The signature, geometric cycles, and $\mathbb{R}$ K-theory

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by Dennis Sullivan

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We record here a theorem about ~~the~~ the K-theory and signature of 4k-manifolds which on the one hand generalizes the ~~the~~ Thom transversality construction of rational Pontryagin classes to capture the odd primary information[1] and on the other hand ~~[illegible]~~ ~~[illegible]~~ provides one of the crucial algebraic ~~[illegible]~~ topological ~~[illegible]~~ steps for our earlier calculations on the Hauptvermutung [ ].

Our theorem roughly stated goes as follows — suppose first that X is a geometric cycle of dimension n which can be included in a suitable collection of geometric cycles and homologies. ~~which~~ ~~[illegible]~~ We suppose the collection contains smooth manifolds and is rich enough (*)

1 For the 2-primary generalization see Morgan and Sullivan [illegible]

geometrically to construct a K-homology theory. ~~and for which there is for each~~

Suppose next that ~~for~~ to each cycle in this collection we can associate an integer ~~[illegible]~~ (the signature) ~~[illegible]~~ ~~generalizing~~ which is a homology invariant, and which ~~extends~~ to the classical signature invariant for the smooth manifolds in the collection.

Then each cycle in our collection, in particular X, has a natural K-homology orientation class that is

$$\mu_X \in K_n X.$$

~~[illegible]~~ Here $K_i X$ is the homology theory defined by Alexander duality from the real K-~~[illegible]~~ cohomology theory of Bott Atiyah and Hirzebruch ~~[illegible]~~ ~~[illegible]~~ after ~~[illegible]~~ suppressing the prime 2 by tensoring with the

[illegible]

We were ~~[illegible]~~ originally interested in 19[illegible] in the case when X was a triangulable manifold. Here we found for such manifolds natural K-theory orientations which ~~[illegible]~~ define a Poincaré duality in K-theory and provide geometric invariants of ~~[illegible]~~ piecewise linear structure. The last step in our work on the Hauptvermutung was to show that ~~the~~ this natural K-duality ~~[illegible]~~ in a piecewise linear manifold was a homeomorphism invariant using Novikov's toroidal method ~~[illegible]~~ in his ~~[illegible]~~ proof ~~of~~ ~~the~~ ~~[illegible]~~ the topological invariance of rational Pontryagin classes.

The new use of the torus by Kirby and Siebenmann in the ~~theory of topological~~ study of homeomorphisms shows for example there is enough transversality to briefly [illegible] consider a ~~[illegible]~~ compact topological manifold [illegible] homology in the dimension. the same argument given above. Thus directly natural K-orientations follow for topological manifolds.

~~Another application of this idea~~

Another example is to take X to be a triangulable space which is a rational homology manifold. Such cycles then receive natural K orientations. The construction shows that if $X \xrightarrow{f} Y$ is a semi-triangulable map[1] with acyclic point inverses for rational coefficients, then

$$f_* \mu_X = \mu_Y.$$

Two other interesting remarks about this case are

i) because they have

[1] Semi-triangulable means the mapping cylinder is triangulable [illegible]

[illegible]

[illegible], rational homology manifolds are not abundant enough to represent all homology classes in spaces – for example the original example of Thom, $u_3 * u_3$ in $H_7(K(\pi),1)$ where $\pi = \mathbb{Z}/3 + \mathbb{Z}/3$ is not so representable. This shows further that the symmetric product construction of $K(\mathbb{Z},n)$ from S^n cannot be used ~~[illegible]~~ directly with transversality to produce a universal cycle with universal singularities in codimension n because everything in sight is a rational homology manifold.

ii) On the other hand this symmetric product construction produces a sequence of canonical elements

$$\rho_i \in K_{ni}(K(\mathbb{Z},n))$$

defined by the natural inclusion of the i^{th} ~~[illegible]~~ symmetric product of S^n with its [illegible] class into the infinite symmetric product [illegible] $K(\mathbb{Z},n)$ [illegible]

One philosophy about geometric cycles and homology theories is described in [ ]. ~~This is...~~

Combining the discussion here with the constructions there which introduce special singularities into manifolds to produce various new homology theories [ ]

And give ~~an~~ one explicit ~~set~~ picture of universal singularities for ordinary homology ~~cycles~~ classes we see that K-homology is in fact ~~...~~ isomorphic to one of these geometric ~~...~~ ~~...~~ cycle theories with a signature invariant. Thus the theorem here can be reformulated as follows:

K-homology is naturally isomorphic to the direct limit of the system of homology theories defined by geometric cycles with a signature invariant.

