Z^* & & \\ & \searrow & \\ & & \hat{Z}_l^* \end{array}$$

and for universal covering space the diagram of spaces

$$\begin{array}{ccc} & & (B_{SG_n})_l \\ & \nearrow & \downarrow \\ B_{SG_n} & & \\ & \searrow & \\ & & (B_{SG_n})_l \end{array}$$

where B_{SG_n} is the classifying space of oriented S^{n-1}-fibration theory.

We obtain then the

COROLLARY. *The stable theory of spherical fibrations is a direct factor in the stable theory of fibrations where the fibers are the completed spheres* $\hat{S}^n$.

We then use this corollary to prove the real and complex Adams conjectures, which we shall now discuss.

The Adams conjectures. These conjectures concern the spherical fiber homotopy type of vector bundles related by the famous Adams' operation ψ^k. For example if η is a complex line bundle then $\psi^k\eta$ is the k-fold tensor power $\eta^k = \eta \otimes \cdots \otimes \eta$. In general ψ^k is the unique ring endomorphism of K-theory extending this operation.

In his work on the order of the image of the Hopf-Whitehead J-homomorphism, [1]–[4],

$$\pi_i Gl(n, \mathrm{C}) \xrightarrow{J} \pi_{2n+i} S^{2n}\ .$$

Adams made use of the natural fiber-preserving map $\eta \xrightarrow{F} \eta^k$ defined on vectors v by $v \to v \otimes \cdots \otimes v$. On each fiber the map can be thought of as $z \to z^k$ in the complex plane.

[1] $R^* =$ units in R.

Thus if we remove the zero sections of η and η^k, F induces a fiber homotopy equivalence provided we suppress the primes dividing k. Adams did this by forming the stable bundle

$$(\eta^k - \eta) \oplus (\eta^k - \eta) \oplus \cdots \oplus (\eta^k - \eta)$$

with k^N-summands, where N depends on the dimension of the base.

Adams proved the corresponding sphere bundle was fiber homotopy trivial (using F) if N was large enough. This gave an upper bound on the order of image J.

At this point Adams conjectured that *for all elements ξ in the K-theory of a finite complex the stable bundle $k^N(\psi^k\xi - \xi)$, N large, has a fiber homotopy trivial associated sphere bundle.*

He then proved the beautiful consequence—*if the relations "$k^N(\psi^k\xi - \xi)$ is fiber homotopy trivial" hold, then all stable fiber homotopy relations between vector bundles can be deduced from them.*

Our formulation and proof of the Adams conjecture use the finite completion of K-theory defined for finite complexes by

$$\hat{K}(X) = \varprojlim_n K(X) \otimes Z/n \ .$$

$\hat{K}(X)$ has a unique extension to the entire homotopy category which is represented by mapping into some classifying space $\hat{B}$. This $\hat{B}$ is just the finite completion of the infinite grassmannian classifying space B of ordinary K-theory.

Now each element $\gamma \in \hat{K}(X)$ has a well-defined stable fiber homotopy type. This follows by continuity from the fact that the group of stable spherical fiber homotopy types over a finite complex is finite.

For each integer k we can define an isomorphism of $\hat{K}(X)$ using ψ^k on the p-components for $(p, k) = 1$ and the identity if $(p, k) > 1$.

Computations made by Adams imply that these operations extend by continuity to an action of the group of units $\hat{Z}^* = \varprojlim_n (Z/n)^*$ on $\hat{K}(X)$.

Our proof of the Adams conjecture is based on describing this symmetry in $\hat{K}$ in terms of algebraic symmetry in the grassmannian varieties which classify K-theory.

Amusingly enough, this symmetry is obtained by transforming the points of the grassmannians discontinuously by a field automorphism of C which moves the roots of unity around. For example, to prove the Adams conjecture for ψ^k we use a field automorphism σ of C which raises n^{th} roots of unity to the k^{th} power for $(n, k) = 1$ and is the identity on all $k^{r\,\text{th}}$ roots of 1, for any r.

We thus refer to the symmetry group $\hat{Z}^*$ which contains the isomorphic part of the Adams' operations as the Galois group. Then we have the

THEOREM (Adams conjecture). *In real or complex $\hat{K}$-theory the stable fiber homotopy type is constant on the orbits of the Galois group $\hat{Z}^*$.*

COROLLARY (Adams). *Any element in $\hat{K}$ which is fiber homotopy trivial is of the form $\eta^\sigma - \eta$ for some $\sigma \in \hat{Z}^*$ and $\eta \in \hat{K}$.*

The theorem follows quickly from the fact that the Galois symmetry in $\hat{K}$ arises from homotopy symmetry in the finite completions of the grassmannian approximations to B. In fact the theory of Section 4 allows us to obtain a *canonical* fiber homotopy equivalence between the appropriate completed spherical fibrations *before* stabilizing.

We note in passing certain other homotopy theoretical results found in the discussion.

i) In the localization section the following principle is described: If a homotopy problem involving only finitely many simply-connected finite complexes has a rational solution, then it has a solution after localization *away* from a certain *finite* set of primes.

ii) In that section we also find that there are no "phantom maps" of a countable complex into a nilpotent space whose homotopy groups are built up from Q-vector spaces of finite rank.

iii) In the completion section we find a Hasse principle for maps—which says that two maps of a finite complex into a nilpotent space of finite type are homotopic if and only if they are homotopic at every (finite) prime.

iv) In Section 3 we use the "coherent genotype" picture of QP^∞ to construct self-mappings of any odd square degree.

v) In Section 4 we discuss principal spherical fibrations and prove for p odd that S_p^{2n-1} is homotopy equivalent to a topological group if n divides $p - 1$.

(It has been known for some time that the conditions of iv) and v) were necessary.)

Now it is time to acknowledge the many debts incurred during the research on this paper. First of all G. Washnitzer first pointed out to me the miracle of a "discontinuous map" acting on the cohomology of a space like the grassmannians. Greg Brumfiel first explained the Adams conjecture to me at an opportune moment in the study of smooth normal invariants.

I am indebted to Dan Quillen who first raised the specter of algebraic geometry in characteristic p in connection with the Adams conjecture. He has also found a proof of the Adams conjectures using another "algebraic

theory of cohomology", the cohomology of finite groups [9], [11], [12]. His methods lead to interesting computations of algebraic K-theory and a general homotopy theoretical attack on certain algebraic questions. Hopefully our proof will lead to a certain understanding of the geometric topology of algebraic varieties.

I am grateful to George Bergman who told me many things about the p-adics and to Saul Lubkin who explained many useful things about his approach to étale cohomology.

Barry Mazur and Mike Artin helped me understand the étale cohomology in characteristic zero which is simpler than characteristic p because of the appealing geometric interpretations and yet sufficient for the Adams conjecture. Also the notion of finite completion used here is evolved from theirs [5].

I am grateful to John Morgan who worked out many thorny technical points and provided a set of notes without which this paper would still be in the "to appear" category.

I am indebted to Dan Kan for explaining the notion of nilpotent space and to his student Emmanuel Dror for pointing out that our original localization arguments worked essentially word for word for this more interesting class of spaces. I am grateful to Ellie Palais for promoting the genetic terminology which is useful for understanding the structure found here.

Finally I offer my thanks and admiration to Frank Adams for discovering this beautiful phenomenon in K-theory.

1. Algebraic constructions

Here we describe the algebraic constructions and objects needed for the later discussion. These include localization and completion of groups and the corresponding ground rings Z_l and $\hat{Z}_l$.

One of the main point is the "arithmetic square" (page 14), introducing one of the major themes of the work.

The constructions and their properties are described in serial order with some indication of proof. Topological examples illustrate the algebra.

Localization.

We shall sometimes want in our calculations to concentrate on a certain subset of primes l and allow division by the primes outside of l. The appropriate *ground ring* for this situation is Z_l, the subring of the rationals consisting of fractions whose denominators are products of primes outside l.

We shall make use of certain properties of this local situation: Z_l and its modules.

First of all there are natural inclusions $Z_l \to Z_{l'}$ if $l' \subset l$, and it is easy to check that

1.1
$$\begin{array}{ccc} Z_{l \cup l'} & \xrightarrow{i} & Z_l \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle l} \\ Z_{l'} & \xrightarrow[k]{} & Z_{l \cap l'} \end{array}$$

is a "fiber square". That is, the sequence

1.1′
$$0 \longrightarrow Z_{l \cup l'} \xrightarrow{i \oplus j} Z_l \oplus Z_{l'} \xrightarrow{l-k} Z_{l \cap l'} \longrightarrow 0$$

is exact.

Z_l is used to *localize Abelian groups* by tensoring

1.2
$$G \longrightarrow G_l \quad \text{where} \quad G_l \equiv G \otimes_Z Z_l \,.[1]$$

We refer to G_l as the localization of G with respect to the set of primes l. Of course $G_{\{\text{all primes}\}} = G$ and we write $G_\varnothing = G_Q$.

Examples.

a) If G is a finitely generated Abelian group $G_l \cong Z_l \oplus \cdots \oplus Z_l \oplus l$-torsion G.

b) $(Q/Z)_l = \bigoplus_{p \in l} Z/p^\infty \quad (Z/p^\infty = \varinjlim_\alpha Z/p^\alpha)$.

The natural localization map $G \to G_l$ has kernel the torsion of G prime to l.

More generally, we think of G_l as formed from G by making the operations of multiplication by integers prime to l into isomorphisms. Thus we have

1.3
$$G_l \cong \varinjlim_{(n,l)=1} \{G \xrightarrow{\cdot n} G\} \,.$$

1.4 Thus we see that G is isomorphic to its localization at l if and only if G is a Z_l-module. In this case we say that G *is local* (with respect to l).

1.5 The direct limit formula 1.3 shows localization preserves exactness, finite products, and commutes with the operation of taking homology (when extended to graded groups in the obvious way).

Thus for all spaces X

1.6
$$H_*(X, Z_l) \cong H_*X \otimes Z_l \,.$$

Also, let $F \to E \to B$ be a fibration of connected spaces.

If the fundamental groups are Abelian, and two of

[1] The homotopy arguments of Section 2 lead directly an extension of localization to nilpotent groups (see also [6]).

1.7 $$\{\pi_* F\},\ \{\pi_* E\},\ \{\pi_* B\}$$

are local then the third is also.

1.7 follows by localizing the exact homotopy sequence,

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_i F & \longrightarrow & \pi_i E & \longrightarrow & \pi_i B & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & (\pi_i F)_l & \longrightarrow & (\pi_i E)_l & \longrightarrow & (\pi_i B)_l & \longrightarrow \cdots . \end{array}$$

The bottom row is exact by 1.5 so the result follows from the 5-lemma.

The result of 1.7 extends to homology in case $\pi_1 B$ acts trivially in $\tilde{H}_*(F, Z/p)$ for primes $p \notin l$. For then if two of

1.8 $$\{\tilde{H}_* F\},\ \{\tilde{H}_* E\},\ \{\tilde{H}^* B\}$$

are local, the third is also.

1.8 follows from the Serre spectral sequence relating the mod p homologies of $F \to E \to B$ and

1.9 "$\tilde{H}_* X$ is local if and only if $\tilde{H}^*(X, Z/p)$ vanishes for primes $p \notin l$."

1.9 follows from the coefficient sequence

$$\cdots \longrightarrow H_i X \xrightarrow{\ p\ } H_i X \longrightarrow H_i(X, Z/p) \longrightarrow \cdots$$

and 1.4.

The analogue of 1.1 for groups is true, namely

1.10 $$\begin{array}{ccc} G_{l \cup l'} & \longrightarrow & G_l \\ \downarrow & & \downarrow \\ G_{l'} & \longrightarrow & G_{l \cap l'} \end{array}$$

is a *fiber square of groups*. For example if $l \cup l' =$ all primes and $l \cap l' = \varnothing$ then we see that *G is the fiber product of its l-localizations and l'-localizations over G_Q*,

1.11 $$G \cong G_l \times_{G_Q} G_{l'} .$$

1.10 follows by tensoring 1.1′ with G since $\operatorname{tor}(G, Z_l) = 0$ for all G and l.

If we localize G at l and then localize the result at l', we obtain the localization at $l \cap l'$,

1.12 $$(G_l)_{l'} \cong G_{l \cap l'} .$$

This follows from 1.3 or the isomorphism

1.13 $$Z_{l \cap l'} \cong Z_l \otimes_Z Z_{l'} .$$

For certain considerations localizing is not drastic enough. We can suppress the influence of the rationals and obtain an interesting compactness phenomenon by completing with respect to a set of primes l.

The ground ring for this situation is the inverse limit of the finite rings,

1.15 $$\hat{Z}_l = \varprojlim_{n \in \{l\}} Z/nZ$$

where $\{l\}$ is the multiplicative set generated by the primes of l. $\{l\}$ is partially ordered by divisibility.

We shall make use of the natural compact topology on $\hat{Z}_l$ coming from the inverse limit and the uniform structure obtained by regarding $\hat{Z}_l$ as the completion of Z with respect to the ideals nZ, $n \in \{l\}$.

Of particular interest for us are the cases

i) $l = \{p\}$. Then $\hat{Z}_l$ is the ring of *p-adic integers*

1.16 $$\hat{Z}_p \cong \varprojlim_n Z/p^n Z .$$

We will use the interesting fact that the group of units in $\hat{Z}_p$ contains the $(p-1)^{\text{st}}$ roots of unity. In fact, we need the natural isomorphisms

$$\hat{Z}_p^* \cong Z/p-1 \oplus \hat{Z}_p , \qquad p \text{ odd}$$

and

$$\hat{Z}_2^* \cong Z/2 \oplus \hat{Z}_2 ,$$

constructed using the p-adic log and exp functions.

ii) $l = \{\text{all primes}\}$. Then $\hat{Z}_l \equiv \hat{Z}$ is the finite completion of Z,

1.17 $$\hat{Z} \cong \varprojlim_n Z/n .$$

It is clear from 1.17 that we have the continuous isomorphism

1.18 $$\hat{Z} \cong \prod_p \hat{Z}_p .$$

We will also encounter the localizations

1.19 i) $(\hat{Z}_p)_Q \equiv \hat{Z}_p \otimes_Z Q$, the field of p-adic numbers, denoted Q_p. Recall that Z is contained in the compact ring $\hat{Z}_p$ as a dense subset, and $\hat{Z}_p$ is contained in the locally compact field Q_p as the closed unit disk (with respect to the natural p-adic metric on Q_p).

1.20 ii) $(\hat{Z})_Q \equiv \hat{Z} \otimes_Z Q$, the ring of "finite Adeles for Q" denoted A^f. A^f is isomorphic to a restricted product over the primes of the p-adic numbers; that is, $A^f \subset \prod_p Q_p$ is the locally compact subring consisting of elements of the product which have integral components at all but finitely many primes.

From the description it is clear that A is a direct limit

1.21 $$A^f = \varinjlim_S A_S$$

where S ranges over finite sets of primes and

$$A_S^f = (\prod_{p \in S} Q_p) \times (\prod_{p \notin S} \hat{Z}_p) .$$

Our main use of A^f concerns the fiber square,

1.22
$$\begin{array}{ccc} Z & \longrightarrow & \hat{Z} \\ \downarrow & & \downarrow \\ Q & \xrightarrow{d} & A^f \end{array}$$
"arithmetic square" ,

where $Q \xrightarrow{d} A$ is the diagonal embedding

$$\frac{n}{m} \longrightarrow \left(\frac{n}{m}, \frac{n}{m}, \cdots\right) \in A^f \subset \prod_p Q_p .$$

The exactness of

1.22′
$$0 \longrightarrow Z \longrightarrow \hat{Z} \oplus Q \longrightarrow A^f \longrightarrow 0$$

is checked fairly easily.

The study of $Q \rightarrow A^f$ is certainly enhanced by combining it with the real completion $Q \rightarrow R$. The product embedding of Q in the full ring of Adeles for Q, $A = A^f \times R$, is now *discrete with a compact quotient.* This property is only used as a provocation in this work.

These considerations may be extended to groups in several ways. First of all we shall make some use of the formal completion of *Abelian* groups

1.23
$$G \longrightarrow G_f^{\wedge} \quad \text{where} \quad G_f^{\wedge} \equiv G \otimes_Z \hat{Z}_l .$$

The formal completion enjoys many of the algebraic properties of localization, especially 1.5.

A more interesting construction is the l-finite completion of an *arbitrary* group G. We have

$$G \longrightarrow \hat{G}_l \quad \text{where} \quad \hat{G}_l \equiv \varprojlim_\alpha G/H_\alpha ,$$

and H_α ranges over those normal subgroups of G so that G/H_α has finite order, a product of primes in l.

We are most interested in the case where l is {all primes}. Then we have *the finite completion*

$$G \longrightarrow \hat{G} \quad \text{where} \quad \hat{G} = \varprojlim_\alpha G/H_\alpha, \quad G/H_\alpha \text{ finite} .$$

We shall make use of the natural compact topology on these profinite groups given by their expression as an inverse limit of finite groups.

1.24 For example, each map of G into a finite group has a unique continuous extension to $\hat{G}$.

1.25 If G is a finitely generated Abelain group, the finite completion and formal completion agree. The topology on the former is intrinsic to the algebraic structure of $\hat{G}$.

In this case we also have the fiber square of groups

1.26
$$\begin{array}{ccc} G & \xrightarrow{\wedge} & \hat{G} \\ \downarrow & & \downarrow l \\ G_Q & \xrightarrow[c]{} & G \otimes_Z A^f , \end{array}$$
"arithmetic square for G" .

Because of 1.25 we can check the exactness property 1.26 by tensoring 1.22′ with G since $\mathrm{tor}\,(G, A^f) = 0$.

Since c and l are defined intrinsically in terms of G_Q and $\hat{G}$ (namely formal completion and localization respectively) we can view 1.26 as a natural recovery formula for G in terms of its localization G_Q and its finite completion $\hat{G}$.

Also for Abelian groups it is clear that

1.27
$$\hat{G} \cong \prod_p \hat{G}_p \ .$$

So the arithmetic square 1.26 binds together the rational type of G, G_Q and its rather independent p-adic completions $\hat{G}_p$ using the natural isomorphism $(G_Q)\hat{f} \cong (\hat{G})_Q$ (each is naturally isomorphic to $G \otimes_Z A^f$). We shall generalize this picture to homotopy in Section 3.

1.28 We shall make use of the fact that $G \xrightarrow{\wedge} \hat{G}$ gives a *dense embedding* of a finitely generated Abelian group G into its finite completion $\hat{G}$. In fact finite completion is characterized by this property for finitely generated Abelian groups.

Examples.

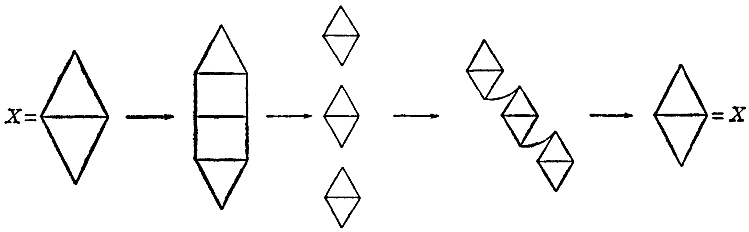
a) If G is a finitely generated Abelian group $\hat{G} \cong \hat{Z} \oplus \cdots \oplus \hat{Z} \oplus$ torsion G.

b) $Q^\wedge = 0$, but $Q_f^\wedge = A^f$.

c) If G is a vector space over Z/p with infinite cardinality α then $\hat{G}$ is a vector space over Z/p with cardinality 2^{2^α}. The topology on $\hat{G}$ is clearly important in this case.

2. Localization in homotopy theory

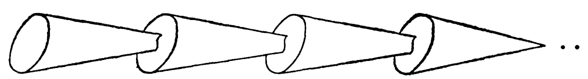
Let l be a set of primes in Z, vacuous or not. There are two examples of localization of spaces at l which come to mind easily. First, if X is the suspension of a connected space, then for each integer n there is a natural map $X \xrightarrow{n} X$. Pictorially, $X \xrightarrow{3} X$ is given by



It is clear that this map multiplies homology classes in positive dimensions by n.

We can form the *infinite mapping telescope* of the maps

$$X \xrightarrow{n} X \xrightarrow{m} X \longrightarrow \cdots$$

2.1 

where n, m, etc. range through a cofinal set of those integers obtained from primes not in l by multiplication.

We obtain a new space X_l containing X. An easy argument using the compactness of cycles and homologies shows $X \to X_l$ localizes homology,

2.2 $$H_i X_l \cong (H_i X)_l \ .$$

In the second example Y is any space with a continuous multiplication of its points. $Y \xrightarrow{n} Y$ can be defined by raising points of Y to the n^{th} power. This map multiplies homotopy classes by n if Y has a unit. Again let Y_l denote the infinite mapping telescope of

$$Y \xrightarrow{n} Y \xrightarrow{m} Y \longrightarrow \cdots ,$$

n, m, etc., as above. The compactness of the sphere and disk implies $Y \to Y_l$ localizes homotopy,

2.3 $$\pi_i Y_l \cong (\pi_i Y)_l \ .$$

The main theorem of this section on localization asserts 2.2 and 2.3 are equivalent for many such spaces. So we can deduce

for the suspension example, $\pi_i X_l \cong (\pi_i X)_l$, and
for the Hopf space examples, $H_i X_l \cong (H_i X)_l$.

Thus suspensions and H-spaces have good localizations from the point of view of the invariants of algebraic topology.

The suspension construction for spheres was used by Adams in studying multiplications on spheres (away from 2).

The H-space construction was needed in [16] to express the results determining the structure of G/PL, the classifying space for surgery problems in geometric topology. The structure found there was very different "at 2" from that "away from 2". It was natural to wonder what more general localization of spaces was possible. Interestingly enough there is quite a general theory.

