SINGULARITIES IN SPACES

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Introduction

In this note we hope to outline a few results and intuitions about singularities among various classes of spaces.

We will work in the context of topological spaces with some extra geometrical structure. We may start with a piecewise linear structure, an analytic structure, or some stratification of the space into equisingular manifolds.

We will be concerned with various geometrical and algebraic problems associated to the singularities. For example we consider the singularities of geometric cycles and their possible stratified structure. One theorem gives a canonical form for the singularities after resolving. These are join-like singularities based on an a-priori sequence of almost-complex manifolds.

Another theorem gives an implicit description of the singularities in a generic embedded cycle.

Finally, there is a geometric obstruction theory for reducing the dimension of singularities in a given situation - the most powerful application being to homology manifolds.

These last two topics are very elementary and mostly interesting because of their geometric appeal. The discussion of canonical forms is at the same time geometrical and algebraic. One is led simultaneously to a geometric approach to generalised homology theory and to certain difficult questions about the algebraic significance of certain singularities.

Resolving Singularities. Canonical forms

First consider the general problem of resolving singularities. We assume that our space with singularities V is a <u>geometric cycle</u>, that is for some triangulation V is the union of its top-dimensional simplices and that these can be oriented so that their sum is a cycle.

By a <u>blowing up</u> of V we mean an onto stratifiable map of geometric cycles

$$W \xrightarrow{f} V$$

such that

i) f (singularity W) $\subseteq$ singularity V

ii) f induces an isomorphism

$$f^{-1} (V - \text{singularity } V) \rightarrow V - \text{singularity } V .$$

(We can also usually assume that the left hand side of ii) is dense in W.)

Since f has degree one we know from the classic paper of Thom on cobordism that for some V there is no non-singular blow-up. Thom shows that it is not true that every homology class in a manifold say contains a non-singular cycle.

The investigations below stemmed from our curiosity about these <u>innately singular homology classes</u> discovered by Thom. What do they look like geometrically?

The first "innately singular" example occurs in dimension seven, for example the torsion product

$$x_1 * x_5 \in H_7 (K(Z/3 \times Z/3, 1)) \quad \text{(Thom)} .$$

The theory below implies that this class contains a geometric cycle V whose singularity Σ_V is a two-dimensional equisingular submanifold of V. In fact, a neighbourhood of Σ_V in V is isomorphic to

$$\Sigma_V \times \text{cone } \mathbb{C}P^2 ,$$

$\mathbb{C}P^2$ the complex projective plane.

Any such V in an innately singular homology class cannot be completely resolved. However, any seven dimensional geometric cycle can be blown up so that the result has only this untwisted $\mathbb{C}P^2$ - singularity.

In order to give the theorem we need to discuss join-like singularities. First of all, let me say that these singularities are not canonical for this problem - other geometric ideas might work. However they provide a fairly elegant means of cancelling the Thom phenomenon. Also their simplicity (and success) is based on a deep theorem from algebraic topology - the structure of the complex cobordism ring <u>over Z.</u>

The product structures and <u>a-priori</u> description of the links of strata for these singularities allow these cycles to be treated as simply as manifolds in some geometric contexts.

Now we look at these join-like singularities. Consider any sequence of distinct closed manifolds

$$\mathbb{C}_1 \,,\; \mathbb{C}_2 \,,\; \ldots \,,\; \mathbb{C}_i \,,\; \ldots \quad .$$

If $I = (i_1, \ldots, i_r)$ is any finite set of distinct indices, consider the join

$$\mathbb{C}_I = \mathbb{C}_{i_1} * \mathbb{C}_{i_2} * \ldots * \mathbb{C}_{i_{r+1}} \; .$$

Recall that the points of $\mathbb{C}_I$ are the points of all possible r - simplices whose vertices lie (respectively) in the disjoint union $\mathbb{C}_{i_1} \cup \mathbb{C}_{i_2} \cup \ldots \cup \mathbb{C}_{i_{r+1}}$. Suppose that one of the manifolds in the sequence (say the last) is a positive dimensional sphere.

Then $\mathbb{C}_I$ is singular at those boundary points of the simplex not in the open face opposite the vertex in $\mathbb{C}_{i_{r+1}}$.

We can stratify $\mathbb{C}_I$ according to the natural stratification scheme of a closed quadrant in Euclidean space of r-dimensions.