This ~~...~~ raises the problem of describing in a ~~[illegible]~~ direct geometric way a geometrical [illegible] of geometric cycles with a signature invariant. ~~[illegible]~~ These would be the cycles of K-theory.

All the above discussion involving the signature takes place over $\mathbb{Z}[\frac{1}{2}]$ because even smooth manifolds do not admit K orientations over $\mathbb{Z}$ in general. There is however a completely analogous discussion and theory ~~[illegible]~~ over $\mathbb{Z}$ based on geometric cycles ~~[illegible]~~ having an integral invariant generalizing the Todd genus of stably almost complex manifolds. Then we find a geometric cycle description of complex K-homology (over $\mathbb{Z}$). This theory is not directly applicable in the theory of ~~[illegible]~~ homeomorphism invariants for manifolds but ~~[illegible]~~

arithmetic genus of complex algebraic varieties (allowing singularities) and the Grothendieck Riemann Roch theorem (see for example ~~and especially~~ the recent construction of Baum and MacPherson). ~~The problem is to~~ ~~modify~~ The interesting problem seems to define an appropriate notion of geometric homology between algebraic varieties ~~th~~ which preserves the arithmetic genus and converts the ~~class~~ resonance into concrete mathematics.

~~The~~

Section 2 contains the definitions and proofs. The paper ends with corollaries about some of the classifying spaces of geometric topology.

Section 2 Definitions and Proofs

Recall the bordism theory defined by associating to each space Y the abelian group of bordism classes of maps $M^n \xrightarrow{f} Y$ where M^n is an oriented n-manifold and a sum $\Sigma(M_i^n, f_i) = 0$ if we find the disjoint union of the (M_i, f_i) as the complete boundary of $W^{n+1} \xrightarrow{F} Y$ for some compact $(n+1)$ manifold W mapping to Y. (~~and~~ the basic reference is [ ])

Using manifolds with boundary the bordism groups of a pair are defined and the basic axioms of Eilenberg Steenrod homotopy (invariance), exactness, excision, and the functorial properties are all ~~[illegible]~~ satisfied. The normalization axiom about the ~~first~~ groups of a point is of course only ~~[illegible]~~ ~~is satisfied~~ for the relation by asking homology theory. The verification of the axioms is either formal ~~geometric~~ except for one geometric point. ~~Excision and~~ ~~and part of exactness~~ require the following geometric axiom: ~~if K is a compact set in one~~

~~is necessary to~~ ~~because~~ ~~the fact that~~ The ability to form the cartesian product ~~of~~ ~~a~~ with the unit interval and paste ~~~~ manifolds together along good pieces of the boundary suffices ~~for~~ for most of the ~~except~~ verification. ~~for the geometric point.~~ ~~The~~ ~~For the point~~

Then transversality is used for the nontrivial geometric step ~~to~~ to cut out a compact set from an open set in a manifold to prove excision and the non-formal part of exactness. ~~Thus~~ With the all the axioms of a generalized homology so are proved.

Without being more precise we will say we have a geometric homology theory if we have a collection of ~~[illegible]~~ relative cycles (like compact manifolds with boundary) in abundant supply to carry out the ~~[illegible]~~ bordism type construction ~~[illegible]~~ ~~[illegible]~~ of a generalized homology theory. Thus in practice we need a boundary, to be able to paste along relative cycles in the boundary, to form the product with the unit interval, and finally to be able to cut out arbitrary compact sets fairly precisely.

There are many examples of this situation:

go to page A1

If V denotes a collection of geometric cycles such as the above, let $V(X)$ denote the corresponding homology theory. Then $V(pt)$ are cobordism groups of closed cycles of V regarded up to homology.