In this section we generalize the suspension example to a cell by cell localization of a simply-connected cell complex. Each cell of the original space X is replaced by a "local cell" of the new space X_l.

The H-space example generalizes to a stage by stage localization of any principal Postnikov system X. The homotopy groups and k-invariants of X are replaced by the localized groups and k-invariants of X_l. These constructions are very easy, given Theorem 2.1.

In either case we obtain for a simple Postnikov space X a localization map $X \xrightarrow{l} X_l$ which localizes homology and homotopy. Obstruction theory shows X_l is universal for maps of X into spaces with local homotopy groups. Thus localizations are unique and we have a good localization functor on the homotopy category[1] generated by simple Postnikov spaces.

We discuss briefly certain localizations of more twisted spaces. The section ends with some examples and certain propositions about the *rational type of* X which is obtained by inverting all the primes.

Recall that $K(\pi, n)$ denotes the space with one homotopy group π in dimension n. Call a space X a simple Postnikov space if it can be approximated up to any dimension by spaces constructed inductively from a point by *principal* fibrations whose fibers are $K(\pi, n)$'s, where the π are Abelian.

We refer to such a sequence of approximations as a Postnikov decomposition (or a simple Postnikov decomposition) of X.[2] Such spaces were first emphasized by E. Dror and Dan Kan who aptly call them *nilpotent spaces*.

PROPOSITION (Kan and Dror). *X has a simple Postnikov decomposition as above if and only if each π_1-module $\pi_n X$ has a finite filtration*

$$e = \Gamma_k \pi_n \subset \Gamma_{k-1}\pi_n \subset \cdots \subset \Gamma_1 \pi_n = \pi_n$$

so that the action of π_1 is trivial on the successive quotients.

Thus π_1 is nilpotent and the action of π_1 on π_n is nilpotent. Their point is this—these spaces behave for many homotopy purposes like simply con-

[1] The objects are spaces having the homotopy type of simple Postnikov spaces. The morphisms are homotopy classes of maps between these spaces.

[2] We refer to the more general decomposition with non-Abelian $K(\pi, 1)$'s and non-principal fibrations as twisted Postnikov decompositions.

nected spaces. For example we have ordinary obstruction theory for maps into these spaces, the Whitehead theorem relating ordinary homology isomorphism and homotopy equivalence, and so on. In fact they comprise a kind of maximal class of spaces where these "simply-connected" or homological techniques work.

Unless otherwise stated we shall work only with these simple Postnikov (or nilpotent) spaces in this section. We say that a simple Postnikov space is local if each π occurring in some approximating Postnikov system is local; that is, π is a Z_l-module for the set of primes l under discussion.

We refer to the π as the "homotopy groups" of X. We say that a map of some space X into a local space X_l, $X \xrightarrow{l} X_l$, is a localization of X if it is universal for maps of X into local spaces; i.e., given f there is a unique f_l making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;\;l\;\;} & X_l \\ & {}_{f}\searrow \quad \swarrow_{f_l} & \\ & \text{local space} & \end{array}$$

commutative.[1]

Local spaces and localization are characterized by

THEOREM 2.1. *For a map*

$$X \xrightarrow{\;l\;} X'$$

of simple Postnikov spaces (nilpotent spaces) the following are equivalent:

i) *l is a localization,*

ii) *l localizes integral homology,*

iii) *l localizes "homotopy".*[2]

Taking $l =$ identity we have the

COROLLARY. *For a simple Postnikov space the following are equivalent:*

i) *X is its own localization,*

ii) *X has local homology,*

iii) *X has local "homotopy".*

We note here that a map induces an isomorphism of local homology if and only if it does so on rational homology and on mod p homology for $p \in l$.

The proof of the theorem is not uninteresting but it is long so we defer it to the end of this section.

[1] Maps are considered up to homotopy.

[2] We give a definition of $\pi_1 X \otimes Z_l$ for $\pi_1 X$ nilpotent in the remark after Theorem 2.3.

We go on to our constructions of the localizations which make use of Theorem 2.1. Recall the local sphere described above as an infinite mapping telescope. The inclusion of $S^i \xrightarrow{l} S^i_l$ as the first sphere in the telescope clearly localizes homology. In positive dimensions we have

$$0 \xrightarrow{l} 0 , \qquad j \neq i ,$$

$$Z \xrightarrow{l} \varinjlim Z = Z_l , \qquad j = i .$$

Thus by Theorem 2.1, l also localizes homotopy and l is a localization. This homotopy situation is interesting because the map induced on homotopy by a degree d map of spheres is not the obvious one; e.g.,

i) $S^4 \xrightarrow{d} S^4$ induces multiplication by d^2 on $Z \subset \pi_7 S^4$. (H. Hopf)

ii) $S^4 \xrightarrow{2} S^4$ induces a map represented by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\pi_8 S^4 = Z/2 \oplus Z/2$. (David Frank)

COROLLARY. *The map on d torsion of $\pi_j(S^i)$ induced by a map of degree d is nilpotent.*

Definition. A local CW complex is built inductively from a point or a local l-sphere by attaching cones over the local sphere using maps of the local spheres S_l into the lower "local skeletons".

Note. Since we have no local 0-sphere we have no local 1-cell.

THEOREM 2.2. *If X is a* CW *complex with one zero cell and no one-cells, there are a local* CW *complex X_l and a "cellular" map*

$$X \xrightarrow{l} X_l$$

such that

i) *l induces a bijection between the cells of X and the local cells of X_l.*

ii) *l localizes homology.*

COROLLARY. *Any simply-connected space has a localization.*

Proof of Corollary. Choose a CW decomposition with one zero cell and zero one-cells and consider

$$X \xrightarrow{l} X_l$$

constructed in Theorem 2.2. By Theorem 2.1, l localizes homotopy and is a localization.

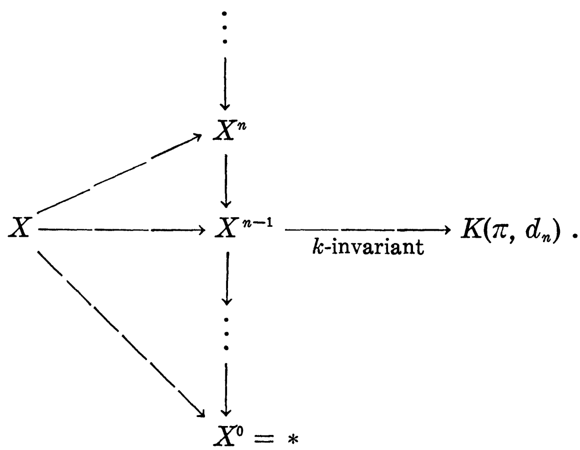
Proof of 2.2. The proof is by induction over the skeletons $\{X^n\}$. Assume we have constructed $X^n \xrightarrow{l^n} X^n_l$ satisfying i) and ii), and let $S \xrightarrow{a} X^n$ be an attaching map for X^{n+1}. Consider the diagram

$$\begin{array}{ccc} S^n & \xrightarrow{a} & X^n \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle l^n} \\ S_l^n & \underset{a'}{\dashrightarrow} & X_l^n \end{array} .$$

By Theorem 2.1, we have a unique factoring a'. We can then attach the appropriate local cells to X_l^n to obtain $X^{n+1} \xrightarrow{l^{n+1}} X_l^{n+1}$ satisfying i). l^{n+1} satisfies ii) by an easy exact sequence argument.

Then let $X_l = \bigcup_n X_l^n$ to complete the proof of the theorem.

There is a construction dual to the cellular localization for simple Postnikov spaces. Let X be a Postnikov decomposition



We say X is a "local Postnikov decomposition" if each π is local.

THEOREM 2.3. *If X is any Postnikov decomposition there is a local Postnikov decomposition X_l and a Postnikov map*

$$X \longrightarrow X_l$$

which localizes "homotopy groups" and k-invariants.

Proof. We induct over the number of stages in X. Assume we have a map of the n-stages $X^n \to X_l^n$ localizing homotopy groups, and let $X^n \xrightarrow{k} K$ be the primary obstruction (or k-invariant) of the $(n+1)$-stage $X^{n+1} \to X^n$. By Theorem 2.1 we have a unique extension k' in the diagram

$$\begin{array}{ccc} X^n & \xrightarrow{l^n} & X_l^n \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k'} \\ K & \longrightarrow & K_l \end{array}$$

where $K_l = K(A_l, n+2)$ if $K = K(A, n+2)$.[1] The induced map on the fibers gives the desired $X^{n+1} \to X_l^{n+1}$ and we are done.

COROLLARY. *Any simple Postnikov space has a localization.*

Proof. Apply Theorem 2.3 to a Postnikov decomposition of the space.

Remark. This argument shows us what $\pi \otimes Z_l$ should mean for a nilpotent group π. Let $X = K(\pi, 1)$ with the Postnikov decomposition corresponding to the lower central series of π. We localize the Postnikov system using ii) $\Rightarrow$ iii) of Theorem 2.1 with the inductive interpretation of $\pi_1 X_n \otimes Z_l$. Then $\pi \otimes Z_l$ is the fundamental group of the resulting space X_l.

In $\pi \otimes Z_l$, $x \to x^n$ is a bijection for $(n, l) = 1$. In fact $\pi \to \pi \otimes Z_l$ is universal for maps of π into such "l-divisible groups". Also the fiber square 1.10 is still valid. These points are easy to see using induction over the lower central series of π.

So choose for each cell complex X a particular cell complex X_l and a particular localization map $X \xrightarrow{l} X_l$, say an inclusion. Then given a map $X \xrightarrow{f} Y$ the composition $X \xrightarrow{f} Y \xrightarrow{l} Y_l$ has a unique extension up to homotopy over X_l by obstruction theory (see proof of 2.1). In particular we have

THEOREM 2.4. *There is a canonical localization functor on the homotopy category generated by simple Postnikov or nilpotent spaces.*

Actually the obstruction theory argument allows much more precise functorial statements about localization in a given situation. For example, some function spaces and fibrations are localized (function by function) and (fiber by fiber) in Section 4.

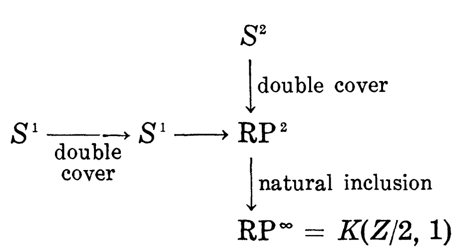
THEOREM 2.5. *The localization functor $X \to X_l$*

a) *preserves fibrations of connected, simple spaces,*

b) *preserves cofibrations of simply-connected spaces.*

Proof. a) and b) follow respectively from the homotopy and homology properties of localization using Theorem 2.1.

We note here that no extension of the localization functor to the entire



[1] Using the universal coefficient sequence over Z_l one sees that a map which induces an isomorphism of Z_l homology also induces an isomorphism of Z_l cohomology.

homotopy category can preserve fibrations and cofibrations. For this, consider the diagram. The vertical sequence is a fibration (up to homotopy) and the horizontal sequence is a cofibration. If we localize "away from 2", i.e., if l does not contain 2, we obtain

$$\begin{array}{ccccc} & & & & S^2_l \\ & & & & \downarrow \\ S^1_l & \xrightarrow{\cong} & S^1_l & \longrightarrow & \mathrm{RP}^2_l \\ & & & & \downarrow \\ & & & & \mathrm{RP}^\infty_l \cong * . \end{array}$$

If cofibrations were preserved RP^2_l should be a point. If fibrations were preserved RP^2_l should be S^2_l (which is not a point).

It is interesting to ask what localizations are possible for more general spaces.

Note (localization of twisted Postnikov spaces).

A) The fiberwise localization of Section 4 allows some treatment of non-simple spaces X. For example, let $X' \to X$ be a covering corresponding to a normal nilpotent subgroup π of $\pi_1 X$ which acts in a nilpotent manner on the higher homotopy groups. Then we can localize the fibers in the fibration

$$X' \longrightarrow X \longrightarrow K(\pi_1/\pi, 1)$$

to obtain a partial localization of X. The new space X^l_π is the total space of a fibration

$$X'_l \longrightarrow X^l_\pi \longrightarrow K(\pi_1/\pi, 1) .$$

Thus the higher homotopy groups of X^l_π are those of X localized; while $\pi_1 X^l_\pi$ is the natural extension

$$1 \longrightarrow \pi_l \longrightarrow \pi_1 X^l_\pi \longrightarrow \pi_1/\pi \longrightarrow 1 ,$$

a partial localization of $\pi_1 X$.

B) There is a more algebraic view of localization due to Bousfield and Kan. First of all Kan has a construction for converting an arbitrary space into a tower of nilpotent spaces. Then ordinary localization may be employed [6]. In this approach the first step is to form the nilpotent completion of the Kan group complex for the space of loops on X. Then "algebraic localization" is possible. This "algebraic localization" is the natural extension via inverse limit of localization of nilpotent groups (discussed above) to inverse limits of nilpotent groups.

This construction gives our localization on nilpotent spaces by uniqueness and has the advantage of being defined in a very rigid way for all semi-simplicial complexes. For non-nilpotent spaces the change in homotopy groups presents an interesting new problem in homotopy theory.

C) For localization at Q, discussed specifically in the section, a completely different approach is possible. On a triangulated space there lives a natural rational de Rham algebra of differential forms [17]. From this algebra a tower of Q-nilpotent spaces can be constructed. This tower agrees with the Kan-Bousfield localization of general spaces.

The rational type of X.

When l is vacuous we refer to X_l as the *rational type* of X and denote it X_Q. Of course X_Q is the most drastic localization of X since its higher homotopy and homology groups are converted into rational vector spaces, while π_1 becomes a group in which extraction of n n^{th}-roots is always possible and unique.

$$\pi_i X_Q \cong (\pi_i X) \otimes Q \ , \quad H_i X_Q \cong (H_i X) \otimes Q \ .$$

One use of X_Q is that of a base for melding the more subtle p-primary structures of X.

If $l \cup l'$ is a partition of the primes into two disjoint sets, we can form the homotopy commutative square

$$\begin{array}{ccc} X & \longrightarrow & X_l \\ \downarrow & & \downarrow \\ X_{l'} & \longrightarrow & X_Q \ . \end{array}$$

Taking homotopy groups gives a fiber square by 1.10 so X has the homotopy type of the path space fiber product of

$$\begin{array}{ccc} & & X_l \\ & & \downarrow \\ X_{l'} & \longrightarrow & X_Q \ . \end{array}$$

Example. Consider G/PL, the union of the homogeneous spaces constructed from the spaces G_n of proper homotopy equivalences of Euclidean space $\mathbf{R}^n$ and the spaces PL_n of PL-homeomorphisms of $\mathbf{R}^n$ as n approaches infinity.

G/PL classifies surgery problems. Treating the classical surgery obstructions as homotopy invariants leads to a complete homotopy analysis of this space. These calculations can be expressed in terms of the localizations of G/PL at 2 and away from 2. In fact

$$(G/\mathrm{PL})_{\mathrm{odd}} \simeq (B_O)_{\mathrm{odd}}$$
$$(G/\mathrm{PL})_2 \simeq Y \times \prod_{i=6}^{\infty} K(P_i, i)$$

where B_O is the classifying space for the stable orthogonal group, Y is the total space of the fibration over $K(\mathbf{Z}/2, 2)$ with fiber $K(\mathbf{Z}_{(2)}, 4)$ and k-invariant δSq^2, and P_i is the period-four sequence of groups,

$$0,\ \mathbf{Z}/2,\ 0,\ \mathbf{Z}_{(2)},\ 0,\ \mathbf{Z}/2,\ 0,\ \mathbf{Z}_{(2)},\ \cdots .$$ [1]

Thus $(G/\mathrm{PL})_Q \simeq \prod_{i=1}^{\infty} K(\mathbf{Q}, 4i)$, and we have that G/PL has the homotopy type of the fiber product of the diagram

$$\begin{array}{ccc} & & Y \times \prod K(P_i, i) \\ & & \big\downarrow n \\ (BO)_{\mathrm{odd}} & \xrightarrow[p]{} & \prod_i K(\mathbf{Q}, 4i) \end{array}$$

where p is the Pontrjagin character and n is the natural map induced by tensoring $\{P_i\}$ with the rationals.

This analysis applies to G/Top using the triangulation work of Kirby and Siebenmann. Here we have

$$(G/\mathrm{Top})_{\mathrm{odd}} \simeq (B_O)_{\mathrm{odd}}$$
$$(G/\mathrm{Top})_2 \simeq \prod_{i=1}^{\infty} K(P_i, i) .$$

We shall pursue this discussion and the relation of G/PL and G/Top to the classification of manifolds in Part II. (See also [18].)

In the next section we shall consider more drastic decompositions of X in terms of X_Q and p-adic completions. To set the stage for this discussion we shall prove two propositions about rational homotopy types.

As a corollary to the first (Lemma 2.6) we have the following principle: *If a homotopy problem involving only finitely many simply-connected finite complexes can be solved for the corresponding rational types, then it can be solved for the localizations away from a certain finite set of primes.*

For example, the problem "does the finite complex X have a continuous multiplication with unit?" is expressible in terms of the finite diagram of finite complexes,

$$\begin{array}{ccc} X \times X & & \\ \text{inclusion} \big\uparrow & \searrow \mu & \\ X \cup X & \xrightarrow[\text{fold}]{} & X . \end{array}$$

Here the map μ is the desired solution.

[1] $Z_{(2)}$ means the integers localized at the set of primes $\{2\}$.

From Lemma 2.6 we have an interpolating sequence of finite complexes between X and X_Q,

$$X = X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots X_Q = \bigcup X_n$$

where each map is a Q-equivalence.

If X_Q has a multiplication, consider the diagram

$$\begin{array}{ccc} X_Q \times X_Q & \xrightarrow{\mu_Q} & X_Q \\ \uparrow\cup & & \uparrow\cup \\ X \times X & \underset{\mu_n}{\dashrightarrow} & X_n \\ \uparrow\cup & & \uparrow\cup\, i_n \\ X \vee X & \underset{\text{fold}}{\longrightarrow} & X\,. \end{array}$$

The factoring μ_n exists since $X \times X$ is compact. The map i_n is a Q-equivalence between finite complexes so it is also an equivalence outside some finite set of primes S. If l is the set of primes complementary to S, then it follows that X_l has a natural multiplication.

To solve the original homotopy problem we have to concentrate on the finite singular set of primes for the rational solution. For this work the p-adic completions of the next section are often more useful than localizations. The assembly procedure is explained there by the concept of a coherent genotype.

A proper discussion of this theory is simplified by the second proposition (Lemma 2.7) which asserts that there are no phantom maps into spaces whose "homotopy groups" are rational vector spaces of finite rank.

Note (further remarks on multiplications in spaces).

A) By the work of Hopf and Milnor-Moore the existence of a multiplication for a rational homotopy type is equivalent to the vanishing of the k-invariants. For a finite complex this means X is Q-equivalent to a product of odd-dimensional spheres $\prod S$. The compactness argument above then shows X and $\prod S$ are equivalent when we localize each away from a finite set of primes.

At each prime of this finite set, X may be equivalent to localizations of various Lie groups as recent examples of Zabrodsky and others show.

B) If we ask for an associative multiplication on a space homotopy equivalent to X, the set of singular (bad) primes for a rational solution may not be finite. For this question implicitly involves infinite spaces. This is clear from the reformulation, "find a space B so that X is equivalent to the space of loops on B".

For example, if X is the seven dimensional sphere, $X_Q \simeq K(Q, 7)$ which is equivalent of course to loops on $K(Q, 8)$. Thus we have an associative multiplication on S^7 over the rationals. But it can be shown that the (unique) rational solution of this problem can only be lifted to the localization of S^7 at the set of primes which are congruent to one modulo 4. (See "Principal spherical fibrations" in Section 4 for a more complete discussion of this phenomenon.)

Examples of rational types.

i) $S_Q^{2n+1} \cong K(Q, 2n+1) \cong RP_Q^{2n+1}$

ii) $S_Q^{2n} \cong \text{fiber}\left(K(Q, 2n) \xrightarrow{x^2} K(Q, 4n)\right)$

iii) $CP_Q^n \cong \text{fiber}\left(K(Q, 2) \xrightarrow{x^{n+1}} K(Q, 2n+2)\right)$

iv) $(B_{U_n})_Q \cong \prod_{i=1}^n K(Q, 2i)$

v) Let V be a finite polyhedron.[1] Then there is an integer $k = k(\dim V)$ so that if $x \in H^{2n}(V, \mathbf{Z})$, $k \cdot x$ is *naturally* represented by the Thom class of a subcomplex $V_x \subset V$ with a complex normal vector bundle E_x. In this representation x restricted to V_x gives the n^{th} rational Chern class of E_x while the lower rational Chern classes of E_x are zero.

Proofs. i)–iv) are checked by computing rational cohomology. The isomorphism in iv) is given by the rational Chern classes and implies the universal space for the geometry of v), MU_n, has rational type

$$\text{cofiber}\left(\prod_{i=1}^{n-1} K(Q, 2i) \longrightarrow \prod_{i=1}^{n} K(Q, 2i)\right) .$$

Thus $(MU_n)_Q$ has $K(Q, 2n)$ as a *canonical* retract. By obstruction theory, for any d-skeleton of $K(\mathbf{Z}, 2n)$ there is an integer k_d and a diagram

$$\begin{array}{ccc} K(\mathbf{Z}, 2n)_{d\text{-skeleton}} & \xrightarrow{L} & MU_n \\ \Big\downarrow{\scriptstyle \cdot k_d} & & \Big\downarrow \\ K(Q, 2n) & \xrightarrow{\text{“}c_n\text{”}} & (MU_n)_Q . \end{array}$$

The desired subcomplexes are produced by transversality using L.