The stratification is achieved by removing the closure of that open face from each simplex and identifying the result with the quadrant.

There are certain points to be made about this type of stratification -

i) each point p in $\mathbb{C}_I$ has a neighbourhood isomorphic to

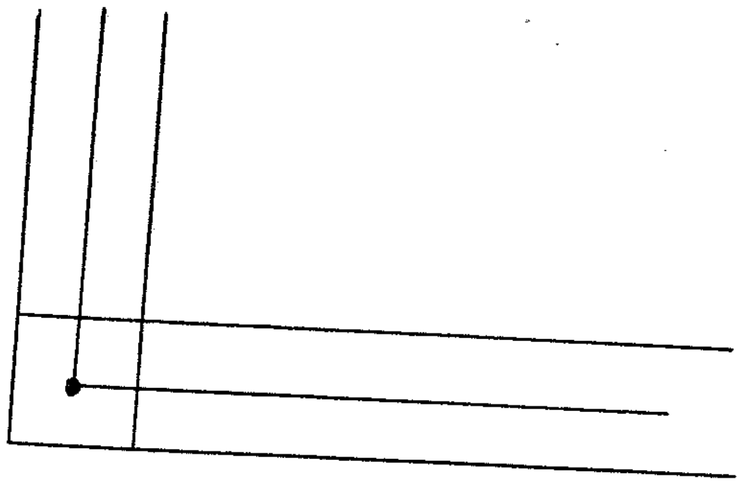
$$(\text{euclidean space}) \times (\text{cone } \mathbb{C}_J)$$

with $J \subset (i_1, \ldots, i_r)$. J is the set of indices for which the natural barycentric coordinates of p (excluding i_{r+1}) vanish.

ii) the natural (cone $\mathbb{C}_J$) bundle giving the neighbourhood of the stratum of p has a given product structure. This bundle and its product structe <u>extend</u> to the closure of the stratum - giving a neighbourhood of the closure.

iii) along a stratum in the adherence of larger strata the various product structures are related by the embeddings

$$\mathbb{C}_J \subseteq \mathbb{C}_L \ , \quad J \subseteq L \ .$$



Actually, these consideration also apply to the inclusion of p and its stratum into any larger stratum.

If W can be stratified so that these properties hold then we say that W has *join-like singularities with respect to the sequence* $\{\mathbb{C}_i\}$.

Now we can state the resolution theorems. First the more precise geometric version.

Theorem A *Let* $\{\mathbb{C}_i\}$ *be an irredundant sequence of almost-complex manifolds generating the complex cobordism ring. Suppose our space with singularities* V *has an almost complex structure on its non-singular points. Then there is blow up*

$$W \xrightarrow{f} V$$

so that W *has only join-like singularities with respect to the sequence* $\{\mathbb{C}_i\}$.

The same proof gives the following representing result for homology.

Theorem B *In a manifold any homology class of less than half the dimension contains an embedded geometric cycle having only join-like singularities for the sequence* $\{\mathbb{C}_i\}$.

We note in passing that the representation of Theorem B is unique up to a cobordism with join-like singularities. Thus if $V \subset M$ represents $x \in H_v(M)$ the closures of strata of V define lower dimensional homology classes in M canonically associated to x. Some of these classes are determined by dual cohomology operations. For example we can assume for p a prime $\mathbb{C}_{p-1} = \mathbb{C}P^{p-1}$, and the closure of the stratum of V with a normal (cone $\mathbb{C}P^{p-1}$) singularity represents

$$\beta_* \mathcal{P}^1_* x \in H_{v-2p+1}(M; Z/p) .$$

Sketch of Proof

Consider the cycles and cycles with bouhdary having join-like singularities based on the sequence $\{\mathbb{C}_i\}$.

From this geometric material we can build a generalized homology theory by forming groups out of the cobordism classes of maps $V \to X$, X an arbitrary space.

One can show that all the axioms of Steenrod are satisfied.

The excision and exactness axioms are naturally proved by inductive transversality arguments.

The dimension axiom is more delicate. It follows by deriving an exact sequence relating this theory to that with one manifold left out of the sequence. We can then peel off the singularities one by one and get back to the vacuous sequence, cobordism theory, and the beautiful complex Thom cobordism ring.

Thus we have integral homology theory represented by quasi-complex manifolds with join-like singularities based on $\{\mathbb{C}_i\}$.

This and general position proves theorem B.