We will say that "V-homology theory has a signature" if V ~~all compact oriented smooth manifolds~~ ~~and there is a homomorphism~~ is closed under the operation of cartesian product with closed smooth manifolds and the identification $V_0(pt) = \mathbb{Z}$ extends to a homomorphism

$$V_*(pt) \xrightarrow{\sigma} \mathbb{Z}$$

satisfying $\sigma(M \times L) = \text{signature } M \cdot \sigma L$ where L is a cycle of V and M is a manifold.

The first part of the assumption means that we can intersect V-cycles with manifold cycles inside an ambient manifold to obtain a V-cycle in the ambient manifold. For example if U is a nice ~~[illegible]~~ open subset of R^n ~~[illegible]~~ then the relative V homology of a neighborhood of the diagonal of U (mod its boundary) is isomorphic by the suspension isomorphism to the V-homology of U ~~[illegible]~~ with an appropriate shift of dimension. Thus if we have $V \to U$ and $M \to U$ ~~[illegible]~~ and we can form $V \times M \to U \times U$ in V-theory we can cut down to the intersection element ~~formally~~ by a formal application to the suspension isomorphism.

We will apply this idea in the following way. Let X be a closed V cycle and let U be the interior of a closed regular neighborhood N of

X [illegible]

above procedure to intersect relative manifold cycles of N and ∂N with X to obtain a homomorphism

$$\cancel{\Omega_*(N, \partial N) \xrightarrow{\sigma_X}}$$

~~via the composition~~

$$\Omega_*(N, \partial N) \xrightarrow{\Lambda_X} V_* N$$

of degree − codimension X.

Now we combine this homomorphism with the second part of the signature assumption with the composition

$$V_* N \xrightarrow{\varepsilon_*} V_*(pt) \xrightarrow{\sigma} \mathbb{Z}$$

where $\varepsilon: N \to pt$ to obtain finally a homomorphism

$$\Omega_*(N, \partial N) \xrightarrow{\sigma_X} \mathbb{Z}$$

$$\cancel{\sigma_X (L \times M \xrightarrow{f \cdot p_1} (N, \partial N))}$$

which is a $\Omega_*(pt)$ module homomorphism relative to the [illegible] $\Omega_* pt \to \mathbb{Z}$ [illegible]

At this point we have enough to construct a K-orientation for X using σ_X. It will however only be determined modulo torsion elements in $K_n(X)$. To make the choice canonical we assume the signature in V-theory can be extended to residue classes mod n. Namely for each n we have a signature mod n

$$V_*(pt, Z/n) \xrightarrow{\sigma_n} Z/n$$

which is compatible with the reductions $Z \to Z/n$ and $Z/kn \to Z/n$ and with the multiplication by closed manifolds.

To clarify this last condition we recall that $V_*(X, Z/n)$ has a natural interpretation in terms of mapping into X relative V cycles whose boundary fields with n disjoint isomorphic copies. (see). Then $V_*(pt, Z/n)$ is an $\Omega_*(pt)$ module and σ_n should be a module homomorphism relative to the signature. Then the intersection above yields

$$\Omega_*(N, \partial N; Z/n) \xrightarrow{\cap X} V_*(N, Z/n)$$

and finally by composing with Γ_* and σ_n we have

$$\Omega_*(N, \partial N; Z/n) \xrightarrow{\sigma_{X,n}} Z/n .$$

Now the compatible collection of $\Omega_*(pt)$ module homomorphisms $\{\sigma_X, \sigma_{X,n}\}$ will determine a canonical K-orientation for X.

For this we need one proposition about K-theory and one proposition relating K-theory and smooth bordism theory.

The first proposition asserts that K^*, the cohomology theory defined by vector bundles and Bott periodicity and K_* the formally defined homology theory defined via Alexander Duality e.g.

$$\tilde{K}_{*i}(Y) \cong K^{*j}(N, \partial N)$$

where Y is any finite complex and N is a regular neighborhood of Y in Euclidean space (and $i+j = \dim N$) satisfy a somewhat unexpected algebraic relationship, K_* and K^* satisfy Pontryagin duality at least if we suppress the prime 2.