LEMMA 2.6. *Let X be a simply-connected space whose reduced integral homology consists of finitely many rational vector spaces of finite rank. Then X is equivalent to an increasing union of finite complexes whose integral homology groups* (mod *torsion*) *embed in those of X as an increasing union of subgroups of maximal rank.*

Proof. X is equivalent to some increasing union of finite complexes $\{Y_n\}$

[1] In case V is an oriented smooth manifold this example shows the smooth submanifolds of V generate the rational homology in every dimension.

which are simply-connected. We can assume each inclusion $Y_n \to X$ is a surjection of rational homology, after discarding the first few spaces.

Suppose $H_i(Y_n; Q) \to H_i(X; Q)$, for some n, is actually an isomorphism for $i < q - 1$. Then consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_q(X, Y_n; Q) & \xrightarrow{j} & H_{q-1}(Y_n; Q) & \xrightarrow{i} & H_{q-1}(X; Q) & \longrightarrow & 0 . \\ & & \uparrow h & & & & & & \\ & & \pi_q(X, Y_n) & & & & & & \end{array}$$

By the Serre form of the relative Hurewicz theorem, image h contains a set of generators. We can then attach a finite number of q-cells to Y_n in X, killing image j and making i into an isomorphism in dimension $q - 1$.

The rational homology picture above dimension $q - 1$ is unchanged. So we can continue in this way to find a new finite complex X_n so that

$$H_q(X_n, Q) \overset{i^*}{\cong} H_q(X, Q)$$

for $q \leqq$ dimension X_n. It follows that i^* is an isomorphism for all q.

Now X_n is contained in some Y_m for $m > n$. We can modify Y_m to get X_m and so on. Then $X = \bigcup X_n$, and the theorem is proved.

LEMMA 2.7. *Let X be a space whose "homotopy groups" are finite dimensional Q vector spaces. Then if Y is a countable complex, two maps $Y \overset{f}{\underset{g}{\rightrightarrows}} X$ are homotopic if and only if they are homotopic on each finite subcomplex of Y.*

Proof. Write Y as the increasing union of connected finite subcomplexes Y_n, $Y = \bigcup_{n=0}^{\infty} Y_n$. Let F_n denote the (based) function space X^{Y_n}. Then we have

i) an inverse system of spaces $\{F_n\}$;

ii) an inverse system of pairs of points $\{f_n \text{ and } g_n\}$ in $\{F_n\}$;

iii) paths in F_n connecting f_n and g_n not necessarily compatible. The homotopy classes of such paths form an inverse system of sets $\{H_n\}$;

iv) an inverse system of groups $\{G_n\} = \{\pi_1(F_n, f_n)\}$ acting on the system of sets $\{H_n\}$. Each action is free with one orbit, and the maps $H_{n+1} \to H_n$ are equivariant with respect to the natural maps $G_{n+1} \to G_n$.

Our tasks are to

a) prove that $\varprojlim_n H_n \neq \varnothing$;

b) realize a formal homotopy in $\varprojlim_n H_n$ by an actual homotopy between f and g.

Now b) is easy if we think of Y as the infinite telescope on the Y_n's. a) is more subtle and makes use of the hypothesis on X.

The main point of the proof is to show that the groups G_n are Q-nilpotent. That is, G_n has a finite ascending central series with successive quotients finite dimensional vector spaces over Q. The argument proceeds in the following way.

i) Consider two finite complexes K and K' where K is obtained from K' by attaching an n-cell. There is an easily derived exact sequence

$$\cdots \longrightarrow \pi_2(X^{K'}, g') \xrightarrow{j} \pi_{n+1}X \xrightarrow{\eta} \pi_1(X^K, g) \xrightarrow{r} \pi_1(X^{K'}, g') \xrightarrow{\nu} \pi_n X .$$

We will assume inductively that we have shown that

a) $\pi_1(X^{K'}, g')$ is Q-nilpotent and

b) $\pi_n(X^{K'}, g')$ is a finite dimensional Q-vector space, $n > 1$.

By exactness, our hypothesis on X, and b), image η is a finite dimensional Q-vector space. A direct geometric argument shows that image η lies in the center of $\pi_1(X^K, g)$.[1] Thus $\pi_1(X^K, g)$ is a central extension of a rational vector space by image r = kernel ν.

Once we have shown kernel ν is Q-nilpotent, we will have established a) for K. b) follows for K using the part of the exact sequence to the left of $\pi_{n+1}X$.

ii) Now Q-nilpotent groups enjoy the properties

a) kernels of maps between Q-nilpotent groups are Q-nilpotent,

b) images of maps between Q-nilpotent groups are Q-nilpotent and

c) any descending sequence of Q-nilpotent groups is finite.

Now a) completes the discussion of i), and b) and c) then imply that the inverse system of groups $\{G_n\} = \{\pi_1(F_n, f_n)\}$ is Mittag-Leffler. That is, for any n, $\bigcap_{k>n}$ image $(G_k \to G_n)$ = image $(G_{k_n} \to G_n)$ for some k_n.

The proof is completed by observing that the inverse system $\{H_n\}$ is Mittag-Leffler because $\{G_n\}$ is and the actions of G_n on H_n are free with one orbit. It follows easily that $\varprojlim \{H_n\} \neq \emptyset$ and we are done given the algebraic properties a), b), and c) of Q-nilpotent groups.

iii) Among nilpotent groups Q-nilpotent groups are characterized by

a′) $(x \mapsto x^n)$ is a bijection for all $n > 0$,

b′) there is a finite set $x_1, \cdots, x_k$ in the group so that every element may be expressed as a word in the $x_1, \cdots, x_k$ with rational exponents.

Proof. An easy induction using the central series shows a Q-nilpotent group satisfies a′) and b′).

[1] The idea is that two elements in $\pi_1(X^K, g)$ commute if they have disjoint support. Elements in image η are concentrated on the n-cell while any element in $\pi_1(X^K, g)$ is supported off the n-cell.

Calculating with commutators shows that *the lower central series* of a group satisfying a′) and b′) has successive quotients which are finite dimensional Q-vector spaces.

This characterization of Q-nilpotent groups makes b) clear. c) is proved by induction on the nilpotent complexity of the ambient group.

If G' denotes the commutator subgroup of G and $\cdots \hookrightarrow G_{i+1} \hookrightarrow G_i \hookrightarrow \cdots \hookrightarrow G$ is a descending sequence of Q-nilpotent groups, then

$$\cdots \hookrightarrow G'_{i+1} \hookrightarrow G'_i \hookrightarrow \cdots \hookrightarrow G'$$

is a descending sequence of smaller nilpotent complexity. By induction $\cdots = G'_{i+1} = G'_i$ for some i. But then the original sequence can have no more than

$$i + \dim_Q (G_i/G'_i)$$

distinct members. This proves c).

To prove a) note that a kernel $K \subset G$ satisfies a′). Then it is easy to see that $K \cap$ (ascending Q-series for G) is an ascending Q-series of K. The successive quotients are clearly finite dimensional Q-vector spaces so K is Q-nilpotent. This completes the proof of the lemma.

Proof of Theorem 2.1. First we show ii) and iii) are equivalent. For this we need three general remarks. Remark a): For studying the map

$$X \xrightarrow{l} X'$$

we can consider an associated map of Postnikov decompositions

$$\text{I} \qquad \begin{array}{ccc} X^n & \xrightarrow{l_n} & (X')^n \\ \downarrow & & \downarrow \\ X^{n-1} & \xrightarrow{l_{n-1}} & (X')^{n-1} \\ \downarrow & & \downarrow \scriptstyle{n^{\text{th}}\ k\text{-invariant of } X'} \\ K(\pi, d_n) & \xrightarrow{k^n(l)} & K(\pi', d_n) \end{array} \qquad n = 1, 2, 3, \cdots$$

where $X^0 = X^0_l = *$, the vertical sequences are fibrations, and

$$X \xrightarrow{l} X' = \varprojlim \{X^n \xrightarrow{l_n} (X')^n\}\,.$$

(The use of $\varprojlim$ here is innocuous because of the skeletal convergence of Postnikov systems. In Section 3 we consider a more non-trivial $\varprojlim$ situation and illustrate one of the pitfalls of $\varprojlim$.)

Remark b): For studying the maps

$$K(\pi, d_n) \xrightarrow{k^n(l)} K(\pi', d_n)$$

which are induced by homomorphisms

$$\pi \xrightarrow{k} \pi'$$

we have the diagram

$$\text{II} \qquad \begin{array}{ccc} K(\pi, n) & \longrightarrow & K(\pi', n) \\ \downarrow & & \downarrow \\ P & \longrightarrow & P \\ \downarrow & & \downarrow \\ K(\pi, n+1) & \longrightarrow & K(\pi', n+1) \end{array} .$$

Here P, "the space of paths", is contractible and the vertical sequences are fibrations.

Remark c): If we have a map of principal fibrations

$$\begin{array}{ccc} F & \xrightarrow{f} & F' \\ \downarrow & & \downarrow \\ E & \xrightarrow{g} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{h} & B' \end{array}$$

then

i) if all spaces are connected and nilpotent and two of the maps f, g, and h localize homotopy, then the third does also;

ii) if two of the maps f, g, and h localize homology, the third does also. The proof of i) follows from the exact ladder of homotopy using 1.5 as in 1.7, with a little care for π_1.

The proof of ii) has two points. First, by 1.8 if two of the homologies

$$\tilde{H}_* F' , \quad \tilde{H}_* E' , \quad \tilde{H}_* B'$$

are local the third is also. Second, if we know the homologies on the right are local then to complete the proof of ii) it is equivalent to check that f, g, h induce isomorphisms on

$$\tilde{H}_*(\ , Z_l)$$

since, e.g.,

$$\tilde{H}_*(F') \cong \tilde{H}_* F' \otimes Z_l \cong \tilde{H}_*(F', Z_l) .$$

But this last point is clear since if two of f, g, h induce isomorphisms on

$H^*(\ \ , Z_l)$ the third does also by the spectral sequence comparison theorem.

With these remarks in mind it is easy now to see that a map of simple Postnikov spaces

$$X \xrightarrow{l} X'$$

localizes homotopy if and only if it localizes homology.

Step 1. The case

$$(X \xrightarrow{l} X') = \big(K(\pi, 1) \xrightarrow{l} K(\pi', 1)\big) , \qquad \pi \text{ and } \pi' \text{ Abelian} .$$

If l localizes homology, then it localizes homotopy since $\pi = H_1X$, $\pi' = H_1X'$.

If l localizes homotopy then

$$(\pi \longrightarrow \pi') = (\pi \longrightarrow \pi_l) .$$

So l localizes homology if

a) $\pi = Z$; for l is just the localization

$$S^1 \longrightarrow S^1_l$$

studied above.

b) $\pi = Z/p^n$; for $\pi_l = 0$ if $p \notin l$,
$\pi_l = Z/p^n$ if $p \in l$.

c) general π; take finite direct sums and then direct limits of the first two cases.

Step 2. The case

$$(X \xrightarrow{l} X') = \big(K(\pi, n) \xrightarrow{l} K(\pi', n)\big) , \qquad n > 1 .$$

If l localizes homology, then it localizes homotopy as in Step 1 because $\pi = H_nX$, $\pi' = H_nX'$.

If l localizes homotopy, then we use induction, Step 1, diagram II in Remark b) and Remark c) to see that l localizes homology.

Step 3. The general case $X \xrightarrow{l} X'$.

If l localizes homotopy then apply Step 2 and diagram I inductively to see that each $X^n \xrightarrow{l_n} (X')^n$ localizes homology for all n. Then $l = \varprojlim l_n$ localizes homology.

If $X \xrightarrow{l} X'$ localizes homology consider the following argument. We have now justified the steps of the proof of Theorem 2.3 by the first part of Step 3. Let Y denote the localized Postnikov system of X. Then we have a diagram

$$\begin{array}{ccc} X & \xrightarrow{l} & X' \\ & {\scriptstyle l_p}\searrow & \downarrow{\scriptstyle f} \\ & & Y \end{array}$$

and f exists by obstruction theory (see details below). The maps l and l_p localize homology so f must be a homology isomorphism. Since X' and Y are simple Postnikov spaces f is a homotopy equivalence. Thus l localizes homotopy.

Now we show that i) and ii) are equivalent.

If $X \xrightarrow{l} X'$ is universal for maps into local spaces Y, then by taking Y to be various $K(\pi, n)$'s with π local we see that l induces an isomorphism of $H^*(\ , Q)$ and $H^*(\ , \mathbf{Z}/p)$, $p \in l$. Thus l induces homomorphisms of

$$H_*(\ , Q) \text{ and } H_*(\ , \mathbf{Z}/p)\,, \qquad p \in l$$

which must be isomorphisms because their dual morphisms are. Using the Bockstein sequence

$$\cdots \longrightarrow H_i(\ , \mathbf{Z}/p^n) \longrightarrow H_i(\ , \mathbf{Z}/p^{n+1}) \longrightarrow H_i(\ , \mathbf{Z}/p^n) \longrightarrow \cdots$$

and induction we see that l induces an isomorphism on

$$H_*(\ , \mathbf{Z}/p^n) \text{ for all } n\,.$$

Thus l induces an isomorphism on

$$H_*(\ , \mathbf{Z}/p^\infty)$$

since taking homology and tensoring commute with direct limit, and

$$\mathbf{Z}/p^\infty = \varinjlim_n \mathbf{Z}/p^n\,, \qquad p \in l\,.$$

Finally l induces an isomorphism of

$$H_*(\ , Z_l)$$

using the coefficient sequence

$$0 \longrightarrow \mathbf{Z}_l \longrightarrow Q \longrightarrow Q/\mathbf{Z}_l \longrightarrow 0$$

and the equivalence

$$Q/\mathbf{Z}_l = \bigoplus_{p \in l} \mathbf{Z}/p^\infty\,.$$

Now X' is a local space by definition. Thus the homology of X' is local by what we proved above. This proves i) implies ii).

To see that ii) implies i) consider the obstruction to uniquely extending f to f_l in the diagram

$$\begin{array}{ccc} X & \xrightarrow{l} & X' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f_l} \\ & & Y \end{array} .$$

These lie in

$$H^*(X', X; \text{“}\pi_* Y\text{”}) .$$

Now “$\pi_* Y$” is a $\mathbf{Z}_l$-module and l induces an isomorphism of Z_l homology. Using the natural sequence (over Z_l)

$$0 \longrightarrow \operatorname{Ext}(H_i(\ \ , \mathbf{Z}_l), \mathbf{Z}_l) \longrightarrow H^{i+1}(\ \ , \mathbf{Z}_l) \longrightarrow \operatorname{Hom}(H_{i+1}(\ \ , \mathbf{Z}_l), \mathbf{Z}_l) \longrightarrow 0$$

we see that l induces an isomorphism of $\mathbf{Z}_l$-cohomology. By universal coefficients (over $\mathbf{Z}_l$) the obstruction groups all vanish.

Now think of Y as the actual inverse limit of its n-stages, $Y = \varprojlim Y_n$. The above calculations show

i) each composition $X \xrightarrow{f} Y \rightarrow Y_n$ extends over X',

ii) any two such extensions are homotopic, and

iii) any two homotopies between two such extensions are homotopic.

i) and ii) enable us to find an actual extension of $X \xrightarrow{f} Y$ over X'.

ii) and iii) enable us to construct a homotopy between any two such extensions.[1]

Thus there is a unique extension f_l, and l is a localization. Q.E.D.

3. Completions in homotopy theory

In this section we discuss finite and p-adic completions in homotopy theory. The first motivation is to be able conveniently to treat the information coming from algebraic geometry via étale cohomology theory. Thus algebraic varieties and morphisms will provide a rich supply of p-adic spaces and mappings between them.

The second motivation is to understand how the p-primary and rational information in a homotopy type interact. The last theorem of the section describes an explicit equivalence of categories which shows how the spaces and morphisms of classical homotopy theory are constructed from rational spaces, p-adic spaces, and homotopy classes of maps between them respecting a rational coherence condition.

We begin by constructing the finite completion $X \rightarrow \hat{X}$ of a connected cell complex X. $\hat{X}$ is a cell complex with the extra structure of a compact

[1] This argument shows that the localization construction implicitly admits the construction of associated “infinite homotopies”.

topology on the mapping sets $[Y, \hat{X}]$. In many cases the homotopy theory of $\hat{X}$ is simply related to that of X (Theorem 3.1) and even determines this extra topological structure.

We also define l-adic completions $X \rightarrow \hat{X}_l$ for l a set of primes. Under the appropriate assumptions on X there is a "continuous" equivalence

$$\hat{X} \cong \prod_p \hat{X}_p$$

where p ranges over the set of primes. A formal completion $X \rightarrow \hat{X}_f$ is defined for treating rational homotopy types.

These construction are combined to make the genetic analysis described in the introduction.

One somewhat surprising result of the section is a Hasse principal for maps which asserts that f is homotopic to g if and only if f and g are homotopic after p-adic completion for each prime p (Theorem 3.2).

The construction of the finite completion $\hat{X}$ was motivated by the work of Artin and Mazur. They viewed the completion more formally as the inverse system determined by the category of all maps of X into spaces with finite homotopy groups. We construct the single space $\hat{X}$ from this inverse system using a little theory about "compact Brownian functors".

Construction of the finite completion $\hat{X}$.

We outline the construction and begin with the following observation. Let F denote a space with finite homotopy groups. Then the functor defined by homotopy classes of maps into F, $[\ , F]$ may be *topologized* in a natural way. This (compact) topology arises from the equivalence

$$[Y, F] \longrightarrow \varprojlim_\alpha [Y_\alpha, F]$$

where Y_α ranges over the finite subcomplexes Y_α of Y.

Now given X, consider the category $\{f\}_X$ whose objects are maps

$$X \xrightarrow{f} F, \qquad \pi_1 F \text{ finite, } F \text{ connected}$$

and whose morphisms are homotopy commutative diagrams

$$\begin{array}{ccc} & & F \\ & \nearrow & \\ X & & \downarrow \\ & \searrow & \\ & & F' . \end{array}$$

This category is suitable for forming inverse limits (Proposition 3.3 below) and a functor $\hat{X}$ is defined by

$$\hat{X}(Y) = \varprojlim_{\{f\}_X} [Y, F] .$$

The compact topology on $[Y, F]$ extends via the inverse limit to a compact topology on $\hat{X}(Y)$. Using the compactness we find that the functor $\hat{X}$ is represented by the homotopy functor $[\ ,\hat{X}]$ for some CW complex $\hat{X}$ (property II of compact Brownian functors below).

Assuming all this for the moment, we make the

Definition (*finite completion*). The finite completion of a connected cell complex X is the map of connected cell complexes $X \to \hat{X}$ where $\hat{X}$ is constructed from the inverse system of spaces $\{F\}$ above and has the compact topology on the mapping sets $[\ ,\hat{X}]$ determined by the inverse limit.

As a result of finite completion, $\hat{X}$ can be thought of as a "profinite space" with profinite homotopy groups, profinite mapping sets $[\ ,\hat{X}]$, etc. It is clear from the definitions that the finite completion is a natural construction on the homotopy category.

To justify our definition of the finite completion and to aid future computations we discuss a little theory about "compact Brownian functors".

Recall Brown's theorem that a contravariant functor from the category of cell complexes and homotopy classes of maps to sets (everything based),

$$Y \longrightarrow G(Y) ,$$

is equivalent to $Y \to [Y, B]$ for some cell complex B if and only if two axioms hold. The Brown axioms are

i) the natural map

$$G(\bigvee_\beta Y_\beta) \longrightarrow \prod_\beta G(Y_\beta)$$

is a bijection. ("$\bigvee$" denotes one point union.)

ii) the natural sequence

$$G(X \cup Y) \longrightarrow G(X) \times G(Y) \rightrightarrows G(X \cap Y)$$

is exact.[1] Here X and Y are subcomplexes of $X \cup Y$ (for a proof see [13]).

Motivated by this characterization we make the

Definition. A "compact Brownian functor" is a contravariant functor from the homotopy category to the topological category of compact Hausdorff spaces satisfying the properties i) and ii) of Brown.

Note that a set bijection in property i) implies homeomorphism because the natural map is continuous, $G(\bigvee_\beta Y_\beta)$ is compact and $\prod_\beta G(Y_\beta)$ is Hausdorff.

---

[1] $A \xrightarrow{j} B \overset{f}{\underset{g}{\rightrightarrows}} C$ is exact when $f(b) = g(b)$ if and only if $b = j(a)$ for some $a \in A$.

We will often use the following two properties of compact Brownian functors.

I. (Extension property). Let G be a compact Brownian functor defined only on the subcategory generated by finite cell complexes. (We assume Brown's axioms hold when they make sense.) *Then the underlying set-valued functor has a unique extension*[1] *to the entire homotopy category satisfying Brown's axioms.*

This unique extension is given by the formula

$$(*) \qquad G(Y) = \varprojlim_{\alpha} G(Y_\alpha)$$

where Y_α ranges over the finite subcomplexes of an arbitrary complex Y. Thus *the unique extension is again a compact Brownian functor.*

Proof of I. One can check easily that the right hand side of (*) defines a compact Brownian functor. The first axiom of Brown is checked formally. The second axiom uses the compactness and the basic fact that the inverse limit of non-void compact Hausdorff spaces in a non-void compact Hausdorff space. For more details see the proof of property II below.

This proves there is at least one Brownian extension of G to the entire homotopy category. To prove uniqueness, note that any extension of G maps to this inverse limit extension. This map is clearly an isomorphism for finite complexes and so it is for all complexes by Whitehead's theorem.

II. (Inverse limit property). The arbitrary inverse limit of compact Brownian functors is again a compact Brownian functor.