The first theorem is proved by looking at a nice neighbourhood N of the singularity of V. The inclusion $\partial N \subset N$ is homologous to zero so may be replaced by a join-like homology in N. This homology is glued to the exterior of N to obtain W. There is a natural collapsing map $W \to V$ which may be shifted slightly to obtain the precise properties required of a blow-up.

Geometric Homology Theories

The join-like singularities construction may be considered from another point of view.

Any such construction for any sequence of manifolds leads to a homology theory satisfying all the Steenrod axioms but that of dimension.

For example, consider any irredundant sequence of almost complex manifolds generating the ideal of manifolds having zero Todd genus. In this case we obtain a theory (V_*, V^*) which is a version of connective complex K-theory,

$$V^0(X) \simeq \text{complex } K(X)$$
$$V^n(X) \simeq \{0\}, \quad n > \dim X .$$

Similar remarks apply to oriented manifolds, the signature, and real K-theory (ignoring the prime 2).

In the general case it is possible to compute the groups for a point in case $\{\mathfrak{C}_i\}$ is a regular sequence.

<u>Theorem C</u> Suppose $\{\mathfrak{C}_i\}$ satisfies

$$\mathfrak{C}_{k+1} \text{ is not a zero divisor of } \Omega/(\mathfrak{C}_1, \ldots, \mathfrak{C}_k).$$

Then for the point the homology theory based on $\{\mathfrak{C}_i\}$ join-like singularities has the value

$$\Omega/(\mathfrak{C}_1, \mathfrak{C}_2, \ldots) .$$

(Ω the complex cobordism ring.)

We can generalize all this and contemplate constructing hordes of homology theories from the geometric material of cycles and homologies with specified singularities.

In fact there is a functor

$$\left\{ \begin{array}{c} \text{category of} \\ \text{singularity} \\ \text{schema} \end{array} \right\} \xrightarrow{\ \pi\ } \left\{ \begin{array}{c} \text{generalized} \\ \text{homology} \\ \text{theories} \end{array} \right\} .$$

We specify stratified sets which are the cycles and homologies of the theory by saying what schemes of stratifications are allowed, what the normal cones to strata are, and what the normal cone bundles can be. The specification can be completely explicit as in the join-like singularities above. The description can be rather implicit - for example "the cycles have the local homology properties of manifolds for some coefficients". The point is that the specification be essentially geometric in character.

This geometric approach to generalized homology theory is certainly distinct from the homotopy theoretical one begun in the basic paper of G.W. Whitehead. Geometric considerations on the cycle level seem more precise than constructions with spectra - the objects of stable homotopy theory. However one is led to a Pandora's box of unknown and difficult questions relating the local and semi-local geometry of a space and its global algebraic properties.

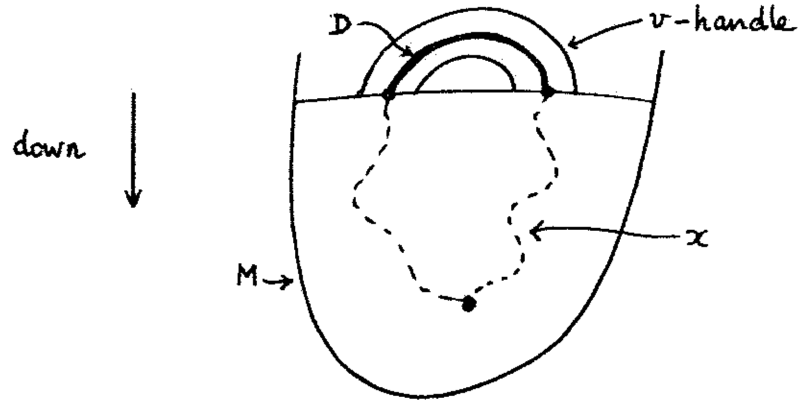
For example one might try to analyse the Zeeman spectral sequence showing how Poincaré duality is affected by the singularities. One might then be able to solve the problem of which singularities do not disturb the homology invariance of the signature of a cycle (K-theory is the limit of all geometric theories based on these singularities).

Another example is the functor π itself - what are the global properties of a singularity schema as manifested in the corresponding homology theory?