We will formulate this conveniently for our purposes here. Let $K_i(X, Q/Z_{(\text{odd})})$ denote the direct limit of $K_i(X, Z/n)$ as n ranges over the multiplicative set of odd integers. Let $K_i(X; Q) = K_i X \otimes Q$.

the Pontryagin duality assertion for real K-theory asserts that $K^i(X) \otimes \mathbb{Z}[\frac{1}{2}]$ is naturally isomorphic to the group of commutative diagrams

$$\begin{array}{ccc} K_i(X; \mathbb{Q}) & \xrightarrow{\sigma} & \mathbb{Q} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ K_i(X, \mathbb{Q}/\mathbb{Z}[\frac{1}{2}]) & \xrightarrow{\tau} & \mathbb{Q}/\mathbb{Z}[\frac{1}{2}] \end{array}$$

where i and j are the natural maps and σ and τ are to be given.

This isomorphism follows somewhat formally for finite complexes from

i) the elementary exact sequences relating $K^i(X)$ and $K^i(X, \mathbb{Z}/n)$ and $K_i X$ and $K_i(X, \mathbb{Z}/n)$

ii) that $K_i X$ and $K^i X$ are finitely generated Abelian groups for a finite complex

iii) the elementary fact that $K_i(X) \otimes \mathbb{Q}$ is isomorphic to $K^i(X) \otimes \mathbb{Q}$, and

iv) the natural [illegible]

$\mathrm{Hom}(K_i(X,\mathbb{Z}/n),\mathbb{Z}/n) \simeq K^i(X,\mathbb{Z}/n)$
for n odd.

Only the last point really needs comment (for more details on the formal argument see [ ]). ~~An~~ natural evaluation maps $K^i(X,\mathbb{Z}/n) \xrightarrow{e} \mathrm{Hom}(K_i(X,\mathbb{Z}/n),\mathbb{Z}/n)$ can be constructed for n odd because ~~[illegible]~~ $K^i(X,\mathbb{Z}/n)$ is still a multiplicative cohomology theory [ ]. ~~[illegible]~~ For the same reasons $K_i(X,\mathbb{Z}/n)$ is a $\mathbb{Z}/n$ module. It follows that e is a natural transformation of cohomology theories since $\mathrm{Hom}(~~[illegible]~~, \mathbb{Z}/n)$ ~~is~~ ~~[illegible]~~ carries exact sequences of $\mathbb{Z}/n$ modules into exact sequences. But e is an isomorphism for a point by Bott periodicity ~~[illegible]~~. This proves iv).

The second proposition relates bordism and K-homology after tensoring with $\mathbb{Z}[\frac{1}{2}]$ (which we assume has been done from now on).

The relationship between Ω_* and K_* was inspired by [ ] and employs a certain multiplicative transformation $\Omega_* \xrightarrow{L} K_*$ which for a point is just the signature.

Because L is multiplicative there are induced transformations of $\mathbb{Z}/2$-graded theories

$$\Omega_*(\,) \underset{\Omega_*(pt)}{\otimes} \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{L^{\otimes}} K_*(\,)$$

for $\mathbb{Z}$ and $\mathbb{Z}/n$ coefficients. The second of our propositions asserts that the induced maps $L^{\otimes}$ are isomorphisms.

We outline the construction of L and the proof that $L^{\otimes}$ are isomorphisms.

to B1

9.

Now we can combine the two propositions with the "torus of X" homomorphisms $\{\sigma_X, \sigma_{X,n}\}$. Let

$\cancel{\Omega_*(N,\partial N)\otimes Q}$

$\Omega_*(Y,Q) = \Omega_*(Y)\otimes Q$ and $\Omega_*(Y, Q/Z[\tfrac{1}{2}])$

be the direct limit over n odd of $\Omega_*(X, Z/n)$. Using $\{\sigma_X, \sigma_{X,n}\}$ we have a diagram of $\Omega_*(pt)$ module homomorphisms

$$\begin{array}{ccc}
\Omega_*(N,\partial N; Q) & \xrightarrow{\sigma_X} & Q \\
\downarrow & & \downarrow \\
\Omega_*(N,\partial N; Q/Z[\tfrac{1}{2}]) & \xrightarrow{\sigma_{X,\infty}} & Q/Z[\tfrac{1}{2}]
\end{array}$$

where $\sigma_{X,\infty}$ is the union of all the $\sigma_{X,n}$ for n odd.