Note on II. In property II we may allow ourselves[2] to form inverse limit over any small filtering category $\mathcal{C}$, namely,

i) (smallness of $\mathcal{C}$) the objects of $\mathcal{C}$ form a set,

ii) (directedness of $\mathcal{C}$) every pair of objects c, d of $\mathcal{C}$ can be embedded in a diagram

$$\begin{matrix} c & \searrow & \\ & & e\,, \\ d & \nearrow & \end{matrix}$$

iii) (essential uniqueness of maps in $\mathcal{C}$) if $c \rightrightarrows d$ is a pair of maps of $\mathcal{C}$ there is a map $d \to e$ so that the composed maps $c \rightrightarrows e$ are equal.

Proof of II. If $\{G_\alpha\}$ is an inverse system of compact Brownian functors, define the inverse limit by

[1] Up to canonical equivalence.

[2] See the appendix to Artin-Mazur "Étale Homotopy" [5] for the formal discussion of general limits.

$$(\varprojlim_\alpha G_\alpha)(Y) = \varprojlim_\alpha \left(G_\alpha(Y)\right) .$$

The values are naturally compact Hausdorff spaces and the first axiom of Brown is again checked formally. The second axiom is checked as follows: Let $G_\infty = \varprojlim_\alpha G_\alpha$. Then if $(x, y) \in G_\infty(X) \times G_\infty(Y)$ and x and y agree in $G_\infty(X \cap Y)$, we have for each index α at least one μ_α in $G_\alpha(X \cup Y)$ so that μ_α restricts to the projections of x and y in $G_\alpha X$ and $G_\alpha Y$ respectively. The set of such μ_α is determined by two closed conditions and so forms a compact subspace of $G_\alpha(X \cup Y)$. The inverse limit of these subspaces is again non-void so we have at least one element μ_∞ in $G_\infty(X \cup Y)$ restricting to x and y in $G_\infty(X)$ and $G_\infty(Y)$ respectively. This proves the non-obvious half of exactness in

$$G_\infty(X \cup Y) \longrightarrow G_\infty(X) \times G_\infty(Y) \rightrightarrows G_\infty(X \cap Y) .$$

Examples.

I. *Profinite K-theory.* Consider one of the natural K-theory functors defined on finite complexes by associating to Y the group of stable equivalence classes of vector bundles over Y. Since $K(Y)$ is a finitely generated Abelian group the new functor $\hat{K}$ defined by

$$\hat{K}(Y) = \varprojlim_n K(Y) \otimes \mathbf{Z}/n$$

is a compact Brownian functor on finite complexes. By property I, $\hat{K}$ has a unique extension satisfying Brown's axioms, and we have a natural definition of "profinite K-theory".

II. *Spaces with finite homotopy groups.* Consider a space F with finite homotopy groups. It is easy to see by obstruction theory that

$$[Y, F] \text{ is finite}$$

for each finite complex Y. Thus $[\ , F]$ is a compact Brownian functor when restricted to finite complexes.

By property I we have that the total functor $[\ , F]$ must be the unique extension of this partial Brownian functor given by the inverse limit formula

$$[Y, F] \cong \varprojlim_\alpha [Y_\alpha, F] .$$

So the functor $[\ , F]$ *has a natural compact topology.*

III. *Spaces with profinite homotopy groups.* If $\{F_\alpha\}$ is any inverse system[1] of spaces with finite homotopy groups, we can use the above remarks to form an inverse limit space $\hat{F}$ so that

[1] Over a suitable indexing category.

$$(*)\pi_i\hat{F} = \varprojlim_\alpha \pi_i F_\alpha \,.$$

By Example II the homotopy functor $[\ ,F_\alpha]$ has a natural compact topology. By property II of compact Brownian functors

$$\varprojlim_\alpha [\ ,F_\alpha]$$

is again a compact Brownian functor and is thus represented set theoretically by $[\ ,\hat{F}]$ for some cell complex $\hat{F}$. Putting in S^i gives the equation (*) and we see that $\hat{F}$ has profinite homotopy groups.

In case the index category contains a countable cofinal subcategory it is possible to represent the topology on $[\ ,\hat{F}]$ by a second topology on the cell complex $\hat{F}$. The second topology on $\hat{F}$ induces, via the compact open topology on the function spaces $\hat{F}^Y$, the above compact topology on $[Y, \hat{F}]$. For example, for an infinite product the product topology on

$$\hat{F} = \prod_\alpha F_\alpha$$

plays the role of the second topology.

IV. *A counterexample.* The essential nature of compactness for forming inverse limits of homotopy functors is easily illustrated by an example. Let L denote the inverse limit of the homotopy functor $[\ ,S^2]$ using the endomorphism induced by the degree 3 map of S^2. It is easy to check that $L(S^1)$ and $L(S^2)$ each have one element but $L(\mathrm{RP}^2)$ has two elements. Thus L is not represented by $[\ ,B]$ for any space B.

V. *Cohomology with profinite coefficients.* If A is the profinite group $\varprojlim_i A_i$, A_i finite, then from Example III it is clear that

$$K(A, n) \cong \varprojlim_i K(A_i, n)$$

as compact Brownian functors. Thus for all spaces X

$$H^*(X, A) = \varprojlim_i H^*(X, A_i)\,,$$

and the A-cohomology has a natural topology. For example,

$$H^*(X, \hat{\mathbf{Z}}) \cong \varprojlim_i H^*(X, \mathbf{Z}/i)$$

for all spaces X.

For finite complexes it follows from the universal coefficient formulae that we also have

$$H^*(X, \mathbf{Z})^\wedge \cong \varprojlim_i H^*(X, \mathbf{Z}/i)\,.$$

VI. *The classifying space for profinite K-theory.* The profinite K-theory

$Y \to \hat{K}(Y)$ constructed in Example I is classified by maps into some space $\hat{B}$ because $\tilde{K}$ is a compact Brownian functor. This space $\hat{B}$ has homotopy groups

$$\hat{Z}\ 0\ \hat{Z}\ 0\ \hat{Z}\ 0\ \hat{Z} \cdots$$

for the complex case and

$$Z/2\ Z/2\ 0\ \hat{Z}\ 0\ 0\ 0\ \hat{Z}\ Z/2\ Z/2\ 0\ \hat{Z}\ 0\ 0\ 0\ \hat{Z} \cdots$$

for the real case using the classical results of Bott on $K(S^i)$. This follows from

$$\pi_i \hat{B} \equiv [S^i, \hat{B}] \cong \hat{K}(S^i) \equiv \varprojlim_n K(S^i) \otimes Z/n .$$

In either case $\hat{B}$ may be described in two ways,

i) $\hat{B}$ is just the finite completion of the classifying space B of the usual K-theory, the union of Grassman manifolds,

$$B = \bigcup_{n,k} G_{n,k} \quad (\text{over } R \text{ or C}) .$$

(See Theorem 3.7 below.)

ii) $\hat{B}$ is also the inverse limit of the classifying spaces B_n for K-theory with coefficients in Z/n,

$$\hat{B} = \varprojlim_n B_n \quad \text{and} \quad \hat{K}(Y) = \varprojlim_n K(Y, Z/n) .$$

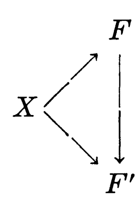
This relation follows because for finite complexes we have the isomorphism

$$\hat{K}(Y) \cong \varprojlim_n K(Y, Z/n)$$

because of the long exact sequence for coefficients.

Our justification and discussion of the definition of the finite completion will be completed by

PROPOSITION 3.3 (Artin-Mazur). *The category $\{f\}_X$ whose objects are maps $X \to F$, $\pi_* F$ finite, and whose morphisms are commutative diagrams*

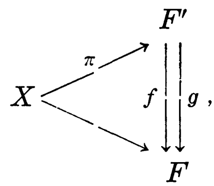


is suitable for forming inverse limits.

Proof. First, $\{f\}_X$ contains an equivalent *small* sub-category. We can find one by picking a representative from each homotopy type of the F's.

Second, the directedness of $\{f\}_X$ is clear. To $X \xrightarrow{f_1} F_1$ and $X \xrightarrow{f_2} F_2$ we associate $X \xrightarrow{f_1 \times f_2} F_1 \times F_2$.

Third, the essential uniqueness of maps in $\{f\}_X$ is clear from the co-equalizer construction in homotopy theory. Given



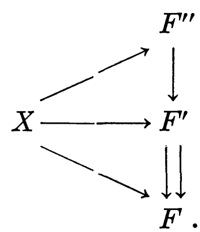
let F'' be an appropriate component of

$$\{(p, x) \in F^I \times F' \colon p(0) = f(x),\ p(1) = g(x)\}\ .$$

Then F'' has finite homotopy groups using the exact sequence

$$\cdots \longrightarrow \pi_i F'' \longrightarrow \pi_i F' \xrightarrow{f_* - g_*} \pi_i F \longrightarrow \cdots\ ;$$

and using a homotopy between $f \circ \pi$ and $g \circ \pi$ we can form the commutative diagram



This completes the proof that $\{f\}_X$ is equivalent to a small filtering category.

General remarks, computations, and examples.

We study some properties of the cell complex $\hat{X}$, its extra structure, and its relationship to X.

Property i). For the space X, the finite completion $X \to \hat{X}$, and the inverse system $\{F\}$ over $\{f\}_X$ we have

a) $\hat{X} = \varprojlim_{\{f\}_X} F$ by definition,

b) $(\pi_1 X)^\wedge \cong \varprojlim_{\{f\}_X} \pi_1 F$,

c) $H^*(X, M) \cong \varinjlim_{\{f\}_X} H^*(F, M)$

for all finite coefficient systems M.

Conversely, the map $X \to \hat{X}$ and the topology on $[\ \ ,\hat{X}]$ are characterized by this situation; namely, if $\{F_\alpha\}$ is any inverse system of spaces with finite homotopy groups into which X maps so that $(\pi_1 X)^\wedge \cong \varprojlim_\alpha \pi_1 F_\alpha$ and

$H^*(X, M) \cong \varinjlim_\alpha H^*(F_\alpha, M)$ then there is a "continuous" homotopy equivalence

$$\hat{X} \cong \varprojlim_\alpha F_\alpha \, .$$

Proof of property i). b) follows easily from the definition by covering space arguments. For example, for any map $X \xrightarrow{f} F$ consider a lifting $X \xrightarrow{\tilde{f}} \tilde{F}$ where $\tilde{F}$ is the covering of F corresponding to the image of f on π_1. Now $\tilde{f}$ is surjective on π_1. Also any finite quotient π of π_1 occurs. For example, take the map $X \to K(\pi, 1)$. So the sub-category of $\{X \xrightarrow{f} F\}$ where f is surjective is cofinal and b) clearly holds.

c) is also easy to prove. Any class in $H^i(X, M)$, for M a finite coefficient system, is induced by a map $X \to F$ where F is a space with (at most 2 non-zero) finite homotopy groups. So the left map of c) is onto. Conversely if a class in $H^i(F, M)$ goes to zero in X, use the class as a k-invariant to construct a (twisted) fibration over F into which X maps. This shows the class is zero in the direct limit group, and c) is proved.

For the converse, first note that we can replace the big category $\{X \to F\}$ by the sub-category $\bigcup_n \{X \to F_n\} = \{f\}^\#_X$ where F_n has zero homotopy groups above n. It is clear that the natural map $\hat{X} \to \varprojlim_{\{f\}^\#_X} F_n$ is a "continuous" homotopy equivalence.

Now let F^n_α be the space obtained from F_α by killing homotopy above n. Then clearly $F_\alpha = \varprojlim_\alpha F^n_\alpha$ and $\varprojlim_\alpha F_\alpha = \varprojlim_{n,\alpha} F^n_\alpha$. We may regard the $\{F^n_\alpha\}$ as a subsystem of the system $\bigcup_n \{F_n\}$ above.

Properties b) and c) for the $\{F_\alpha\}$ hold also for the $\{F^n_\alpha\}$ and elementary obstruction theory may be used to show that $\{X \to F^n_\alpha\}$ is cofinal in $\bigcup_n \{X \to F_n\} = \{f\}^\#_X$. So the converse is proved.

Main example. Let V_t denote the underlying topological space of a (finite type) complex algebraic variety V. So V is made up of a finite number of affine varieties $V_i \subset \mathbf{C}^n$ defined by polynomial equations,

$$V_i = \{(z_1, \cdots, z_n) \in \mathbf{C}^{n_i}: p_{\alpha_i}(z_1, \cdots, z_n) = 0\} \, .$$

Then the Zariski open sets, the complements of subvarieties of V, and finite covering spaces of these have algebraic meaning. Of course the first fact is a tautology while the second is a deep theorem of complex analysis.

Anyway, we can treat finite étale coverings of V,

$$\begin{array}{ccc} U_1 & U_2 & U_n \\ \searrow & \downarrow & \swarrow \\ & V & \end{array} , \qquad \bigcup_i \text{image } U_i = V$$

where each U_i is a finite covering space of some Zariski set open in V, in a Čech-like manner and construct associated nerves which record the combinatorial information in the cover; that is, in the category whose objects are the U_i and the morphisms are maps $U_i \to U_j$ which commute with the projections to V([10]).

We obtain an inverse system of complexes $\{N_\alpha\}$ indexed by the étale coverings of V. Using the corresponding complexes associated to coverings of V by tiny contractible sets—say the stars of vertices in triangulations of V—we obtain a map of V_t into the system of étale nerves $\{N_\alpha\}$.

The basic comparison theorem of Artin-Grothendieck [5] is that under mild assumptions on V (say, V is normal[1]), the homotopy groups of N_α are finite and

$$(\pi_1 V_t)^\wedge \cong \varprojlim_\alpha \pi_1 N_\alpha \quad \text{and} \quad H^*(V_t, M) \cong \varinjlim_\alpha H^*(N_\alpha, M)$$

for all finite π_1-modules M.

Thus $\varprojlim_\alpha N_\alpha$ is the finite completion $\hat{V}_t$ of V_t described above and defined now here in a purely algebraic way.

Remarks on the proof. The fundamental group statement follows immediately from the statement that a finite covering of a complex algebraic variety is algebraic.

The cohomology statement for the case when V is a non-singular projective variety depends on the following geometric fact: If $p \in V$ is contained in a Zariski open set $U \subsetneqq V$, then there is a smaller Zariski open set $W \subsetneqq U$ containing p which is a $K(\pi, 1)$. Moreover, we can choose W so that the group π is built up inductively from a point by extending finitely generated free groups.

For $\dim V = 1$, this fact is easy since a Riemann surface minus a finite number of points is a $K(\pi, 1)$ where π is an f.g. (free group).

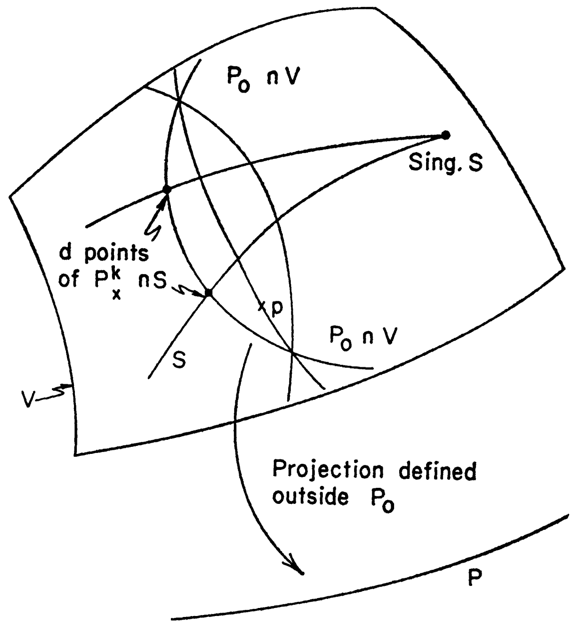
For $v = \dim V > 1$, let $S = V - U$ and consider P the $(v - 1)$ dimensional projective space of planes P^k containing a given $k - 1$ plane P_0, where $k + v = 1 +$ ambient dimension.

If P_0 intersects V transversely and then only in the complement of S and p, a Zariski open set X in the space P of the k-planes P^k will i) intersect V transversely; ii) intersect S only in $S -$ sing S and there transversely in d points where d is constant on X; iii) be such that, for some $x_0 \in X$, the corresponding $P^k_{x_0}$ passes through p.

If we remove, for each $x \in X$, $(P_0 \cap P^k_x) \cup (P^k_x \cap S)$, we are removing (degree V) $+ d$ points from each smooth curve $P^k_x \cap V$.

[1] Roughly, the singularities have codimension at least two.

The family of these deleted curves forms a Zariski open W' in V containing p and contained in U. Moreover W' is patently fibered by punctured Riemann surfaces over X. By induction we may choose a Zariski $K(\pi, 1)$ in X containing x_0. Then the desired W is the restricted fibration over this $K(\pi, 1)$.



AMBIENT PROJECTIVE SPACE

This geometric fact leads rather quickly to the eventual finite acyclicity required in the nerves N_α to get the correct cohomology.

Example 2. Let $X = K(Q/Z, 1)$. Then $(\pi_1 X)^\wedge = 0$ and a cohomology check shows $X \xrightarrow{\text{Bockstein}} \{K(Z/n, 2)\}$ is suitable for computing $\hat{X}$. Therefore $\hat{X} = \varprojlim_n K(Z/n, 2) = K(\hat{Z}, 2)$. Similarly $K(Q/Z, n)^\wedge \cong K(\hat{Z}, n + 1)$.

Example 3. If X is the classifying space B_{O_2} of the orthogonal groups O_2, then the twisted fibrations $K(Z/n, 2) \to E_n \to K(Z_2, 1)$ where $Z/2$ acts on Z/n by $x \mapsto -x$ are suitable for computation, and $\hat{X} = \varprojlim E_n$, the total space of the fibration

$$K(Z, 2) \longrightarrow \hat{X} \longrightarrow K(Z/2, 1)$$

with the action $x \mapsto -x$ on $\hat{Z}$. Theorem 3.1 below generalizes this situation.

Example 4. Let X be a $K(\pi, 1)$ where π is the direct limit of the finite symmetric groups. Let F denote the direct limit under suspension of the spaces of degree zero maps of S^n to itself.

We can approximate X by the space, {sets of n-distinct points in R^N; n, N large}. Given n points $x_1, \cdots, x_n$ in R^N, define a degree n map of S^N by mapping little disjoint islands around the x_i onto S^N with degree 1, with the boundary of each island and the sea in between mapping to $\infty \in S^N$.

We can translate these maps to the component of degree zero, and it has been checked[1] that

$$K(\pi, 1) \longrightarrow F$$

induces an isomorphism of $\hat{\pi} = Z/2 \cong \pi_1 F$ and of the cohomology.

Thus $K(\pi, 1)^\wedge \cong F$, and the infinite symmetric group π is converted by finite completion into the stable homotopy groups of spheres, $\pi_* F$.

Property ii) (The cohomology and higher homotopy groups of $\hat{X}$). In the previous paragraph we found $(\pi_1 X)^\wedge \cong \pi_1 \hat{X}$ as topological groups. What about the higher homotopy groups of X?

We also found $\varinjlim_\alpha H^*(F_\alpha, M) \cong H^*(X, M)$. What about $H^*(\hat{X}, M)$?

To study these questions we formulate the notion of "good homotopy groups" of a space. Then we prove the

THEOREM 3.1. i) *Groups commensurable*[2] *with solvable groups of finite type*[3] *are "good" fundamental groups. Finitely generated Abelian groups are "good" higher homotopy groups.*

ii) *If X has "good" homotopy groups then*

$$(\pi_i X)^\wedge \cong \pi_i \hat{X} \quad \text{and} \quad H^i(X, M) \cong H^i(\hat{X}, M)$$

for all finite coefficients system M.

COROLLARY. *If X has "good" homotopy groups every map $X \to F$ extends uniquely over $\hat{X}$. Thus X and $\hat{X}$ have isomorphic finite completions, and the topology on $[\ ,\hat{X}]$ is intrinsic to the homotopy type of $\hat{X}$.*

Proof of Corollary. This follows immediately by obstruction theory.

To begin our discussion we note that the statement $(\pi_k X)^\wedge \cong \pi_k \hat{X}$ of the theorem is not at first sight the natural statement.

Consider the diagram

$$\begin{array}{ccc} \pi_n X & \longrightarrow & (\pi_n X)^\wedge \\ \downarrow & & \downarrow p \\ \pi_n \hat{X} & \underset{q}{\longleftarrow} & (\pi_n X)^\wedge_{\pi_1} \end{array}$$

[1] This was contributed to by Barratt, Kahn, Milgram, Priddy, Quillen, et al.

[2] Two groups are commensurable if there is a third group which is isomorphic to a subgroup of finite index in each one.

[3] Such groups can be described as solvable groups in which every subgroup is finitely generated.

where $(\pi_n X)_{\pi_1}$ denotes the "equivariant profinite completion of π_n", the inverse limit of the finite π_1-quotients of π_n. q exists because $\pi_n \hat{X}$ is by definition an inverse limit of finite π_1-modules.

Now p is always onto but not always an isomorphism. Thus it is more natural to ask when q is an isomorphism.

Example. Let $X = S^1 \vee \sum \mathrm{RP}^2$, the one point union of a circle with the suspension of the real projective plane. One can check that $\{K(Z/n, 1) \vee \sum \mathrm{RP}^2\}$ is an appropriate system for determining $\hat{X}$ by computing cohomology. The universal covering space of $K(Z/n, 1) \vee \sum \mathrm{RP}^2$ is the infinite sphere with n-copies of $\sum \mathrm{RP}^2$ attached along symmetric points of a great circle. So $\pi_2(K(Z/n, 1) \vee \sum \mathrm{RP}^2)$ is the direct sum of $n(Z/2)$'s with the cyclic action of $\pi_1 = Z/n$. It follows that $\pi_2 \hat{X} = \varprojlim_n (\prod_{i=1}^n Z/2) = \prod_{-\infty}^{\infty} Z/2$ with $(\pi_1 X = Z) \subsetneqq (\pi_1 \hat{X} = \hat{Z})$ acting by shift of coordinates.