The Generic Cycle in a Homology class

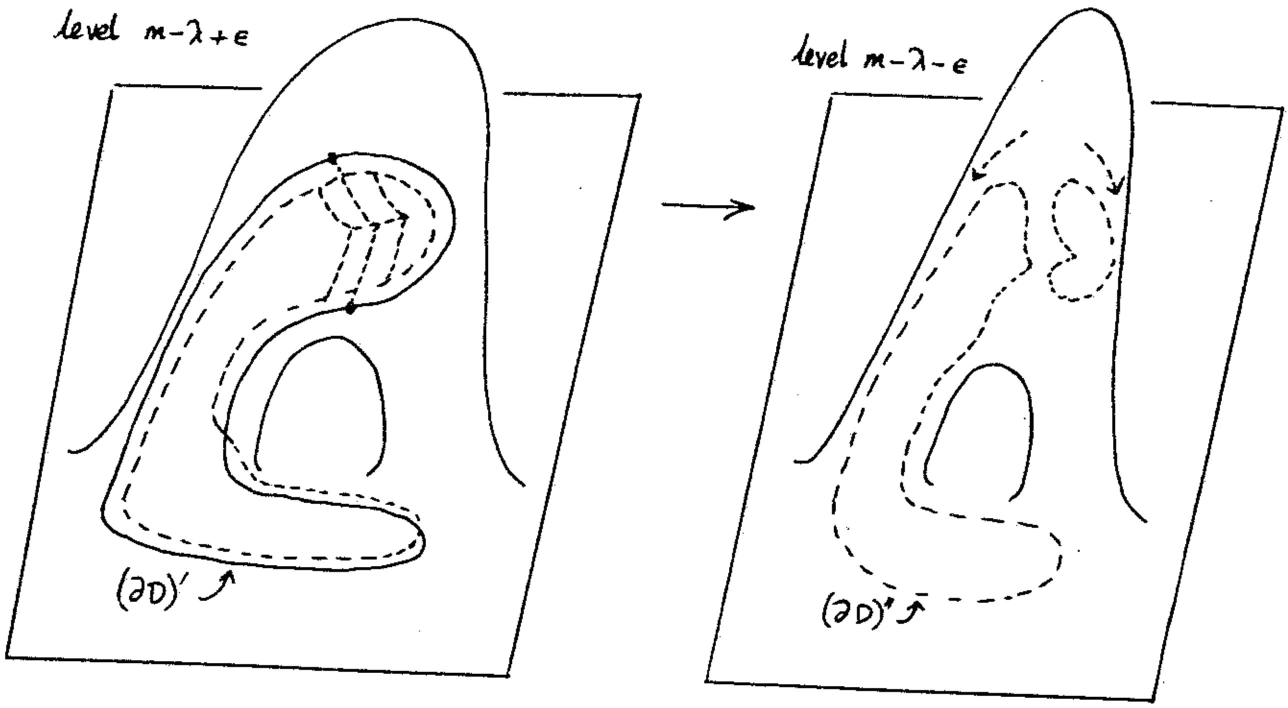
We can try to find an embedded cycle in a homology class of high dimension. We cannot dictate the singularities as in Theorem B because the algebraic obstructions to a non-singular cycle are not understood at all - there is only the homotopy theoretical criterion of Thom in terms of a pair of finite Grassmannians. However, there is a kind of generic representation theorem. The result and method is completely naive but they lead to a certain pictorial intuition for singularities - even the idea of a stratified space.

Suppose $x \in H_v(M)$ corresponds to a handle in some handlebody decomposition of the manifold M.



If D represents the v-disc core of the handle, we try to construct a cycle representing x by dragging (homologically) ∂D down through M to a point. The inductive obstruction to proceeding past a lower $(m-\lambda)$ - handle turns out to be the transversal intersection of $(\partial D)'$ (the inductively constructed edge of the deformation) with the boundary of the transverse disc to the handle.

Any homology in D^λ of this intersection gives the core of a deformation of $(\partial D)'$ allowing it to slip past the handle. If we use a cone on the intersection (which creates the singularities) we obtain Theorem B' below.



If we define a generalized handle to be -

"attach cone $V \times$ disc to Q along an embedding of $V \times$ disc in ∂Q"

then we have

Theorem B' Any homology class of M contains an embedded geometric cycle constructed inductively from a disc by attaching generalized handles.

The singularities in such a representation are almost general singularities. We may assume that the singularities have codimension at least three and that each transverse cone to a stratum has boundary irreducible geometric cycle of the appropriate dimension. The theorem gives an inductively explicit description of what singularities "look like".