Using the second ~~of~~ proposition we have ~~[illegible]~~ a diagram

$$\begin{tikzcd} K_*(N,\partial N;\mathbb{Q}) \arrow[r, "\sigma_X"] \arrow[d] & \mathbb{Q} \arrow[d] \\ K_*(N,\partial N;\mathbb{Q}/\mathbb{Z}[\tfrac{1}{2}]) \arrow[r, "\sigma_X"] & \mathbb{Q}/\mathbb{Z}[\tfrac{1}{2}] \end{tikzcd}$$

which by the first proposition is precisely an element in $K^{\cdot}(N,\partial N)\cancel{\otimes\mathbb{Z}[\tfrac{1}{2}]}$. ~~which~~ This latter group is by definition $\cong K_n(X)\cancel{\otimes\mathbb{Z}[\tfrac{1}{2}]}$. where X is ~~[illegible]~~ our original geometric cycle of dimension n. So we have our canonical orientation

$$\mu_X \in K_n X.$$

We can take for our relative cycles[1]

i) compact oriented piecewise linear manifolds

ii) ~~Z~~ homology manifolds with coefficients in $\mathbb{Z}$, $\mathbb{Q}$ or any intermediate subring (~~[illegible]~~

iii) any collection of relative n-circuits so that the links of interior points lie in a class of polyhedra satisfying suitable properties e.g. closed under suspension and desuspension etc (the reader can work out this very pleasant exercise which was first done by George Cooke (unpublished))

iv) collections of n-[illegible] circuits which satisfy semi-local geometric conditions e.g. the Thom's natural generic

1 We are restricting ourselves for analogues of oriented or $\mathbb{Z}$-homology for the moment

p' stratifications only have links in even codimension, or the singularities are "like" those of the finite joins of some collection of manifolds (see [ ]). Or one can work with a suitable certain collection of Thom stratified sets.

The possibilities are enormous. For example, join like singularities on specific manifolds are sufficient for ordinary homology, real K-homology tensor $\mathbb{Z}[\frac{1}{2}]$, ~~and~~ Cayley K-homology, and many new intermediate theories between bordism and K-theory (see [ ]).

One amusing example is singularities which are locally the product of Euclidean space and based on a product a number of cone CP^{p-1} ~~singularities~~ for p a fixed prime [illegible]. Then we [illegible] of [illegible] all classes [illegible]

· homology localized at p. For $p = 2$ we have the PP' operations which are not C^∞ [illegible] invariants. and the ^statement this reduces to the rep [illegible] Thom theorem which asserts [illegible] about representing cycles by manifolds at the prime 2. (see [ ], and [ ])

First we describe the transformation L. If M^{4k} is a compact smooth manifold, let N be a ~~tu~~ tubular neighborhood of M in a high Euclidean space. We can think of N as the total space of the bundle $(\nu_M \otimes \mathbb{C}) \oplus (\tau_M)$ where ν_M is the normal bundle of dimension $4j$ in a lower Euclidean space and τ_M is the tangent bundle. If E^+ denotes the Thom space or one point compactification of the vector bundle E, then we have two elements

$$\gamma \in K_U(\nu_M \otimes \mathbb{C}^+) \text{ and } \sigma \in K_O(\tau_M^+).$$

~~defined as follows~~ γ is the natural K orientation of the complex bundle $\nu_M \otimes \mathbb{C}$ and σ is the symbol of the differential

Now we list some corollaries that can be deduced rather formally from these K-elements. (all over $\mathbb{Z}[\frac{1}{2}]$)

i) as mentioned before piecewise linear, in fact topological manifolds, have natural K-orientations and thus natural K-dualities.

ii) thus the corresponding bundles theories have natural K-theory Thom isomorphisms.

iii) one can check that that natural map of stable pl or top bundles to stable spherical fibrations with K orientation is an isomorphism at odd primes (eg the induced map on classifying spaces is an isomorphism [illegible] $\longrightarrow \Lambda$ [illegible]

[illegible] of M^{4k}. We can regard σ as lying in real K-theory $\otimes \mathbb{Z}[\frac{1}{2}]$ and form $\gamma \otimes \sigma$ in $K(N, \partial N)$, which is by definition $L_{4k}(M)$. This construction determines the transformation L. That $\Omega_*(pt) \to K_*(pt)$ is the signature is the Atiyah-Singer index theorem![1]

The induced transformation $L^{\otimes}$ is shown to be an isomorphism by the following steps —

2) tensor everything that comes up with $\mathbb{Z}[\frac{1}{2}]$.

ii) convert the problem to cohomology using Alexander duality.