Now the universal cover of X is the real line with a copy of $\Sigma \mathrm{RP}^2$ attached at each integer point. Thus $\pi_2 X$ is $\bigoplus_{-\infty}^{\infty} Z/2$ with $\pi_1 X$ acting by the shift again.

Then an algebraic calculation shows $(\pi_2 X)^{\wedge}_{\pi_1} = (\bigoplus_{-\infty}^{\infty} Z/2)^{\wedge}_{Z} \cong \prod_{-\infty}^{\infty} Z/2 = \pi_2 \hat{X}$. Thus q is an isomorphism.

On the other hand, one can compute the finite completion of the countable vector space $V = \pi_2 X$ to be the double dual of V,

$$(\pi_2 X)^{\wedge} \cong \mathrm{Hom}\,(\mathrm{Hom}\,(V, Z/2), Z/2)$$

which has a greater cardinality than $(\pi_2 X)^{\wedge}_{Z} = \pi_2 \hat{X}$. So p is not an isomorphism.

We do not understand this phenomenon in general, but we can impose conditions that insure that both p and q are isomorphisms.

We say that π_1 is a "good" fundamental group if the system of finite quotients $\{\pi_\alpha\}$ of π_1 has a countable, cofinal subsystem and

$$H^i(\pi_1, M) \cong \varinjlim_\alpha H^i(\pi_\alpha, M) \cong H^i(\varprojlim_\alpha \pi_\alpha, M) < \infty$$

for each finite π_1-module M, and all i.

This condition is equivalent to the analogous condition using only simple coefficients for all subgroups of finite index in π_1 (see below and [5]).

For any π_1 we say that the π_1-module π_n is a "good" n^{th} homotopy group if the system of finite π_1-quotients $\{\pi_\alpha\}$ has a countable, cofinal subsystem and

$$H^i(\pi_n, A) \cong \varinjlim_\alpha H^i(\pi_\alpha, A) \cong H^i(\varprojlim_\alpha \pi_\alpha, A) < \infty$$

for all finite coefficient groups A, and all i.

Now we prove Theorem 3.1 with the more refined statement

$$(\pi_n \hat{X}) \overset{q}{\cong} (\pi_n X)^\wedge_{\pi_1} \overset{p}{\cong} (\pi_n X)^\wedge ,$$

as topological groups.

Proof. We have already treated the case $n = 1$. For $n > 1$ the p isomorphism is easy. Since $\pi = \pi_n$ is a "good" homotopy group we can take $i = 1$, $A = Z/k$ to deduce,

$$\text{finite group} \cong \operatorname{Hom}(\pi, Z/k) \cong \varinjlim_\alpha \operatorname{Hom}(\pi_\alpha, Z/k) \cong \operatorname{Hom}(\varprojlim_\alpha \pi_\alpha, Z/k) .$$

From this it is not hard to see that $\varprojlim_\alpha \pi_\alpha \cong \hat{\pi}$. But $\varprojlim_\alpha \pi_\alpha \cong (\pi)^\wedge_{\pi_1}$ by definition of $\{\pi_\alpha\}$. Thus p is an isomorphism.

Now we consider q. Let X_n denote the space with the first n-homotopy groups of X and $\pi_i X_n = 0$, $i > n$. Consider the inductive statement: there is a countable simply-ordered inverse system, $\{F^n_\alpha\}$ of spaces with $(\leqq n)$ finite homotopy groups into which X maps such that

i) $(\pi_i X_n)^\wedge_{\pi_1} \cong \varprojlim_\alpha \pi_i F^n_\alpha$,

ii) for each finite π_1-module M,

$$H^i(X_n, M) \cong \varinjlim_\alpha H^i(F^n_\alpha, M) \cong H^i(\varprojlim_\alpha F^n_\alpha, M) < \infty .$$

This is true for $n = 1$ since π_1 is a good fundamental group.

Assume we have this for n. Let k denote the k-invariant of $X_{n+1} \to X_n$ in $H^{n+2}(X_n, \pi)$, where π denotes the coefficient group $\pi_{n+1} X_{n+1} = \pi_{n+1} X$ twisted by $\pi_1 X_{n+1} = \pi_1 X$. Let k_β denote the image of k in $H^{n+2}(X_n, \pi_\beta)$ where π_β ranges through some linearly ordered cofinal family of finite π_1-quotients of π. By the surjectivity of ii) k_β comes from $F^n_{\alpha_\beta}$ for some α_β. Use this class over $F^n_{\alpha_\beta}$ to form a fibration

$$K(\pi_\beta, n + 1) \longrightarrow E^n_{\alpha_\beta} \longrightarrow F^n_{\alpha_\beta} .$$

We do this for every β and use the injectivity of ii) to fit these together to form a linear inverse system $\{E_\beta\}$ mapping to $\{F_\beta\}$ where the β's form a cofinal family contained in the α's, and the fibers are $\{K_\beta\} = \{K(\pi_\beta, n + 1)\}$.

From our construction it is clear that $X_{n+1} \to \{E_\beta\}$ and that i) is true for $i = n + 1$.

Now consider a finite π_1-module M. M defines a local system over a cofinal subset of the β's, whose elements we also denote β. We have the Leray-Serre spectral sequence $\{E_r\}$,

$$E_2 = H^p\big(F_\beta, H^q(K_\beta, M)\big) \Longrightarrow H^{p+q}(E_\beta, M) = E_\infty$$

for the fibration $K_\beta \to E_\beta \to F_\beta$.

We can compare the direct limit spectral sequence to those for the fibrations

$$K(\pi, n+1) \longrightarrow X_{n+1} \longrightarrow X_n$$

and

$$K((\pi)^{\wedge}_{\pi_1}, n+1) \longrightarrow \varprojlim_\beta E_\beta \longrightarrow \varprojlim_\beta F_\beta \ .$$

We compute the E_2-term of the limit sequence,

$$\begin{aligned} \varinjlim_\beta H^p(F_\beta, H^q(K_\beta, M)) &\cong \varinjlim_\beta (\varinjlim_{\alpha \geqq \beta} H^p(F_\alpha, H^q(K_\beta, M))) \\ &\cong \begin{cases} \varinjlim_\beta H^p(X_n, H^q(K_\beta, M)) \ , \text{ and} \\ \varinjlim_\beta H^p(\varprojlim_\alpha F^n_\alpha, H^q(K_\beta, M)) \end{cases} \end{aligned}$$

by property ii) applied to the finite π_1-module $H^q(K_\beta, M)$.

Now by the Addendum below, $\varinjlim_\beta H^q(K_\beta, M)$ is a finite group isomorphic to each of $H^q(K(\pi, n+1), M)$ and $H^q(K((\pi)^{\wedge}_{\pi_1}, n+1), M)$.

Since this limit is finite we may interchange the operations of taking cohomology and direct limit (because $\varinjlim_\alpha \operatorname{Hom}_R(C, M_\alpha) \cong \operatorname{Hom}_R(C, \varinjlim_\alpha M_\alpha)$ if $\{M_\alpha\}$ and $\varinjlim_\alpha M_\alpha$ are finite R-modules and C is a free R-module).

Since this limit is isomorphic to the groups mentioned, we have an isomorphism of the E_2-term of all three spectral sequences. The isomorphisms needed in the analogue of ii) for X_{n+1} and $\{E_\beta\}$ follow from the spectral sequence. The induction is thus completed and the second part of the theorem is proved given the fact proved below.

Addendum. The desired statements follow by induction on $n+1$ using the path fibrations $K(A, n+1) \to * \to K(A, n+2)$ for $A = \pi_\beta, \pi$, and $(\pi)^{\wedge}_{\pi_1}$. The statement for $n = 0$ is our hypothesis on the π_1-module π. The induction is again proved by forming the limit of the spectral sequences for the fibrations with $A = \{\pi_\beta\}$ and by comparing it with the two spectral sequences for $A = \pi$ and $(\pi)^{\wedge}_{\pi_1}$.

Now it is sufficient to treat the case when M is a field of coefficients, say F_p, the field of p elements. Then the E_2-terms with coefficients in F_p are respectively

$$\begin{gathered} \varinjlim_\beta \left(H^p(K(\pi_\beta, n+2)) \otimes H^q(K(\pi_\beta, n+1))\right) , \\ H^p(K(\pi, n+2)) \otimes H^q(K(\pi, n+1)), \end{gathered}$$

and

$$H^p(K((\pi)^{\wedge}_{\pi_1}, n+2)) \otimes H^q(K((\pi)^{\wedge}_{\pi_1}, n+1)) \ .$$

The first is isomorphic to

$$\varinjlim_\beta H^p(K(\pi_\beta, n+2)) \otimes \varinjlim_\beta H^q(K(\pi_\beta, n+1)) .$$

So the E_2-terms of these multiplicative spectral sequences split into tensor products of algebras with isomorphic (fiber) factors by induction. They all converge to the same thing (zero) so the other factors must be isomorphic by the comparison theorem. This completes the induction.

Corollary of proof. If X has "good" homotopy groups and $X \xrightarrow{c} Y$ is a map which is finite completion on π_1 and satisfies either of

i) the higher homotopy groups of Y are profinite π_1-modules and c induces an isomorphism of cohomology with finite π_1-coefficients, or

ii) c is π_1-finite completion of the higher homotopy,

then c is equivalent to finite completion.

Now there are abundant examples of "good" homotopy groups. To check whether π_1 is good, it is equivalent to check whether each subgroup of finite index in π_1 satisfies the "good" condition using only ordinary coefficients M. This follows from the same sort of spectral sequence argument used in the proof of the theorem. For example, to see that Z is a good fundamental group we need only check the condition for $M = Z/k$. A direct computation shows the first isomorphism if we take $\{\pi_\alpha\} = \{Z/n\}$. The second isomorphism follows from the first using the fibration

$$K(Z, 1) \longrightarrow K(\hat{Z}, 1) \longrightarrow K(\hat{Z}/Z, 1)$$

and the fact that $\hat{Z}/Z$ is a Q-vector space. Thus $K(\hat{Z}/Z, 1)$ is acyclic for finite coefficients.

Now it follows easily that any finitely generated Abelian group is a good fundamental group. But then any finitely generated Abelian group π is a good n^{th} homotopy group for any π_1. This follows since the $\{\pi \otimes Z/n\}$ form a natural system of π_1-quotients satisfying the condition (which was just verified in the statement that π is a good fundamental group).

Now we can deduce that any solvable group of finite type is a good fundamental group; that is, if π has a normal series with successive quotients finitely generated Abelian[1] then consider $K(\pi, 1)$ built up inductively from a point by fibrations with fiber $K(A, 1)$ for A finitely generated Abelian.

We can regard these A's as the "homotopy groups" in the proof of the theorem. Since these are "good" the proof goes through to construct a countable family of finite quotients of π satisfying the cohomology properties desired.

[1] One of the characterizations of these groups.

From what we said above, a group commensurable with a "good" fundamental group is also "good."

Property iii) (topology on $\hat{X}$). To study $\hat{X}$ topologically one can put a "countable type" hypothesis on X (Hom ($\pi_1 X$, finite group) and $H^i(X, M)$ are countable) and find that $\hat{X}$ can be constructed from a simple inverse system $(\circ\circ\circ \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow \circ\circ\circ \longrightarrow F_0)$.[1]

We can make the maps in the simple system into fibrations and consider the inverse limit space F_∞. One can show that $\hat{X}$ is homotopy equivalent to the singular complex of F_∞ and the topology on $[Y, \hat{X}]$ is determined by the compact open topology on the function F_∞^Y. Thus $\hat{X}$ is "represented by a space with two topologies".

There is perhaps a more geometric approach to the completion. In this picture $\hat{X}$ is a cell complex with a coarser topology in which

$$(\hat{X})_{n\text{-skeleton}} - (\hat{X})_{(n-1)\text{-skeleton}}$$

might be homeomorphic for example to

$$R^n \times \text{cantor set} \qquad \text{(E. Dror)} .$$

Finally, there is a further structure on $\hat{X}$. Each homotopy set $[Y, \hat{X}]$ has natural *uniform structure* coming from the inverse limit expressions

$$[Y, \hat{X}] = \varprojlim_{\{f\}_X} [Y, F] , \quad \text{or} \quad [Y, \hat{X}] = \varprojlim_{\{f\}_X^\#} [Y, F] .$$

The uniform topology is associated to the discrete topology on $[Y, F]$.

More examples.

i) If G is finitely generated and Abelian then

$$K(G, n)^\wedge \cong K(\hat{G}, n) \cong K(G \otimes \hat{Z}, n) .$$

ii) If $\pi_* X$ is finite, $X \cong \hat{X}$.

iii) $\hat{S}^n$ is *not* the Moore space $M(\hat{Z}, n)$. $\hat{S}^n$ has non-zero rational homology in infinitely many dimensions (Bousfield).

A faithfulness property of finite completion.

Now we show that no information about maps of finite complexes into nice spaces is lost by passing to the completion.

THEOREM 3.2. *Let Y be a finite complex*[2] *and let B be a simple Postnikov space with finitely generated homotopy groups.*[3] *Then if two maps* $Y \underset{g}{\overset{f}{\rightrightarrows}} B$

[1] The sub-category $\{f\}_X^\#$ is then essentially countable, and so has a linearly ordered cofinal subsystem (Lucy Garnett).

[2] Actually cohomological finiteness conditions are sufficient.

[3] That is, B is nilpotent of finite type.

are homotopic in the completion of B

$$Y \xrightarrow{\hat{f}} \hat{B} \sim Y \xrightarrow{\hat{g}} \hat{B}\ ,$$

f and g are actually homotopic in B,

$$Y \xrightarrow{f} B \sim Y \xrightarrow{g} B\ .$$

Proof. Since Y is finite we can prove the theorem by induction over the stages in the Postnikov system of B. Consider a commutative diagram

$$\begin{array}{ccccc} K & \longrightarrow & B_{n+1} & \xrightarrow{r} & B_n \\ & & \uparrow f_{n+1} & \nearrow f_n & \\ & & Y\ , & & \end{array}$$

where

f_{n+1} and f_n are constructed from f,
r is a principal fibration with fiber K,
K is the $K(\pi, n+1)$ with $\pi = \pi_{n+1}B_{n+1}$,
B_n has the first n-homotopy groups of B_{n+1}.[1]

We can apply the based function space and obtain another principal fibration

$$K^Y \longrightarrow B_{n+1}^Y \longrightarrow B_n^Y\ .$$

The tail of the exact homotopy sequence with f_{n+1} and f_n as base points looks like

$$\pi_1(B_n^Y, f_n) \xrightarrow{I} \pi_0 K^Y \overset{*f_{n+1}}{\dashrightarrow} \pi_0 B_{n+1}^Y \xrightarrow{r} \pi_0 B_n^Y$$

where

$*f_{n+1}$ denotes the map given by the action of $\pi_0 K^Y$ on $(f_{n+1}) \in \pi_0 B_{n+1}^Y$;
image $I = I(f_{n+1})$ is the isotropy subgroup for the point $(f_{n+1}) \in \pi_0 B_{n+1}^Y$;
r precisely collapses the orbits of the action of $\pi_0 K^Y = H^{n+1}(Y, \pi)$.

There is a similar situation for maps into the completions $\hat{B}_n$ and $\hat{B}_{n+1}$ and we can extract from all this a ladder,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(f_{n+1}) & \longrightarrow & H^{n+1}(Y, \pi) & \longrightarrow & \text{orbit}\, f_{n+1} & \longrightarrow & 0 \\ & & \downarrow c_0 & & \downarrow c & & \downarrow c_1 & & \\ 0 & \longrightarrow & I(\hat{f}_{n+1}) & \longrightarrow & H^{n+1}(Y, \hat{\pi}) & \longrightarrow & \text{orbit}\, \hat{f}_{n+1} & \longrightarrow & 0\ . \end{array}$$

We know that c is just finite completion by Property I of compact

[1] Here again we think of the homotopy groups as the successive quotients in a nilpotent filtration of $\pi_* B$.

Brownian functors (or by direct computation).

The burden of the argument is to show that c_0 is finite completion. Assuming this, the theorem is proved as follows:

i) Since the upper row is a sequence of finitely generated groups and finite completion is exact for these, c_1 is finite completion. In particular c_1 is an injection.

ii) Assume the theorem true for B_n. Then if $\hat{f}_{n+1} \cong \hat{g}_{n+1}$, we must have $f_n \cong g_n$. But then g_{n+1} lies in the orbit of f_{n+1}. Since c_1 is an injection we have that f_{n+1} is homotopic to g_{n+1}.

iii) As far as the finite complex Y is concerned, $B = B_n$ for n large enough so we are done.

To study c_0, consider the square

$$\begin{array}{ccc} \pi_1(B_n^Y, f_n) & \longrightarrow & I(f_{n+1}) \\ \big\downarrow d & & \big\downarrow c_0 \\ \pi_1(\hat{B}_n^Y, f_n) & \xrightarrow{I} & I(\hat{f}_{n+1}) \end{array} .$$

We claim that I is in a natural way a continuous surjection of topological groups. Furthermore, with respect to this topology d is just finite completion of a finitely generated nilpotent group.

It follows that $I(\hat{f}_{n+1})$ is a compact group and c_0 imbeds the finitely generated group $I(f_{n+1})$ as a dense subgroup. Thus c_0 is finite completion, given our statements about d and I.

Discussion of I: The topological structure on the situation arises as usual from the approximation by "finite" spaces.

The proof of Theorem 3.1 shows there is an inverse system of principal fibrations with finite homotopy groups

$$K_i \longrightarrow E_i \longrightarrow F_i$$

approximating

$$K \longrightarrow B_{n+1} \longrightarrow B_n .$$

If we can assume the maps

$$K_{i+1} \longrightarrow K_i , \quad E_{i+1} \longrightarrow E_i , \quad F_{i+1} \longrightarrow F_i$$

are fibrations, then $\hat{K} \to \hat{B}_{n+1} \to \hat{B}_n$ is obtained by taking the singular complexes of the inverse limit spaces, $K_\infty \to E_\infty \to F_\infty$, as we remarked above.

Since $\varprojlim_i (X_i^Y) = X_\infty^Y$ as topological spaces it follows that the sequence

$$\pi_1(\hat{B}_n^Y, f_n) \xrightarrow{I} \pi_0 \hat{K}^Y \underset{\text{action}}{\dashrightarrow} \pi_0 B_{n+1}^Y$$

is the inverse limit of the finite sequences

$$\pi_1(F_i^Y,\, f_n^i) \xrightarrow{\ I\ } \pi_0 K_i^Y \xrightarrow[\text{action}]{} \pi_0 E_i^Y\ .$$

This explains the topological structure in the map I above.

Discussion of d: Now we use the exact sequence of Lemma 2.7, induction on the number of cells of Y, and the fact that finite completion is exact for these groups to compare B_n^Y and $\hat{B}_n^Y$ and prove that $\pi_*(B_n^Y)_{f_n} \to \pi_*(\hat{B}_n^Y)_{\hat{f}_n}$ is finite completion, and $\pi_1(B_n^Y, f_n)$ is finitely generated, nilpotent.

Remark. The proof has a corollary. The argument shows B^Y has "good" homotopy groups and $\pi_* B^Y \to \pi_* \hat{B}^Y$ is finite completion. Thus $B^Y \to \hat{B}^Y$ is finite completion.

l-finite completion.

One can carry out the preceding discussion replacing finite groups by l-finite groups. (l is a set of primes and l-finite means the order of the group is a product of primes in l.) The constructions and propositions go through without essential change. If we take $l = \{p\}$ we have $X \to \hat{X}_p$, the p-finite or p-adic completion.

Now spaces whose homotopy groups are finite p-groups are nilpotent so $\hat{X}_p = \varprojlim_\alpha F_\alpha^p$ is "nilpotent at infinity" in the sense that there is an (infinite) nilpotent filtration of the homotopy groups giving a neighborhood base at the identity.[1] So the p-completions can be treated like nilpotent spaces, with (untwisted) homological methods.

For example, any inverse limit of finite p-spaces F_β^p such that

$$(\pi_1 X)_p^\wedge = \varprojlim_\beta \pi_1 F_\beta^p \quad \text{and} \quad H^*(X, Z/p) \cong \varinjlim_\beta H^*(F_\beta^p, Z/p)$$

gives $X_p^\wedge$; that is, $\hat{X}_p \cong \varprojlim_\beta F_\beta^p$. We do not need twisted coefficients to characterize $\hat{X}_p$.

This follows as in the argument above for $\hat{X}$. Now, however, only ordinary obstruction theory is needed. For example if X is the classifying space of O_2, B_{O_2}, the 2-completion is the total space of the fibration $K(\hat{Z}_2, 2) \to \hat{X}_2 \to K(Z/2, 1)$ with the twist $x \to -x$ in $\hat{Z}_2$.

However, the p-completion for p odd is the same as the p-completion of infinite quaternionic projective space, $QP^\infty =$ classifying space of S^3. Thus the homotopy groups of $\hat{X}_p$ are the p-completions of the homotopy groups of the three-sphere. This can be seen using the map $O_2 \to S^3$ relating O_2 to

[1] Such "pro-nilpotent spaces" are studied by a former student of Dan Kan, Emmanuel Dror.

the normalizer of the maximal torus in S^3.

This lack of *coherence* in the p-completions arises from the fact that $Z/2$ is an "l-good" fundamental group precisely when l contains the prime 2. For example, for p odd, $H^0(Z/2, M) = M^{Z/2} \neq M = H^0((Z/2)^{\wedge}_p, M))$ for general $Z/2$-modules M.