From a more algebraic point of view the "dragging down" process allows a geometric interpretation of the differentials, indeterminacy, filtration, etc. in the bordism spectral sequence

$$H_*(X, \Omega_*) \Rightarrow \Omega_* X .$$

It might be interesting to compare this cycle with that obtained by forming the closure of the union of the descending trajectories through the critical point corresponding to D for a nice Morse function giving the handle decomposition used above.

The Local Obstruction and Homology Manifolds

There is a geometric procedure for reducing the dimension of the singularities in a space V. The process is obstructed in general and the value group of the obstruction depends on the context.

Suppose V is triangulated and Σ_V the singularity locus has dimension s. For each s-simplex of Σ_V we have its link, a well-defined (v-s-1)-manifold. The link determines an element in the appropriate cobordism group. The sum of the singular simplices with these link coefficients is a cycle and defines an obstruction

$$\vartheta_V \in H_s(\Sigma_V ; \Omega) .$$

__Theorem D__ __The singularity class in__ $H_* Q$ __is zero, where__ $\Sigma_V \subseteq Q \subseteq V$, __if and only if there is a blow-up of__ V __(in the context)__

$$W \xrightarrow{f} V$$

__so that__ f __is an isomorphism outside__ Q __and the singularity of__ W __has dimension smaller that__ s.

So, given this W we can look at its (s-1) dimensional singularity obstruction and so on.

__Example i)__ (General singularities - oriented case)

If V is a geometric cycle, the natural obstructions lie in

$$H_s(V; \Omega_r) \qquad r + s + 1 = \dim V$$

where Ω_r is the oriented cobordism group.

If V is a complex variety, the first obstruction vanishes because the chain vanishes identically. The links are quasi-complex manifolds of odd dimension and therefore cobordant to zero.

Example ii) (Homology manifolds)

Let R be a subring of Q with unit and suppose V has the local homology properties of a manifold (coefficients in R).

Then we can consider blow-ups

$$W \xrightarrow{f} V$$

where W is also a "homology manifold" and $f^{-1}(p)$ is "R-acyclic" for each point of V.

The local obstructions lie in

$$H_s(V, \vartheta_r) \qquad r + s + 1 = \dim V$$

where ϑ_r is the group of H-cobordism classes of r-dimensional homology spheres (namely r-manifolds having the R-homology groups of S^r). H-cobordism means oriented cobordism where the cobordism is R-homologically like $S^r \times$ unit interval.

a) In the case R is the ring of integers, we have ordinary homology manifolds. Then the coefficient groups are all zero (by surgery arguments) except when r is three. The group ϑ_3 is unknown except for a famous surjection

$$\vartheta_3 \xrightarrow{r} Z/2 .$$ [1]

If V happens to be a topological manifold the dual cohomology class

$$(r_* \vartheta) \in H^4(V, Z/2)$$

is the obstruction to a combinatorial triangulation discussed by Kirby and Siebenmann. [2]

b) In case R = Q we have rational homology manifolds. The obstruction groups in this case are not even finitely generated in all dimensions of the form 4k-1. This is seen by using the determinant invariant in $Q^*/(Q^*)^2$ and the Brieskorn singularities.

This Kervaire-Milnor-Rochlin invariant is one-eighth of the signature of W reduced mod 2, where ∂W is the homology sphere in question and W is assumed to be parallizable.

Historically the classes ϑ and Bockstein $(r_* \vartheta)$ were studied in 1967 by the [illegible] in work on the Hauptvermutung. In fact the desire to understand the [illegible] Bockstein obstruction led to the discussion of this section.

Sketch of Proof of Theorem D

To see that the singularity chain is a cycle we look at the link of each $(s-1)$ - simplex. The non-singular part of this link provides just the cobordism needed to deduce that the coefficient of the boundary for this simplex is zero.[1]

The resolution $W \xrightarrow{f} V$ is constructed by replacing the normal cone to various s-simplices by a cobordism of its link to zero. (This assumes the chain is zero. If it is only homologous to zero there is some initial preparatory replacement along various $(s+1)$ - simplices.) The rest of W is constructed by coning. f is constructed by a natural collapse.

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[1] This cycle argument has natural extensions down through the singularities. It seems that there is a host of a-priori obstructions with complicated coefficients somehow related to the higher order obstructions encountered in this process.