[1] Actually, it is at the proof of the [illegible] this case is [illegible] by [illegible]

iii) the argument of 2.5 can be directly adapted to see that $L^{\otimes}$ is an injection. $*$

ii) to see that $L^{\otimes}$ is onto we study the map ~~equivalent localization classifying~~ $\Omega^0 \xrightarrow{L^0} K^0$ in degree zero cohomology. ~~after applying Alexander duality~~ This corresponds to a map of classifying spaces $M \xrightarrow{l} B$ which in homotopy is essentially the signature ~~Ω~~ $*$ Since $B = BO$ has ~~no torsion~~ no odd torsion in its cohomology we see that l has a cross section by elementary obstruction theory after localizing away from 2 ~~[illegible]~~ Thus L^0 and a fortiori $L^{\otimes}$ is onto.

Thus the homotopy theoretical fibre only has homotopy in even dimensions.

$*$ The point is the universal spaces for bordism, the Thom spaces of the universal oriented vector bundles have no odd torsion in the cohomology. Thus we can check directly that $L^{\otimes}$ is

follows by the above [illegible] BO [illegible] on homotopy groups $\otimes \mathbb{Z}[\frac{1}{2}]$.)

iv) the universal spaces for such top or pl bundles with a fibre homotopy trivialization G/PL or G/Top have two K-orientations in their respective universal bundles. One is ~~the one~~ that constructed above by transversality and the other ~~way the~~ is induced by the fibre homotopy trivialization of the universal bundle. These two differ in either bundle by the multiplication by an element in K of the base ~~K(base)~~ and we obtain a diagram of $\mathbb{Z}[\frac{1}{2}]$ homotopy equivalences

$$G/PL \sim BO \sim G/Top.$$

Thus we have a diagram of $\mathbb{Z}[\frac{1}{2}]$ homotopy equivalences

$$B_{PL} \sim B_{KG} \sim B_{Top}$$

The details of these calculations can be found in [ ]. ~~[illegible] detailed account of the [illegible] the~~ But the proper discussion of ~~[illegible]~~ ~~[illegible]~~ these results and ~~the~~ others about PL/O, G/O, B_O, B_{PL} etc obtained ~~[illegible]~~ by combining the above with the Adams conjecture ~~will appear elsewhere~~. ~~([illegible])~~ is really the subject of another paper.

Let X be a triangulable space which locally has the ~~sette~~ homology with rational coefficients of a manifold. If X is compact of dimension n and is a cycle, then we have Poincaré duality

$$H^i(X, \mathbb{Q}) \cong H_{n-i}(X, \mathbb{Q}).$$

In [ ] I have constructed characteristic classes $L^i \in H^{4i}(X, \mathbb{Q})$ using ~~the~~ transversality and the signature invariant ~~of~~ of the symmetric intersection pairings

$$H^{2j}(Y) \xrightarrow{x \cup x} H^{4j}(Y) \simeq \mathbb{Q}$$

where Y are various rational homology submanifolds of X of dimension $4j$.

These classes can be equally considered in the homology of X using the duality —

$$\ell_0 \in H_n X, \quad \ell_1 \in H_{n-4} X, \ldots$$

and $\ell_{n/4} \in H_0 X$, ~~[illegible]~~ if $n \equiv 0\,(4)$, the signature of X.

These Thom homology classes are equivalent to the rational Pontryagin classes if X is a smooth manifold. In this note we mention ~~this~~ integral refinements of the ℓ_i classes *in homology* which do not carry over to cohomology because the Poincaré duality is only over $\mathbb{Q}$. These refinements are important in the classification theory of manifolds cycles of course are for duality over $\mathbb{Z}$. These refinements have their natural expression however in this rational homology context.