Now consider the natural map

$$\hat{X} \xrightarrow{h} \prod_p \hat{X}_p .$$

Since the right hand side is essentially nilpotent $\hat{X}$ must be essentially nilpotent if h is an isomorphism. For example:

THEOREM. *If X is nilpotent of finite type then $\hat{X} \xrightarrow{h} \prod_p \hat{X}_p$ is an isomorphism.*

This follows from the fact that the finite approximations constructed in the proof of Theorem 3.1 are nilpotent if we use a simple Postnikov system for X. But a Postnikov argument shows such finite nilpotent spaces are precisely the ones which factor over the primes:

$$F = \prod_p F_p \text{ if and only if } F \text{ is nilpotent} .$$

As a corollary to Theorem 3.2, we have the *Hasse principle* for maps of a finite complex into a nilpotent space of finite type: two maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are homotopic if and only if f and g are homotopic at every prime. (That is, for each p the compositions to $Y^{\wedge}_p$, $X \rightrightarrows Y \to Y^{\wedge}_p$, are homotopic.)

This theorem has the simple reformulation: if two map $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are not homotopic there is a space P with only finitely many non-zero finite p-groups for homotopy and a map $Y \to P$ so that the compositions $X \rightrightarrows Y \to P$ are still not homotopic.

Note also that in this situation of nilpotent spaces of finite type, each map $\hat{X} \xrightarrow{f} \hat{Y}$ canonically splits into a product (over the primes) of maps, $\hat{X}_p \xrightarrow{f_p} \hat{Y}_p$, by obstruction theory.

The formal completion.

Now we consider a modified form of the finite completion called formal completion. This construction will be used in the next section where the problem of characterizing the relation of the category of profinite homotopy types to the classical homotopy category is considered.

We need the formal completion to treat rational homotopy types. If the homotopy groups of a space X are Q-vector spaces, then any map

$$X \longrightarrow \text{space with finite homotopy groups}$$

is trivial. Thus the finite completion of X is a point.

However, we can construct a non-trivial completion of X by finitely completing its finite subcomplexes and fitting these together. For convenience we treat only countable complexes X. Then X is an increasing union of finite complexes

$$X = \bigcup_n X_n .$$

If we apply the finite completion construction to the system

$$X_0 \longrightarrow \cdots \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \cdots$$

we obtain a system

$$\hat{X}_0 \longrightarrow \cdots \longrightarrow \hat{X}_n \longrightarrow \hat{X}_{n+1} \longrightarrow \cdots .$$

We define the formal completion $\hat{X}_f$ of X to be the infinite mapping telescope of the complexes $\hat{X}_n$. This formal completion $\hat{X}_f$ inherits additional structure from the $\{\hat{X}_n\}$. Any map of a finite complex into $\hat{X}_f$ goes into a finite part of the telescope, which has the homotopy type of some $\hat{X}_n$. This shows that

$$[K, \hat{X}_f] \cong \varinjlim_n [K, \hat{X}_n]$$

for K a finite complex.

It follows that the functor $[\ , \hat{X}_f]$ has a natural topology on the subcategory of finite complexes. Furthermore, the higher homotopy groups of $\hat{X}_f$ have a natural $\hat{Z}$-module structure since the homotopy groups of $\hat{X}_n$ do (being profinite).

Finally note that since X is equivalent to the telescope of $X_0 \to X_1 \to X_n \to X_{n+1} \to \cdots$, we have a natural map $X \to \hat{X}_f$.

We conclude our discussion with the

PROPOSITION. i) *The formal completion $X \to \hat{X}_f$ is natural on the homotopy category generated by countable complexes.*

ii) *If X is nilpotent, then $\hat{X}_f$ is nilpotent and for the "homotopy groups" we have "$\pi_* X$" $\otimes \hat{Z} \cong$ "$\pi_* \hat{X}_f$".*

iii) *If X is nilpotent of finite type, then $\hat{X}_f \cong \hat{X}$.*

Proof. The first part follows easily from the compactness idea used above.

For the second part consider the simple Postnikov system $\cdots \to X_{n+1} \to X_n \to \cdots \to *$ for X. Suppose we have constructed nilpotent spaces of finite type Y^α such that $X_n = \bigcup_\alpha Y^\alpha_n$. Let $\{\pi^j\}$ be a sequence of finitely generated groups whose union is the homotopy group π of the fiber $X_{n+1} \to X_n$. Now $H^j(Y_n^\alpha, \pi) = H^i(Y_n^\alpha, \varinjlim_j \pi^j) = \varinjlim_j H^i(Y_n^\alpha, \pi^j)$ since the homology of Y_n^α can be constructed from finitely generated chain complexes. Thus we can restrict the k-invariant for constructing $X_{n+1} \to X_n$ to Y_n^α and realize it in one of the $H^i(Y_n^\alpha, \pi^j)$. Such classes enable us to construct finite type approximates to X_{n+1}. Extend this replication process to infinity to construct nilpotent spaces of finite type Y^α so that $\varinjlim_\alpha Y^\alpha = X$.

Now consider the skeletons Y_β^α of Y^α. The Y_β^α are nilpotent of finite type up to a certain dimension, so finite completion first completes the homotopy groups at least up to that dimension by Theorem 3.1. For the successive quotients we obtain "$\pi_* Y_\beta^\alpha$" $\otimes \hat{Z}$.

Since $\hat{X}_f = \varinjlim_{\alpha,\beta} \hat{Y}_\beta^\alpha$, and "$\pi_* X$" $= \varinjlim_{\alpha,\beta}$ "$\pi_* Y_\beta^\alpha$", part ii) is done.

Part iii) is also done because this construction shows the natural map $\hat{X}_f \to \hat{X}$ is an isomorphism of "homotopy groups".

Example. The formal completion of $K(Q, n)$, the space with one non-zero homotopy group Q in dimension n, is $K(Q \otimes \hat{Z}, n)$.

Using the $\hat{Z}$-module structure we can regard the homotopy group as the free module on one generator over the ring of finite Adeles,

$$A^f = Q \otimes \hat{Z} = \text{restricted product (over all primes) of the } p\text{-adic numbers.}$$

The genetics of a homotopy type.

Now we shall try to reconstruct a homotopy type X from its irreducible pieces. Recall that we have the rational type X_Q constructed by localization and the finite type $\hat{X}$ constructed by finite completion. To allow these spaces to interact we need a common medium. The medium is dictated by each.

From X_Q's point of view the medium is the formal completion $\hat{X}_Q^f$ constructed above. Recall $\hat{X}_Q^f$ was obtained by fitting together the finite completion of finite subcomplexes X_Q^α of X_Q,

$$(X_Q)\hat{}_f \equiv \hat{X}_Q^f = \varinjlim_\alpha \hat{X}_Q^\alpha .$$

From $\hat{X}$'s point of view the medium is obtained by localizing finite collections of the l-adic types and fitting these together,

$$X_{A^f} = \varinjlim_S \left(\prod_{l \in S} (\hat{X}_l)_Q \times \prod_{l \notin S} \hat{X}_l\right)$$

where S ranges over the finite sets of primes. X_{A^f} might be called the finite Adele type of X.

Now suppose X is nilpotent of finite type. Then we know that the finite type splits completely,

$$\hat{X} \cong \prod_l \hat{X}_l$$

where l ranges over the primes.

We refer to the natural array of spaces $\{X_Q, \hat{X}_2, \cdots, \hat{X}_p, \cdots\}$ as *the genotype of* X.

Also in this case all of our constructions are easily followed by what happens on the "homotopy groups". For example the natural map $\hat{X} \to \hat{X}_f$ is an equivalence and $\hat{X} \to \hat{X}_{Af}$ is equivalent to localization $\hat{X} \to \hat{X}_Q$.

So if we apply formal completion to $X \to X_Q$ we obtain a natural equivalence

$$\mathcal{C}_X \colon (X_Q)\hat{}_{\text{formal}} \sim X_{Af} .$$

$\mathcal{C}_X$ is a class of homotopy-equivalences which preserves the $\hat{Z}$-module structure on homotopy. We refer to $\mathcal{C}_X$ as the coherence of the homotopy type.

If we adjoin $\mathcal{C}_X$ to the genotype we obtain the *coherent genotype* of X,

$$\{X_Q, \hat{X}_2, \hat{X}_3, \cdots; \mathcal{C}_X\} .$$

Now we can use these conditions to formulate the notion of an (abstract) coherent genotype $\{X_Q, X_2, X_3, \cdots; \mathcal{C}\}$, and these form a category $\mathcal{G}_c$ in a natural way.

An object of $\mathcal{G}_c$ is an array of nilpotent spaces whose "homotopy groups" are finitely generated over the appropriate ground rings together with a class of homotopy equivalences

$$\mathcal{C} \colon (X_Q)\hat{}_{\text{formal}} \sim \left(\prod_p X_p\right)_Q$$

preserving this module structure.

A morphism of $\mathcal{G}_c$ is an array of homotopy classes of maps $\{f_Q, f_2, f_3, \cdots\}$ respecting the coherence; that is,

$$\left(\prod f_p\right)_Q \circ \mathcal{C}_{\text{domain}} = \mathcal{C}_{\text{range}} \circ (f_Q)\hat{}_{\text{formal}} .$$

Our constructions so far give us a functor from the homotopy category generated by nilpotent spaces of finite type to this category of coherent genotypes.

We can say a bit more, however, about maps. To $X \xrightarrow{f} Y$ we associate the array $\{f_Q, \hat{f}_2, \cdots, \hat{f}_p, \cdots\}$. Now by Lemma 2.7, f_Q is determined by the homotopy class of f restricted to each finite subcomplex of X. The same is true for the $\hat{f}_p$ by Property I of compact Brownian functors.

So we can collect the maps between X and Y into their "visible" homotopy classes (i.e., maps homotopic on each finite subcomplex). We obtain the category $\mathfrak{N}$ and a functor to the category of coherent genotypes $\mathfrak{N} \xrightarrow{a} \mathcal{G}_c$, which is a kind of arithmetization.

There is a natural construction in the other direction: interaction. Start with a coherent genotype $\{X_Q, X_2, \cdots; \mathcal{C}\}$. Form $\hat{X} = \prod_l X_l$ and consider X_Q and $\hat{X}$ as subspaces of the mapping cylinder $\mathfrak{M}$ of a representative of the coherence $\mathcal{C}$. An interaction is a path in the medium $\mathfrak{M}$ whose endpoints lie in X_Q and $\hat{X}$ respectively. The space of all interactions is a *nilpotent space of finite type over* Z.

Similarly, given maps of coherent genotypes $\{f_Q, f_2, \cdots\}$ we choose representatives and a map of media which agrees with these representatives. By mapping paths we obtain a map on the corresponding interaction spaces.

The homotopy class of this assembled map on every finite complex is independent of the choices because of the Hasse principle for maps.

So we obtain by interaction a natural functor from the category of coherent genotypes to the category generated by nilpotent spaces of finite type (over Z) and visible homotopy classes of maps, $\mathcal{G}_c \xrightarrow{i} \mathfrak{N}$.

THEOREM. *Arithmetization* $\mathfrak{N} \xrightarrow{a} \mathcal{G}_c$ *and interaction* $\mathcal{G}_c \xrightarrow{i} \mathfrak{N}$ *are mutually inverse equivalences of categories.*

Proof. Let $\{\cdots\}^i$ be the interaction space of the coherent genotype $\{\cdots\}$. Let $\{X\}$ denote the coherent genotype of X. From our constructions we have natural maps $X \to \{X\}^i$, and $\{\{\cdots\}^i\} \to \{\cdots\}$. An inductive argument over the simple Postnikov system using the arithmetic square 1.26 and Theorems 2.1 and 3.1 shows these naturally constructed maps are homotopy equivalences.

At the inductive stage we use the exact sequence of homotopy for a path fiber product of spaces to see that we have isomorphisms of the homotopy groups.

Remarks. I) There is a certain fluidity in the coherence $\mathcal{C}$. If $l_1 \cup l_2 \cup \cdots \cup l_r$ is a partition of the set of primes, then $\mathcal{C}_X$ can be factored into an r-fold product of the

$$\mathcal{C}^i_X\colon (X_Q)\hat{}_{l_i\text{-formal}} \sim \left(\prod_{p \in l_i} \hat{X}_p\right)_Q .$$

If we carry out the interaction on this level we obtain instead of X, the localization of X at l_i. We have the analogous structure theorem for nilpotent spaces whose "homotopy groups" are finitely generated modules over Z_l.

II) The structure of coherent genotypes is rather like that of the spectrum of the ring of integers. We have one space for each point of

$$\text{spec}\, Z = \{Q, 2, 3, \cdots\} .$$

$\mathcal{C}$ binds the rational type to each prime space, $\hat{X}_p$, and from $\mathcal{C}$ we may factor off a finite collection of independent p-coherences. This reflects the topology of spec Z.

III) This "picture" of the nilpotent finite type category clarifies the nature of the information in the étale homotopy type of a complex variety V; that is, if V is nilpotent of finite type, to determine the actual homotopy type of V we must find the Q-type V_Q and the coherence $\mathcal{C}_V$. We add this to the étale homotopy type $\hat{V}$ and we have the homotopy type of V.

IV) A good illustration of the genotype method in a classical homotopy problem is provided by the infinite quaternionic projective space. The problem in its stark form is to construct self-mappings $QP^\infty \to QP^\infty$ with any odd square degree on the four dimensional homology. By the theorem it suffices to construct a coherent family of mappings on the spaces of the genotype of QP^∞. The rational type of QP^∞ is $K(Q, 4)$. So here we have endomorphisms of any rational degree, coming from the endomorphisms of Q.

The p-adic type of QP^∞ can be constructed using étale theory, and the Galois symmetry in the construction leads to automorphisms of any degree equal to the square of a p-adic unit. To do this we regard QP^∞ as the fiber of $B_{U_2} \xrightarrow{c_1} CP^\infty$.

At this point we have no actual endomorphism of QP^∞ beyond the identity. The coherence condition is expressible solely in terms of the degree since

$$K(Q, 4)_f^\wedge \cong K(Q \otimes \hat{Z}, 4) .$$

The requirement is met only by the degree $+1$.

However, for odd p, the p-adic type of QP^∞ has a second description in terms of the classifying space of the orthogonal group of degree 2. In fact, as remarked above

$$(QP^\infty)_p^\wedge \cong (B_{O_2})_p^\wedge , \qquad p \text{ odd} .$$

Now O_2 has endomorphisms for each integer k leading to endomorphisms of $(QP^\infty)_p^\wedge$ of any degree equal to the square of a p-adic *integer*. So for the square of an odd integer k we obtain an endomorphism of QP^∞ using

i) at Q: the endomorphism k^2 of Q,

ii) at 2: some Galois automorphism corresponding to raising the $2^{n\text{th}}$ roots of unity to the k^{th} power,

iii) at p: the endomorphism of O_2 which is raising to the k^{th} power on $SO_2 \subset O_2$.

Note that only these odd squares of integers are visible in the classical context. The endomorphism situation on the spaces of the genotype of QP^∞ is considerably richer.

V) (Varieties in characteristic p.) The étale theory applied to varieties over fields of characteristic p yields q-adic homotopy types for each prime q not equal to p. Thus we have all but two of the spaces in a genotype.

If the variety in question can be lifted to the complex numbers, then the missing spaces X_Q and X_p and the coherence $\mathcal{C}_X$ can be constructed from the complex variety. Thus we have homotopy-theoretical necessary conditions for the existence of a lifting.

The analysis of this obstruction should be interesting, especially the relation between the sought-after X_p and X_Q and the two de Rham cohomology theories: the crystal theory in characteristic p and the analytical theory in characteristic zero.

4. Spherical fibrations

In this section we develop the theory of fibrations where the fiber is the localized or completed sphere.

One of the main points is to relate these theories to the standard spherical fibration theory. For example, we show that the stable profinite theory is isomorphic to the ordinary oriented stable theory "direct product with"-theory of $\hat{Z}$ local coefficient systems. In particular, fiberwise completion (on the stable level) embeds the standard theory of spherical fibrations into the profinite spherical theory as a direct factor. We use this fact to prove the Adams conjecture for real or complex K-theory.

The proof gives an interesting unstable form of the Adams conjecture which can only be formulated in the local or profinite context.

Definition. A Hurewicz fibration[1] $\xi: S \to E \to B$ where the fiber is the local sphere S_l^{n-1}, $n > 1$, is called a *local spherical fibration*. The local fibration is oriented if there is given a class in

$$U_\xi \in H^n(E \longrightarrow B; Z_l)\ ^2$$

which generates $H^n(S_l^{n-1} \to *; Z_l) \cong Z_l$ upon restriction.

When l is the set of all primes the theory is more or less familiar:

[1] $E \to B$ has the homotopy lifting property for maps of spaces into B.

[2] $H^n(E \to B)$ means H^n of the pair (mapping cylinder of $E \to B, E$).

i) the set of fiberwise homotopy-equivalence classes of S^{n-1}-fibrations over X is classified by a homotopy set

$$[X, B_{G_n}] ;$$

ii) B_{G_n} is the classifying space of the associative H-space (by composition)

$$G_n = \{S^{n-1} \xrightarrow{f} S^{n-1} \mid \deg f \in \{\pm 1\} = Z^*\} ;$$

that is, $\Omega B_{G_n} \cong G_n$ as infinitely homotopy associative H-spaces [14];

iii) the oriented theory[1] is classified by the homotopy set

$$[X, B_{SG_n}]$$

where B_{SG_n} may be described in two equivalent ways:

a) B_{SG_n} is the classifying space for the component of the identity map of S^{n-1} in G_n, usually denoted SG_n, or

b) B_{SG_n} is the universal cover of B_{G_n}, where $\pi_1 B_{G_n} = Z/2$;

iv) the involution on the oriented theory obtained by changing orientation $U_\xi \to -U_\xi$ corresponds to the covering transformation of B_{SG_n};

v) there are natural inclusions $G_n \to G_{n+1}$, $B_{G_n} \to B_{G_{n+1}}$, corresponding to the operation of suspending each fiber. The union

$$B_G = \bigcup_{n=1}^{\infty} B_{G_n}$$

is the classifying space for the "stable theory".

The stable theory for finite dimensional complexes is just the direct limit of the finite dimensional theories under fiberwise suspension. This direct limit converges after a finite number of steps, so we can think of a map into B_G as classifying a spherical fibration whose fiber dimension is larger than that of the base.

For infinite dimensional complexes X we can say that a homotopy class of maps of X into B_G is just an element in the inverse limit of the homotopy classes of the skeletons of X. This uses the finiteness of the homotopy groups of B_G (see Section 3). Such an element in the inverse limit can then be interpreted as a compatible family of spherical fibrations of increasing dimension over the skeletons of X.

The involution in the "stable theory" is trivial[2] and there is a canonical splitting

$$B_G \cong K(Z/2, 1) \times B_{SG} .$$

Some particular examples can be calculated:

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[1] The fiber homotopy equivalences have to preserve the orientation.

[2] This is the germ of the Adams phenomenon.

$$B_{G_1} \cong \mathrm{RP}^\infty , \quad B_{SG_1} = S^\infty \cong * $$
$$B_{G_2} \cong B_{O_2} , \quad B_{SG_2} \cong CP^\infty \cong B_{SO_2} .$$

All higher B_{G_n}'s are unknown although the (finite) homotopy groups of

$$B_G = \bigcup_{n=1}^{\infty} B_{G_n}$$

are much studied:

$$\pi_{i+1} B_G \cong \overset{\text{stable}}{i\text{-stem}} \equiv \varinjlim_k \pi_{i+k}(S^k) .$$

Stasheff's explicit procedure does not apply without (semi-simplicial) modification to S_l^{n-1}-fibrations for l a proper set of primes. In this case S_l^n is an infinite complex (although locally compact).

If we consider the *l-adic spherical fibrations*, namely, Hurewicz fibrations with fiber $\hat{S}_l^{n-1}$, the situation is even more infinite. $\hat{S}_l^{n-1}$ is an uncountable complex and therefore not even locally compact.

However, Dold's theory of quasi-fibrations can be used (see [8], p. 16.8) to obtain abstract representation theorems for theories of fibrations with arbitrary fiber.

THEOREM 4.1 (Dold). *There are connected CW complexes B_l^n and $\hat{B}_l^n$ so that*

$$\left\{ S_l^{n-1} \begin{array}{c} \textit{theory of} \\ \textit{fibrations} \end{array} \right\} \cong [\ \ , B_l^n]_{free}$$
$$\left\{ \hat{S}_l^{n-1} \begin{array}{c} \textit{theory of} \\ \textit{fibrations} \end{array} \right\} \cong [\ \ , \hat{B}_l^n]_{free} .$$

Actually, Dold must prove a based theorem first, namely

$$\left\{ \begin{array}{c} \text{based} \\ \text{fibrations} \end{array} \right\} \cong [\ \ , B]_{\text{based}} ,$$

then divide each set into the respective $\pi_1 B$ orbits to obtain the free homotopy statements of the theorem.