We also ~~mention~~ discuss other geometric characteristic classes of X in homology: The first one ~~is the~~ are canonical ~~ch~~ cycles for Stiefel Whitney homology classes ~~and first~~ Finally we describe homology classes which obstruct~~s~~ the desingularization of X by a map $M \xrightarrow{f} X$ where point inverses are acyclic over $\mathbb{Q}$ and M is a manifold. (a rational isomorphism.)

In general ~~they~~ all these classes enjoy the ~~following~~ naturality

$$f_* \, l_X = l_Y$$

for any rational cosmorphism $X \xrightarrow{f} Y$.

<u>The detailed description</u>

~~Hsse of the classes will~~

The first generalization of the Thom ~~th~~ characteristic class

$$l_* \in H_{n-4*} X$$

is a K-orientation class

$$\mu_X \in K_n X \otimes \mathbb{Z}[\tfrac{1}{2}]$$

This class is constructed in [ ] by ~~using~~ ~~following~~ combining the transversality construction of Thom with a K-homology relation at odd primes between bordism and ~~K~~ theory,

$$\Omega_*(X) \otimes \mathbb{Z}[\tfrac{1}{2}] \simeq K_* X \otimes \mathbb{Z}[\tfrac{1}{2}].$$

Ω_{*pt}

In place of the relation ~~relation~~
Thom had used the relation

$$\pi_*^S X \otimes \mathbb{Q} \simeq H_*(X, \mathbb{Q}),$$

where π_*^S is stable homotopy.

In [ ] we exploited the relation at the prime 2,

$$\Omega_*(X, \mathbb{Z}_{(2)}) \simeq H_*(X, \Omega_*(pt, \mathbb{Z}_{(2)}))$$

expressed geometrically with a product analysis to construct 2-integral l-classes for manifolds.
When this method is applied to rational homology manifolds it yields canonical classes

$$l_n, l_{n-4}, l_{n-8}, \ldots \in H_{n-4*}(X, \mathbb{Z}_{(2)})$$

where $\mathbb{Z}_{(2)}$ is the subring of the rationals consisting of fractions with odd denominators. We use the same notation these classes as for those of Thom because they correspond under the coefficient homomorphism $\mathbb{Z}_{(2)} \hookrightarrow \mathbb{Q}$.

~~Now there is for any any cohomology theory~~

Before discussing further relationships between these classes we discuss the others. A rational homology manifold X is an Euler space (see [ ] and [ ]) — the links of points ~~[illegible]~~ have constant Euler characteristic. It follows that if we triangulate X and barycentrically subdivide we can form the chains

$$S_i = \sum_\sigma (-1)^{|\sigma|} \sigma$$

where σ ranges over the i simplices of the subdivision ($\sigma = \tau_0 < \tau_1 < \cdots < \tau_i$, a chain of simplices in the original ~~subdivision~~ triangulation, and $|\sigma| = \dim \tau_0 + \dim \tau_1 + \cdots + \dim \tau_i$).

If $\dim X = n$ ~~[illegible]~~ one can calculate [ ] and [ ] the ∂ relations

$$\partial S_{2i} = 2 S_{2i-1} \quad \text{~~for~~ } n \text{ even}$$
$$\partial S_{2i-1} = 2 S_{2i} \quad n \text{ odd}.$$

So in particular we have the Stiefel homology classes
$s_i \in H_i(X, \mathbb{Z}/2)$

which for a manifold are dual to the Whitney classes. (see [ ] and [ ]).

Now we come to the singularity classes of a rational homology manifold X. Let Ω_i denote the following groups constructed from triangulable i-manifolds which have the same Betti numbers as the i-sphere. Oriented isomorphism classes of these manifolds form a commutative monoid using connected sum. If we add the relation $\Sigma^i \sim 0$ if Σ^i is the boundary of a Q acyclic $i+1$ manifold with some triangulation, we obtain an Abelian group Ω_i.

Consider again X. Let S be the subcomplex of nonmanifold X

consisting of points which don't have Euclidean neighborhoods. Elementary reasoning on the links shows the codimension of S is at least four. This reasoning also shows $\Theta_0 = \Theta_1 = \Theta_2 = e$.

If $s = \dim S$ consider the chain

$$\vartheta_s(X) = \sum_{\sigma \in S} (\text{link}\,\sigma)\cdot\sigma$$

where σ ranges over the s-simplices of S and link σ is the oriented link[1] regarded as an element in Θ_s. ~~St~~ Studying the links of $(s-1)$ simplices yields the

[1] We assume X is oriented for convenience.

information that $\partial \theta_s = 0$
so we have a canonical element

$$\vartheta_s \in H_{n-s}(X, \Theta_s)$$

constructed from the generic singularity of X. A pleasant geometrical argument (outlined in [ ]) shows that ~~ϑ_s equals zero iff there~~ if $\vartheta_s = \partial c$ ~~and $n-s+1$ chain~~ we can modify X near the support of c and obtain a new homology manifold X' with singularities S' of dimension $s-1$, and a rational isomorphism

$$X' \longrightarrow X.$$

Conversely given such a space X' and map $X' \to X$ we can study the mapping cylinder as a relative rational manifold and deduce that $\mathcal{V}_s = 0$ in $H_{n-s}(X, \Theta_{s-1})$.

Thus these classes $\mathcal{V}_s$ form a sequence of obstructions for (rationally) resolving the singularities in a rational homology manifold.

Now we note a certain homomorphism which was ~~pointed out to the author by~~ arose in discussions with John Milnor. Let Θ_s' denote the subgroup of Θ_s ~~which~~ consisting of elements which

are boundaries. So we have
an exact sequence

$$0 \to \Theta'_5 \to \Theta_5 \to \cancel{\Omega^{PL}_5}\ \text{torsion}\ \Omega^{PL}_5$$

using the fact that ~~[illegible]~~ the rational Pontryagin numbers of $\Sigma \in \Theta_5$ are zero.

If $\Sigma \in \Theta'_{4k-1}$ equals ∂W^{4k} let $\sigma(\Sigma)$ denote the quadratic form associated to ~~[illegible]~~

$$H^{2k}(W,\partial W) \xrightarrow{x \cup x} H^{4k}(W,\partial W)$$

by the fundamental class of W. ~~[illegible]~~ ~~[illegible]~~ We think of $\sigma(\Sigma)$ ~~[illegible]~~ as an element in the Witt group of ~~[illegible]~~ ~~[illegible]~~ forms over Q [ ] ~~[illegible]~~ modulo the subgroup generated by (1). This group W_Q is isomorphic to

$$\mathbb{Z}/2 \oplus \bigoplus_{p\equiv 1(4)} \mathbb{Z}/4 \oplus \bigoplus_{p\equiv 3(4)} \mathbb{Z}/2\oplus\mathbb{Z}/2$$

where p ranges over the odd primes [ ]

One checks easily that $\Sigma \mapsto T(\Sigma)$

defines a ~~[illegible]~~ homomorphism

$\Theta'_{4k-1} \xrightarrow{T} W_Q$ since two choices

of W can be pasted along Σ

to obtain a closed manifold

with a unimodular quadratic

form which lies in the subgroup generated by

(1).

~~[illegible]~~

~~An examination of Brieskorn~~

~~singularities~~

~~We conjecture that~~ A, B

Say that two groups are

commensurable if they contain isomorphic

subgroups of finite index and

write $A \sim_c B$. Then we conjecture

the following structure theorem about the groups $\textcircled{H}_i$,

$$\textcircled{H}_i \sim_c \begin{cases} 0 & i \not\equiv -1\ (4) \\ W_Q & i \equiv -1\ (4) \end{cases}.$$

The commensurability with W_Q will ~~be~~ be given by $\mathcal{T}$. The surjectivity of $\mathcal{T}$ can probably be deduced directly by examining the Brieskorn singularities (I've checked — the composition

$$\textcircled{H}_i' \xrightarrow{\mathcal{T}} W_Q \xrightarrow{\det} Q^*/(Q^*)^2 \cong \bigoplus_p \mathbb{Z}/2$$

is onto.) The "injectivity" of $\mathcal{T}$ can probably be deduced by following the argument in [ ] with a "rational" surgery argument. Finally