The Main Theorem. We shall use the idea of fiberwise localization and completion to construct a canonical diagram of theories ($n > 1$),

$$(*) \qquad \begin{array}{ccccc} & & \left\{ \begin{array}{c} S_l^{n-1}\text{-fibration} \\ \text{theory} \end{array} \right\} & & \\ & {\scriptstyle \text{fiberwise localization}} \nearrow {\scriptstyle l} & & \searrow {\scriptstyle c} \ {\scriptstyle \text{fiberwise completion}} & \\ \left\{ \begin{array}{c} S^{n-1}\text{-fibration} \\ \text{theory} \end{array} \right\} & & \xrightarrow[\text{fiberwise completion}]{c} & & \left\{ \begin{array}{c} \hat{S}_l^{n-1}\text{-fibration} \\ \text{theory} \end{array} \right\} . \end{array}$$

Using Dold's theorem we have a corresponding diagram of classifying spaces

(**)
$$\begin{array}{ccc} & B_n^l & \\ \nearrow & & \searrow \\ B_{G_n} & \longrightarrow & \hat{B}_n^l \end{array} .$$

THEOREM 4.2. *If $n > 1$,*

i) *the diagram* (*) *exists;*

ii) *for the corresponding diagram of classifying spaces* (**) *we have*

a) *the diagram of fundamental groups is isomorphic to*

$$\begin{array}{ccc} & Z_l^* & \\ \nearrow & & \searrow \\ Z^* & \longrightarrow & \hat{Z}_l^* \end{array} ,$$

b) *the diagram of universal covering spaces is equivalent to*

$$\begin{array}{ccc} & (B_{SG_n})_l & \\ {\scriptstyle localization}\nearrow & & \searrow{\scriptstyle completion} \\ B_{SG_n} & \xrightarrow[\text{completion}]{} & (B_{SG_n})_l\hat{} \end{array} ;$$

iii) *the latter diagram classifies the diagram of oriented theories with the covering-space symmetry corresponding to the action of the appropriate group of units on the orientations.*

Before giving the proof of Theorem 4.2 we shall discuss some of its corollaries.

Let Aut X denote the singular complex of automorphisms of X: a simplex σ is a homotopy equivalence $\sigma \times X \to X$.

COROLLARY 1. *We have the table of computations:*

	Aut S_l^{n-1}	Aut $\hat{S}_l^{n-1}$
group of components	Z_l^*	$\hat{Z}_l^*$
homotopy type of the component of the identity	$(SG_n)_l$	$(SG_n)_l\hat{}$

Remark. i) The fiberwise construction of the proof of the main theorem gives a natural class of maps

$$G_n \xrightarrow{f_l} \text{Aut } S_l^{n-1} .$$

The construction shows f_l has all the homotopy multiplicative properties one might desire. For example f_l is equivalent to the loop of a map of classifying spaces.

ii) Similar remarks apply to the monoid of all self maps of S^{n-1} of degree prime to l, $G_n(l)$. Fiberwise localization again gives a natural map

$$G_n(l) \xrightarrow{f_l} \operatorname{Aut} S_l^{n-1} .$$

This map is equivalent to the natural notion of group completion for topological monoids. Thus these monoids have homotopy equivalent classifying spaces. This common classifying space is the classifying space for the localized spherical fibration theory discussed in the theorem,

$$B_l^n \cong B_{G_n(l)} \cong B_{\operatorname{Aut}(S_l^{n-1})} .$$

We will not use these facts although they explain somewhat the connection of our computations with those of Adams in the $J(X)$ papers [1]–[4].

Symmetry in the unstable theories. We have exhibited the spaces $(B_{SG_n})_l$ and $(B_{SG_n})_l^\wedge$ as universal covering spaces with Galois (or covering transformation) groups Z_l^* and $\hat{Z}_l^*$. Thus we have interesting self-homotopy equivalences of these universal spaces.

For $l = \{p\}$, Z_l^* is isomorphic to $Z/2 \oplus$ (the free Abelian group generated by the primes other than p). The corresponding automorphisms of $(B_{SG_n})_l$ determine automorphisms of the completion $(B_{SG_n})_l^\wedge$ which coalesce topologically as a dense subgroup of the compact group of automorphisms, $Z_l^* \cong Z/p - 1 \otimes \hat{Z}_p$ for p odd (or $\hat{Z}_2^* \cong Z/2 \oplus \hat{Z}_2$ for $p=2$).

We shall see below (Corollary 3) that the homotopy groups of B_{SG_n} are finite except for one dimension,

$$\pi_n B_{SG_n} = Z \oplus \text{torsion for } n \text{ even} ,$$
$$\pi_{2n-2} B_{SG_n} = Z \oplus \text{torsion for } n \text{ odd} .$$

The first segment of finite homotopy groups corresponds to the first segment of stable homotopy groups of spheres.

Then $(B_{SG_n})_l^\wedge$ has for homotopy the l-torsion of these groups plus one $\hat{Z}_l$ (in dimensions n or $2n-2$, respectively).

The units $\hat{Z}_l^*$ act trivially on the low dimensional, stable groups but non-trivially on the higher groups. For example, for n even we have the natural action of $\hat{Z}_l^*$ on

$$\pi_n(B_{SG_n})_l^\wedge/\text{torsion} \cong \hat{Z}_l .$$

On the higher groups the action measures the effect of the degree α map on the homotopy groups of a sphere. This action is computable up to extension in terms of Whitehead products and Hopf invariants. It seems especially interesting at the prime 2.

The Rational Theory. If l is vacuous, the local theory is the "rational theory". Using the fibration

$$[(\Omega^{n-1}S^{n-1})_1 \longrightarrow SG_n \longrightarrow S^{n-1}]_{\text{localized at } l=\phi}\ ,$$

it is easy to verify

COROLLARY 3. *Oriented S_Q^{n-1} fibrations are classified by*
i) *an Euler class in H^n (base, Q) for n even,*
ii) *a Hopf class in H^{2n-2} (base, Q) for n odd.*

It is also not too difficult[1] to see an equivalence of fibration sequences

$$\begin{array}{ccccccccccc}
[\cdots \longrightarrow & SG_{2n} & \longrightarrow & SG_{2n+1} & \longrightarrow & SG_{2n+1}/SG_{2n} & \longrightarrow & B_{SG_{2n}} & \longrightarrow & B_{SG_{2n+1}}]_Q & \\
 & \cong\Big\downarrow \text{evaluation} & & \Big\downarrow\cong & & \Big\downarrow\cong & & \Big\downarrow \text{Euler class} & & \Big\downarrow \text{Hopf class} & \\
\cdots \longrightarrow & S_Q^{2n-1} & \longrightarrow & S_Q^{4n-1} & \xrightarrow[\text{Whitehead product}]{} & S_Q^{2n} & \longrightarrow & K(Q, 2n) & \xrightarrow[\text{cup square}]{} & K(Q, 4n)\ . &
\end{array}$$

Corollary 3 has a "twisted analogue" for unoriented bundles. Stably the oriented rational theory is trivial. The unoriented stable theory is just the theory of Q coefficient systems, $H^1(\ \ , Q^*)$.

Note that Corollary 3 (part i), twisted or untwisted, checks with the equivalence

$$S_Q^{2n-1} \cong K(Q, 2n-1)\ .$$

The group of units in Q,

$$Q^* \cong Z/2 \oplus \text{free Abelian group generated by the primes}\ ,$$

acts in the oriented rational theory by the obvious action for n even and by the square of the obvious action for n odd.

The Stable Theory. As remarked above we construct a stable theory of spherical fibrations by considering compatible[2] families of spherical fibrations of increasing fiber dimension over the skeletons of the base.

The stability of the homotopy groups in the direct systems

$$\{B_{G_n}\}\ ,\quad \{B_n^l\}\ ,\quad \{\hat{B}_n^l\}$$

[1] It is convenient to compare the fibrations

$$SO_n \longrightarrow SO_{n+1} \longrightarrow S^n\ , \qquad (\Omega^n S^n)_1 \longrightarrow SG_{n+1} \longrightarrow S^n\ .$$

[2] This means compatibility under fiberwise suspension (followed by fiberwise completion in the l-adic case).

shows that such stable spherical fibrations are classified by a single map of the base into the appropriate direct limit space,

$$B_G\ ,\quad B^l_\infty\ ,\quad \hat{B}^l_\infty\ .$$

A further simplification results from the computations below. The compactness phenomenon discussed in Section 3 implies for each of the spaces B above

$$[X, B] \cong \varprojlim_{\text{skeletons } X_k \text{ of } X} [X_k, B]\ .$$

Thus a stable fibration is determined up to isomorphism by the isomorphism classes of the various fibrations over the skeletons of X, and the compatibility isomorphisms are relevant only in that they exist.

Similar remarks apply to the stable oriented theories which are classified by mapping into the appropriate universal covering spaces.

COROLLARY 4.

i) *For the stable oriented theories, we have the isomorphisms*

$$\begin{matrix}\textit{oriented stable}\\ \textit{l-local theory}\end{matrix} \cong \begin{matrix}\textit{oriented stable}\\ \textit{l-adic theory}\end{matrix} \cong [\quad , \prod_{p\in l} (B_{SG})_p]\ .$$

ii) *The unoriented stable theory is canonically isomorphic to the direct product of the oriented stable theory and the theory of Z_l or $\hat{Z}_l$ coefficient systems.*

$$\begin{matrix}\textit{stable l-local}\\ \textit{theory}\end{matrix} \cong [\quad , K(Z_l^*, 1) \times \prod_{p\in l} (B_{SG})_p]\ ,$$

$$\begin{matrix}\textit{stable l-adic}\\ \textit{theory}\end{matrix} \cong [\quad , K(\hat{Z}_l^*, 1) \times \prod_{p\in l} (B_{SG})_p]\ .$$

iii) *The action of the Galois group is trivial in the stable oriented theory.*

Proof. i) Because of the rational structure,

$$\varinjlim_n \left((B_{SG_n})_l \xrightarrow{\ \wedge\ } (B_{SG_n})_l^\wedge\right)$$

is an isomorphism. This proves the first part of i) since these direct limit spaces classify the stable oriented theories.

On the other hand, since B_{SG} has finite homotopy groups,

$$\Pi_{p\in l}(B_{SG})_p = (B_{SG})_l \cong \varinjlim_n (B_{SG_n})_l$$

which completes the proof of i).

ii) Consider the local case. A coefficient system, $\tilde{\alpha} \in H^1(\quad , Z_l^*)$, determines an S_l^1-fibration α by letting the units act on some representative of S_l^1 by homomorphisms. (A functorial construction of $K(Z_l, 1)$ will suffice.)

Now for any oriented S_l^{n+1}-fibration γ, $\alpha * \gamma$ is a (cohomologically twisted) S_l^{n+1}-fibration, where $\alpha * \gamma$ means fiberwise join.

One easily checks (using the proof of Theorem 4.2) that this construction leads to a homotopy equivalence

$$K(Z_l^*, 1) \times \varinjlim_n (B_{SG_n})_l \xrightarrow{\cong} \varinjlim_n B_n^l .$$

The l-adic case is similar. $\alpha * \gamma$ is constructed by fiberwise join followed by fiberwise completion.

iii) The action of the Galois group is clearly trivial using ii). Or, more directly, note that the fibration $S * \gamma$ has automorphisms which change the orientation, when γ is any oriented fibration and S is the trivial S_l^1 or $\hat{S}_l^1$-fibration.

Remark. Part iii) of Corollary 4 is a kind of purely homotopy-theoretical Adams conjecture.

Also the proof of part ii) shows how Whitney join[1] makes the stable theories into additive theories.

The Adams conjecture. Let $\hat{K}$ denote the profinite K-theory discussed in Section 3 (either real or complex). Recall that $\hat{K}$ associated to each cell complex a compact topological ring which for finite complexes is the finite completion of the usual K-ring.

$\hat{K}$ is classified for all spaces by maps into the finite completion $\hat{B}$ where B is the infinite grassmannian classifying ordinary K-theory.

$\hat{K}$ has a beautiful symmetry which we shall discuss from two points of view.

The group of symmetries is the group of units in $\hat{Z}$.

The first description uses the famous operations of Adams

$$K \xrightarrow{\psi^k} K , \qquad k \in Z .$$

Recall that ψ^k is defined in terms of the Newton polynomials in the exterior powers of a vector bundle

$$\begin{aligned} \psi^1 V &= \Lambda^1 V = V \\ \psi^2 V &= V \otimes V - 2\Lambda^2 V \\ &\vdots \end{aligned}$$

and gives ring endomorphisms of K.

The integer k defines an element $(k) \in \hat{Z}^*$ by giving the automorphism of $\hat{Z} = \prod_p \hat{Z}_p$,

[1] Followed by fiberwise completion in the l-adic case.

$$(k)x = \begin{cases} k \cdot x & \text{if } x \in \hat{Z}_p\,, \quad (k, p) = 1 \\ x & \text{if } x \in \hat{Z}_p\,, \quad (k, p) \neq 1\,. \end{cases}$$

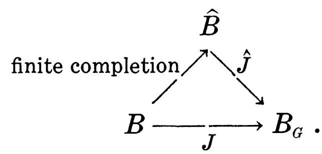
The set of such elements is dense in $\hat{Z}^*$.

Similarly ψ^k defines an automorphism of $\hat{K} = \prod_p \hat{K}_p$ by $\eta \to \eta^{(k)}$ where $\eta \in \hat{K}_p$ and

$$\eta^{(k)} = \begin{cases} (\psi^k)^{\wedge}_p(\eta) & \text{if } (k, p) = 1 \\ \quad \eta & \text{if } (k, p) \neq 1\,. \end{cases}$$

It is possible to show by direct computation [1]–[4] that the family of automorphisms $\{(k) \colon k \in Z\}$ extends by continuity to a continuous group action of the profinite group $\hat{Z}^*$ on $\hat{K}$. This will also follow directly from our second description. We shall refer to this group of symmetries of $\hat{K}$, which embodies the isomorphic part of the Adams' operations, as the Galois group $\hat{Z}^*$ acting on $\hat{K}$.

Now note that an element in $\hat{K}$ has a naturally associated stable fiber homotopy type. This follows from the existence of the natural extension $\hat{J}$ in the diagram of universal spaces



$\hat{J}$ exists because B_G has finite homotopy groups. J is defined as usual by classifying the increasing sequence of canonical sphere bundles over the increasing union of finite grassmannians comprising B.

One of the main points of this work is to prove the

THEOREM (Adams conjecture). *The stable fiber homotopy type of elements in profinite K-theory* (*real or complex*) *is constant on the orbits of the Galois group.*

This theorem follows most naturally from the second description of the symmetry in $\hat{K}$. We will sketch this description which uses ideas from algebraic geometry.

Consider the grassmannians used to construct B. For the complex K-theory these grassmannians are beautiful compact complex varieties naturally embedded in complex projective spaces (via the Plücker coordinates).

For the real K-theory we have the real grassmannians which we think of as homogeneous spaces of the real orthogonal group. For homotopy purposes we may also consider homogeneous spaces of the complex orthogonal group

$$O(n, \mathrm{C}) = \{A \in Gl(n, \mathrm{C}) \colon (Ax, Ay) = (x, y)\}$$

where $x \in \mathrm{C}^n$ and (x, y) is the ordinary inner product $(x, y) = \sum_{i=1}^{n} x_i y_i$. This follows since $O(n, \mathrm{C})$ has the same homotopy type as the real orthogonal group.

Forming homogeneous spaces of these groups, e.g., $O(n+k, \mathrm{C})/O(n, \mathrm{C}) \times O(k, \mathrm{C})$, gives non-singular affine complex algebraic varieties having the homotopy types of the corresponding real grassmannians.

So in either case (real or complex) the classifying space for K-theory may be expressed as an increasing union of complex algebraic varieties. Furthermore, these varieties are defined by equations with coefficients in the field of rational numbers.

This means that transforming the points of C^n or projective space by mapping the coordinates by any field automorphism σ of C

$$\sigma \in \mathrm{Gal}\,(\mathrm{C}/Q)$$

gives rise to (wildly discontinuous but) algebraic automorphisms of these varieties.

Now as remarked above there is an algebraic construction of the finite cohomology of complex algebraic varieties very much like Čech theory in topology. From the Zariski open sets of a variety V and their finite (Überdeckung) coverings an inverse system of nerves N_α (actual complexes) are constructed [10] and these are natural maps $V \longrightarrow \{N_\alpha\}$ giving

$$(*) \qquad (\pi_1 V)^{\wedge} \cong \varprojlim_\alpha \pi_1 N_\alpha \quad \text{and} \quad H^i(V, M) \cong \varinjlim_\alpha H^i(N_\alpha, M)$$

for all finite coefficients M (twisted or untwisted) [5].

Now as noted in Section 3, equation $(*)$ implies that the finite completion of V can be constructed from the nerves N_α, for example

$$V \cong \varprojlim_\alpha N_\alpha$$

in the sense of compact functors if $\pi_i N_\alpha$ is finite.[1]

On the other hand, since the construction of each N_α only involves the abstract algebraic structure of our variety V, each automorphism σ of C (fixing the coefficients field of V) determines a simplicial automorphism of N_α (reflecting the algebraic continuity of the "classically discontinuous" automorphism of the point set of V mentioned above).

Thus Gal (C/Q) *acts on the profinite homotopy type of any complex algebraic variety defined over* Q. For example Gal (C/Q) acts on the classify-

[1] Otherwise $\hat{V} \cong \varprojlim_{\alpha,\beta} \hat{N}_{\alpha\beta}$ where $\hat{N}_\alpha = \varprojlim_\beta N_{\alpha\beta}$.

ing space $\hat{B}$ of profinite K-theory by automorphisms preserving the filtration by completed grassmannians.

Now consider the natural homomorphism (a surjection)

$$\text{Gal}\,(\mathbf{C}/Q) \xrightarrow{A} \hat{Z}^*$$

obtained by letting $\sigma \in \text{Gal}\,(\mathbf{C}/Q)$ act on the roots of unity.

One checks easily that $\text{Gal}\,(\mathbf{C}/Q)$ acts on $\bigcup_n \hat{\mathbf{C}P}^n \cong K(\hat{Z}, 2)$ via A and the natural action of $\hat{Z}^*$ on $K(\hat{Z}, 2)$. (A single computation of the nerves N_α for $\mathbf{C}P^1$ is required.)

It follows by naturality and the splitting principle that $\text{Gal}\,(\mathbf{C}/Q)$ acts through $\hat{Z}^*$ on profinite K-theory and that this action is compatible with the isomorphic part of the Adams' operations discussed above.

Let B_n ($\hat{B}_n$) denote the union over k of the (completed) grassmannians of n-planes in $n + k$-space. Now the total space of the canonical sphere bundle γ_n over B_n is homotopy equivalent to B_{n-1}. Thus by Theorem 3.1 $\hat{B}_{n-1}$ is the total space of a completed spherical fibration $\hat{\gamma}_n$ over $\hat{B}_n$. The restriction of this fibration B_n is clearly the fiberwise completion of γ_n.

The Adams conjecture is proved using the composition

$$\hat{\gamma}_n^\sigma \xrightarrow{\sigma_*} \hat{\gamma}_n \xrightarrow{\sigma^{-1}} \hat{\gamma}_n \ ,$$

where σ_* is the tautological map of $\hat{\gamma}_n^\sigma = \sigma^*\hat{\gamma}_n$, and σ^{-1} is a fiber-preserving inverse of the automorphism σ of $\hat{B}_{n-1} \cong \hat{\gamma}_n$ corresponding to $\sigma \in \text{Gal}\,(\mathbf{C}/Q)$. This composition covers the identity of $\hat{B}_n$. So γ_n and γ_n^σ have the same unstable profinite fiber homotopy type. This is a kind of unstable Adams conjecture for profinite vector bundles.

Let $\hat{\gamma}$ denote the stable profinite spherical fibration over $\hat{B}$ determined by $\{\hat{\gamma}_n\}$ over $\{\hat{B}_n\}$. From the above we have that $\gamma \sim \gamma^\sigma$ as stable profinite spherical fibrations.

Now our computations show that the map from stable fiber homotopy types to profinite stable homotopy types is injective. In fact on the classifying space level we have

$$\begin{aligned} &\text{stable profinite theory:} && \hat{B}_\infty \cong B_{SG} \times K(\hat{Z}^*, 1)\ , \\ &\text{stable theory:} && B_G \cong B_{SG} \times K(Z/2, 1)\ , \end{aligned}$$

so B_G is a factor of $\hat{B}_\infty$, and we are done.

We close with a more precise formulation of the unstable Adams' phenomenon.

Define profinite vector bundle theory in dimension n by mapping into the completion $\hat{B}_n$ of the classifying space for the ordinary vector bundle theory B_n. For example, in the complex case we have

the classifying space: $\hat{B}_n = \bigcup_k \hat{G}^{\mathbf{C}}_{k,n+k} = \hat{B}_{Gl(n,\mathbf{C})}$,
the natural map: $\hat{B}_{Gl(n,\mathbf{C})} \xrightarrow{\hat{J}_n} B^{\wedge}_{SG_{2n}}$,
the symmetry group: Gal (C/Q) for $B^{\wedge}_{Gl(n,\mathbf{C})}$,
the symmetry group: $\hat{Z}^*$ for $B^{\wedge}_{SG_{2n}}$.

The map $\hat{J}_n$ assigns an oriented, completed $(2n-1)$ spherical fibration $[\eta]$ to the profinite complex n-bundle η classified by a map into $B^{\wedge}_n = B^{\wedge}_{Gl(n,\mathbf{C})}$.

If $\sigma \in$ Gal (C/Q), let $(\sigma) = A\sigma \in \hat{Z}^*$ be the element constructed by letting σ act on the roots of unity. The argument above gives the

$$\text{(unstable complex Adams conjecture)}: \quad [\eta^\sigma] = [\eta]^{(\sigma)^n}.$$

There is an analogous statement for real profinite vector bundles, the spherical fibration map $B^{\wedge}_{Gl(n,R)} \xrightarrow{\hat{J}_n} B^{\wedge}_{G_n}$, and the symmetry groups Gal (C/Q)/Gal (C/R)$^{\sim}$ and $\hat{Z}^*/Z^*$.[1] Now we have

$$\text{(unstable real Adams conjecture)}: \quad [\eta^\sigma] = \begin{cases} [\eta]^{(\sigma)^{n/2}} & n \text{ even} \\ [\eta]^{(\sigma)^{(n-1)/2}} & n \text{ odd}. \end{cases}$$

The inertia of intrinsic stable fiber homotopy types.

In order to begin to understand philosophically the proof just given of the Adams conjecture we consider a more general, related phenomenon. Let γ_n denote a compatible sequence of spherical fibrations of increasing fiber dimensions over an increasing union of spaces

$$B_1 \subset B_2 \subset \cdots \subset B_n \overset{i}{\subset} B_{n+1} \subset \cdots .$$

Our basic assumption about the stable bundle $\gamma =$ "$\bigcup_n \gamma_n$" over $B = \bigcup_n B_n$ is that it is *intrinsic* to the filtration of the base B. This is, γ_{n+1} is approximated by the inclusion $B_n \subset B_{n+1}$. More precisely, for arbitrarily large integers n the composition

$$B_n \xrightarrow[\text{cross section}]{} i^*\gamma_{n+1} \xrightarrow{i_*} \gamma_{n+1}$$

is a homotopy equivalence over the $d(n)$-skeleton where

$$d(n) - \text{fiber dim}\, \gamma_{n+1} \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty .$$

In the Lie group examples considered above, this composition was actually a homotopy equivalence.

For the other classifying spaces of geometric topology

[1] Gal (C/R)$^{\sim}$ means the normal subgroup generated by complex conjugation.

$$B_{\mathrm{PL}}\,,\quad B_{\mathrm{Top}}\,,\ \text{and}\ \ B_G$$

the corresponding stable bundles satisfy the more general hypothesis. In fact $d(n)$ is approximately 2 (fiber dimension of γ_{n+1}) for the "block bundle" filtrations of B_{PL} and B_{Top} and the usual filtration of B_G.

We justify our terminology by the

THEOREM. *The stable fiber homotopy type of* γ *can be constructed from the filtration* $\{B_n\}$ *of* B.

Now let $B \xrightarrow{\sigma} B$ be any filtered automorphism of B; that is, $\sigma = \bigcup_n \sigma_n$ where σ_n is an automorphism of B_n. Denote $\sigma^*\gamma$ by γ^σ. Then we have the

COROLLARY. $\gamma^\sigma \sim \gamma$, i.e., γ^σ *is fiber homotopy equivalent to* γ.

Proof of Corollary. Clearly $\gamma^\sigma = \sigma^*\gamma$ is intrinsic if γ is. So $\gamma^\sigma \sim \gamma$ follows from the theorem.

Proof of Theorem. Let d_n be an integer a little less than $d(n)$-fiber dim γ_{n+1}, and consider a fibration approximating $B_n \xrightarrow{i} B_{n+1}$ restricted to the d_n-skeleton of B_{n+1}, say $Y_n \xrightarrow{(i)} X_n$.

Now our hypothesis means that (fiber i)$\rightarrow$(fiber γ_{n+1}) is a homotopy $d(n)$-equivalence. So the induced map over X_n, $Y_n \xrightarrow{j} \gamma_{n+1} \mid X_n$, is a homotopy $d(n)$-equivalence. But $\gamma_{n+1} \mid X_n$ has dimension a little less than $d(n)$.

Thus the part of γ_{n+1} over X_n can be constructed from $B_n \xrightarrow{i} B_{n+1}$ using the composition

$$\gamma_{n+1} \mid X_n \xrightarrow{\ k\ } Y_n \xrightarrow{\ (i)\ } X_n$$

where $j \cdot k \cong 1$. Since j and k are homotopy $d(n)$-equivalences the space $\gamma_{n+1} \mid X_n$ and the map $\gamma_{n+1} \mid X_n \rightarrow X_n$ are well-defined by this procedure.

Note. This proof provides *something like a* converse to our proof of the Adams conjecture; namely, let σ be any automorphism of B, and let e be any stable equivalence $\gamma^\sigma \overset{e}{\sim} \gamma$. Let X_n be a skeleton of B_{n+1} so that e induces an equivalence $\gamma^\sigma_{n+1} \mid X_n \xrightarrow{e_n} \gamma_{n+1} \mid X_n$. We can also assume that X_n is a skeleton of B by the intrinsic hypothesis.

Now we can assume that the automorphism σ preserves the skeleton X_n of B. Then the composition

$$\gamma_{n+1} \mid X_n \xrightarrow{\ e_n^{-1}\ } \gamma^\sigma_{n+1} \mid X_n \xrightarrow{\ \sigma^*\ } \gamma_{n+1} \mid X_n$$

is an automorphism of a high unstable skeleton of $B_n \mid X_n$ covering $\sigma \mid X_n$.

Thus in proving the Adams conjecture over $\hat{B}_U$, say, one has to construct automorphisms of a large unstable subcomplex of $\hat{B}_{U_n}$, containing a stable skeleton.[1]

Remark. One might hope from the corollary to find filtered automorphisms of the geometric theories B_{PL} and B_{Top} leading to homotopy relations between different geometric bundles, as we did for the K-theories above. This turns out to be the case. There is a Galois group and a kind of Adams conjecture (or phenomenon) for PL and Top which we will discuss in a sequel to this paper.

Of course for the spherical fibration theory the corollary means that there are no symmetries preserving the filtration. Thus any compatible sequence of automorphisms of B_{G_n} determines the identity map of B_G.

When is an l-adic fibration the completion of a local fibration?

According to Sections 1 and 3 there are the fiber squares

$$\begin{array}{ccc} Z_l & \longrightarrow & \hat{Z}_l \\ \downarrow & & \downarrow{\scriptstyle \otimes Q} \\ Q & \underset{\otimes \hat{Z}_l}{\longrightarrow} & \hat{Q}_l^f \end{array}, \qquad \begin{array}{ccc} (B_{SG_n})_l & \longrightarrow & (B_{SG_n})_l^{\wedge} \\ \downarrow & & \downarrow \\ (B_{SG_n})_Q & \longrightarrow & ((B_{SG_n})_l^{\wedge})_Q \end{array}.$$

This leads to the

COROLLARY 4. *An oriented $\hat{S}_l^{n-1}$-fibration is the completion of a S_l^{n-1}-fibration if and only if*

a) *for n even, the image of the Euler class under*

$$H^n(base, \hat{Z}_l) \underset{\otimes Q}{\longrightarrow} H^n(base, \hat{Q}_l^f)$$

is rational, namely in the image of

$$H^n(base, Q) \underset{\otimes \hat{Z}_l}{\longrightarrow} H^n(base, \hat{Q}_l^f) ;$$

b) *for n odd, the Hopf class, which is only defined in*

$$H^{2n-2}(base, \hat{Q}_l^f)$$

is rational.

Proof. The fiber square above is equivalent to (n even)

$$\begin{array}{ccc} (B_{SG_n})_l & \longrightarrow & (B_{SG_n})_l^{\wedge} \\ {\scriptstyle \text{rational Euler class}}\downarrow & & \downarrow{\scriptstyle l\text{-adic Euler class}} \\ K(Q, n) & \longrightarrow & K(\hat{Q}_l^f, n) \end{array}.$$

[1] Quillen did this by considering the approximations of $B_{Gl(n,\mathbf{C})} \cong B_{U_n}$ given by the $K(\pi, 1)$ spaces, $B_{Gl(n,k)}$, where k is the algebraic closure of the prime field. The symmetry is given by Frobenius [12].

The corollary is a restatement of one of the properties of a fiber square; that is, in the fiber square of CW complexes,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

maps into C, and B together with a class of homotopies between their images in D determines a class of maps into A.

Addendum. Another way to think of the connection is this: since

$$(B_{SG_n})_l^{\wedge} \cong \prod_{p \in l} (B_{SG_n})_p^{\wedge} ,$$

a local S_l^{n-1}-fibration over a finite complex is a collection of $\hat{S}_p^{n-1}$-fibrations, one for each p in l, together with the coherence condition that the characteristic classes (either Euler or Hopf, with coefficients in Q_p) they determine are respectively in the image of a single rational class.

This uses the Hasse principle of Section 3 which implies that the class of the map into $(B_{SG_n})_l$ is independent of the homotopy used to assemble it.

Principal spherical fibrations.

Certain local (or l-adic) spheres are naturally homotopically equivalent to topological groups. Thus we can speak of principal spherical fibrations. The classifying space for these principal fibrations is easy to describe and maps into the classifying space for oriented spherical fibrations.

PROPOSITION (p odd). *$\hat{S}_p^{n-1}$ is homotopy equivalent to a topological group (or loop space) if and only if n is even and n divides $2p - 2$.*[1]

COROLLARY. *S_l^{2n-1} is homotopy equivalent to a loop space if and only if*

$$l \subseteqq \{p\colon Z/n \subseteqq p\text{-adic units}\} .$$

Remark. Let S_l^{2n-1} have classifying space $P^\infty(n, l)$; then

$$\Omega P^\infty(n, l) \cong S_l^{2n-1} .$$

The fibration

$$S_l^{2n-1} \longrightarrow * \longrightarrow P^\infty(n, l)$$

implies

i) $H^*(P^\infty(n, l), Z_l)$ is isomorphic to a polynomial algebra on one generator in dimension $2n$.

ii) For each choice of an orientation of S_l^{2n-1} there is a natural map

[1] For $p = 2$, it is well-known that only S^1, S^3, and S^7 are H-spaces, and S^7 is not a loop space.

$$P^\infty(n, l) \longrightarrow (B_{SG_{2n}})_l \ .$$

In cohomology the universal Euler class in $(B_{SG_{2n}})_l$ restricts to the polynomial generator in $P^\infty(n, l)$.

Proof of the proposition. A rational argument implies a spherical H-space has to be odd-dimensional.

If $\hat{S}_p^{n-1}$ is a loop space, ΩB_n, it is clear that the mod p cohomology of B_n is a polynomial algebra on one generator in dimension n. Steenrod operations imply λ divides $(p-1)p^k$ for some k, where $n = 2\lambda$. Secondary operations, using Liulevicius' mod p analysis generalizing the famous mod 2 analysis of Adams, show $k = 0$; that is, λ divides $p - 1$.

On the other hand, if λ divides $p - 1$ we can construct B_n directly:

i) Embed Z/λ in $Z/p - 1 \subsetneqq \hat{Z}_p^*$.

ii) Choose a functorial $K(\hat{Z}_p, 2)$ in which $\hat{Z}_p^*$ acts freely by cellular homeomorphisms.

iii) Form

$$B_n = \big(K(\hat{Z}_p, 2)/(Z/\lambda)\big)_p^\wedge \ .$$

We obtain a p-adically complete space which is simply connected, has mod p cohomology, a polynomial algebra on one generator in dimension n, and whose loop space is $\hat{S}_p^{n-1}$.

In more detail, the mod p cohomology of $K(\hat{Z}_p, 2)/(Z/\lambda)$ is the invariant cohomology in $K(\hat{Z}_p, 2)$ under

$$(1, x, x^2, x^3, \cdots) \xrightarrow{\alpha} (1, \alpha x, \alpha^2 x^2, \cdots), \ \alpha^\lambda = 1 \ .$$

This follows since λ is prime to p and we have the spectral sequence of the fibration

$$K(\hat{Z}_p, 2) \longrightarrow K(\hat{Z}_p, 2)/(Z/\lambda) \longrightarrow K(Z/\lambda, 1) \ .$$

Now we can regard B_n as the p-adic completion of the simply connected space

$$\big(K(\hat{Z}_p, 2)/(Z/\lambda)\big) \cup \text{cone}\,(S^1 \cup_\lambda e^2) \ .$$

So B_n is simply connected and by the arguments of Section 3 has the correct mod p cohomology.

The space of loops on B_n is an $(n-2)$ connected p-adically complete space whose mod p cohomology is one Z/p in dimension $n - 1$. By Section 3 we have the p-adic sphere $\hat{S}_p^{n-1}$.

Proof of the corollary. If l is contained in $\{p \colon n \text{ divides } (p-1)\}$[1], construct the "fiber product" of

[1] The case left out is taken care of by S_l^3.

$$\begin{array}{ccc} & & \prod_{p \in l} B_{2n}^p \\ & & \downarrow \\ K(Q, 2n) & \longrightarrow & K(\hat{Q}_l^f, 2n) \end{array}$$

where B_{2n}^p is the de-loop of $\hat{S}_p^{2n-1}$ constructed above.

If we take loop spaces, we get the fiber square

$$\begin{array}{ccc} S_l^{2n-1} & \longrightarrow & \prod_{p \in l} \hat{S}_p^{2n-1} \\ \downarrow & & \downarrow \\ S_Q^{2n-1} & \longrightarrow & (\prod_{p \in l} \hat{S}_p^{2n-1})_Q \,. \end{array}$$

Thom isomorphism.

An S_l^{n-1}-fibration with orientation

$$U_\xi \in H^n(E \longrightarrow X, Z_l)$$

determines a Thom isomorphism

$$H^i(X, Z_l) \xrightarrow[\cong]{\cup U_\xi} H^{i+n}(E \longrightarrow X, Z_l) \,.$$

This is proved, for example, by induction over the cells of the base using Mayer-Vietoris sequences (Thom).

Conversely (Spivak), given a pair $A \xrightarrow{f} X$ and a class

$$U \in H^n(A \longrightarrow X, Z_l)$$

such that

$$H^i(X, Z_l) \xrightarrow{\cup U} H^{i+n}(A \longrightarrow X, Z_l)$$

is an isomorphism, then under appropriate fundamental group assumptions $A \xrightarrow{f} X$ determines an oriented S_l^{n-1}-fibration. For example, if the fundamental group of X acts trivially on the fiber of f, then an easy spectral sequence argument shows that

$$H^*(\text{fiber of } f, Z_l) \cong H^*(S_l^{n-1}, Z_l) \,.$$

If further, the fiber of f is a nilpotent space, a fiberwise localization[1] is possible, and this produces an S_l^{n-1}-fibration over X,

$$\begin{array}{ccccc} \text{fiber } f & \longrightarrow & A & \longrightarrow & X \\ {\scriptstyle\text{localization}}\downarrow & & \downarrow{\scriptstyle\text{fiberwise localization}} & & \| \\ S_l^{n-1} & \longrightarrow & E & \longrightarrow & X \,. \end{array}$$

[1] See proof of Theorem 4.2.

A similar situation exists for l-adic spherical fibrations.

Whitney join.

The Whitney join operation defines pairings between the S_l^{n-1}, S_l^{m-1} theories and S_l^{n+m-1} theories. We form the join of the fibers (S_l^{n-1} and S_l^{m-1}) over each point in the base and obtain an S_l^{n+m-1}-fibration. This of course uses the relation

$$S_l^{n-1} * S_l^{m-1} \cong S_l^{n+m-1} .$$

The analogous relation is not true in the complete context. However, we can say that

$$(\hat{S}_l^{n-1} * \hat{S}_l^{m-1})_l^{\wedge} \cong \hat{S}_l^{n+m-1} .$$

Thus fiber join followed by fiberwise completion defines a pairing in the l-adic context.

Proof of Theorem 4.2 (page 62).

i) The map l is constructed by *fiberwise localization.*

Let ξ be a fibration over a simplex σ with fiber F, a nilpotent space, and let ∂l

$$\begin{array}{ccc} \xi/\partial\sigma & \xrightarrow{\partial l} & \partial\xi' \\ & \searrow \quad \swarrow & \\ & \partial\sigma & \end{array}$$

be a fiber-preserving map which localizes each fiber. Then filling in the diagram

$$\begin{array}{ccccc} \xi/\partial\sigma & \longrightarrow & \xi & \overset{\text{arbitrary trivialization}}{\cong} & \sigma \times F \\ {\scriptstyle \partial l}\downarrow & & & & \downarrow{\scriptstyle \text{projection}} \\ \partial\xi' & \xrightarrow{t} & F_l & \xleftarrow{\text{localization}} & F \end{array}$$

gives an extension of the fiberwise localization ∂l to all of σ; that is,

$$\begin{array}{ccccc} \xi & \xrightarrow{l} & \xi' & = & \text{mapping cone of } t \\ & \searrow \quad \swarrow & & & \wr\| \\ & \sigma & & & \sigma \times F_l . \end{array}$$

But t exists by obstruction theory,

$$H^*(\partial\xi', \xi/\partial\sigma; \pi_* F_l) \cong H^*(\partial\sigma \times (F_l, F); Z_l\text{-module}) \equiv 0 .$$

Thus, fiberwise, we can localize any fibration with nilpotent fiber by proceeding inductively over the cells of the base. We obtain a "homotopically locally trivial" fibration which determines a unique Hurewicz fibration with fiber F_l.

The same argument works for fiberwise completion,

$$F \longrightarrow \hat{F}_l \quad \text{whenever} \quad H^*(\hat{F}_l, F; \hat{Z}_l) \equiv 0 .$$

But this is true, for example when $F = S^{n-1}$ or S_l^{n-1}.

This shows we have the diagram of i) for objects. The argument for maps and commutativity is similar.

To prove ii) and iii) we discuss the sequence of theories

$$U\colon \{\text{oriented } S_R\text{-fibrations}\} \xrightarrow{f} \{S_R\text{-fibrations}\} \xrightarrow{w} H^1(\ \ , R^*)$$

where $S_R = S^{n-1}$, S_l^{n-1}, or $\hat{S}_l^{n-1}$; and R is Z, Z_l, or $\hat{Z}_l$.

The first map forgets the orientation.

The second map replaces each fiber by its reduced R-homology. This gives an R coefficient system classified by an element in $H^1(\ \ , R^*)$.

Now the covering homotopy property implies that an S_R-fibration over a sphere S^{i+1} can be built from a homotopy automorphism of $S^i \times S_R$ preserving the projection $S^i \times S_R \to S^i$. We can regard this as a map of S^i into the singular complex of automorphisms of S_R, Aut S_R. We can assume that a base point of the equator S^i goes to the identity of Aut S_R.

For $i = 0$, the fibration is determined by the component of the image of the other point on the equator. But in the sequence

$$\pi_0 \operatorname{Aut} S_R \longrightarrow [S_R, S_R] \longrightarrow \pi_{n-1}S_R \longrightarrow H_{n-1}S_R$$

the first map is an injection and the second and third are isomorphisms. Thus

$$\pi_0 \operatorname{Aut} S_R \cong R^* \cong \operatorname{Aut}(H_{n-1}S_R) .$$

This proves ii) a) and the fact that oriented bundles over S^1 are all equivalent.

More generally, an orientation of an S_R-fibration determines an embedding of the trivial fibration $S_R \to *$ into it. This embedding in turn determines the orientation over a connected base.

Thus if the orientation sequence U corresponds to the sequence of classifying spaces

$$\tilde{B}_R \xrightarrow{f} B_R \xrightarrow{w} K(R^*, 1) ,$$

we see that $\pi_1 \tilde{B}_R = 0$ and for $i > 0$,

$$\begin{aligned}\pi_{i+1}\tilde{B}_R &\cong [S^{i+1}, \tilde{B}_R]_{\text{free}} \\ &\cong \text{oriented bundles over } S^{i+1} \\ &\cong \text{based bundles over } S^{i+1} \\ &\cong [S^{i+1}, B_R]_{\text{based}} \\ &\cong \pi_{i+1}B_R .\end{aligned}$$

So on homotopy we have

$$* \xrightarrow{f} R^* \underset{\cong}{\xrightarrow{w}} R^* \qquad \text{for } \pi_1 ,$$

$$\pi_{i+1} \underset{\cong}{\xrightarrow{f}} \pi_{i+1} \longrightarrow * \qquad \text{for } \pi_{i+1} .$$

Therefore, U is the universal covering space sequence.

Also the correspondence between based and oriented bundles shows the R^* actions correspond as stated in iii).

We are left to prove the second part of ii). The cell by cell construction of part i) shows we can construct (cell by cell) a natural diagram

$$\begin{array}{ccc} G_n = \text{Aut } S^{n-1} & \longrightarrow & \text{Aut } S_l^{n-1} \\ & \searrow^{c} & \downarrow \\ & & \text{Aut } \hat{S}_l^{n-1} , \end{array}$$

corresponding to fiberwise localization for fibrations over suspensions. The proof is then finished by calculating the induced maps on homotopy groups for the connected components of the identity. For example, to study c look at the diagram

$$\begin{array}{ccc} (S^{n-1}, S^{n-1})_1^{\text{based}} & \underset{\text{completion}}{\xrightarrow{c_1}} & (\hat{S}_l^{n-1}, \hat{S}_l^{n-1})_1^{\text{based}} \\ \downarrow & & \downarrow \\ SG_n & \underset{\text{completion}}{\xrightarrow{c}} & (\text{Aut } \hat{S}_l^{n-1})_1 \\ \downarrow & & \downarrow \\ S^{n-1} & \underset{\text{completion}}{\xrightarrow{c_0}} & \hat{S}_l^{n-1} . \end{array}$$

Now c_0 just tensors the homotopy with $\hat{Z}_l$.

An element in π_i of the upper right hand space is just a homotopy class of maps

$$S^i \times S_R \xrightarrow{f} S_R , \qquad S_R = \hat{S}_l^{n-1}$$

which is the identity on $* \times S_R$ and constant along $S^i \times *$. We can measure the deviation of f from the projection on the second factor $S^i \times S_R \xrightarrow{\pi} S_R$ by the single obstruction to a homotopy between them in

$$H^{i+n-1}\big((S^i, *) \times (S_R, *), \pi_{i+n-1}S_R\big) \cong \pi_{i+n-1}S_R .$$

It is not hard to see that this obstruction determines an isomorphism of groups

$$\pi_i(\text{Aut } S_R)_1 \cong \pi_{i+n-1}S_R .$$

The naturality of this obstruction shows that c_1 l-adically completes the homotopy groups.

Since the vertical sequences are fibrations, it follows that c completes the homotopy groups.

This proves that $B_{SG_n} \to$ universal cover $B_n^\wedge$ is l-adic completion.

The localization statement of ii) b) is treated similarly, and the proof of Theorem 4.1 is complete.

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