COMBINATORIAL INVARIANTS OF ANALYTIC SPACES

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We will discuss some corollaries of the following observation :

Let V be a compact space which is stratified by odd dimensional manifolds (in the sense of Thom). Then the Euler characteristic of V is zero.

The proof proceeds by induction over the number of strata. At the n^{th} stage one is attaching the boundary of an odd dimensional manifold to the previously constructed space to obtain a new stratum X_n. The attaching map is a union of fibrations over various spaces which have zero Euler characteristic by induction. It follows that ∂X_n and thus X_n each has zero Euler characteristic.

The proof actually shows that one can build up (starting from the lower strata) a fixed point free flow on V. †

Corollary 1 If X is a stratified space with only even dimensional strata then X is locally homeomorphic to the cone over a space with zero Euler characteristic.

Proof. It is easy to see that locally X is the cone over a stratified space with only odd dimensional strata.

Example i) Suppose that X can be defined locally in $\mathbb{C}^n$ by complex analytic equations. Then according to Thom and Whitney X may be stratified by complex analytic manifolds. These strata are of course even dimensional so that X satisfies the local topological condition of the corollary.

† This point owes much to remarks of A. Borel and the work of J. Mather.

Example ii) Let X be defined locally in $\mathbb{R}^n$ by real analytic equations. Consider the complex points in $\mathbb{C}^n$ defined by these equations $X_{\mathbb{C}}$ and the natural involution τ on these induced by complex conjugation of the co-ordinates in $\mathbb{C}^n$. Using the Lojasiewicz theorem we can triangulate $X_{\mathbb{C}}$ so that X is a subcomplex and τ is simplicial. It follows that locally

$$X \sim \text{cone } V$$

$$X_{\mathbb{C}} \sim \text{cone } V_{\mathbb{C}}$$

where τ is an involution of $V_{\mathbb{C}}$ with fixed subcomplex V. Since the number of simplices in $V_{\mathbb{C}}$ is clearly congruent modulo two to the number of simplices in V we have

Corollary 2 A real analytic space is locally homeomorphic to the cone over a polyhedron with even Euler characteristic.

Example a) For a one dimensional space to be real analytic there must be an even number of branches at each singular point.

Example b) The natural singularity type

$$(\text{cone over } \mathbb{C}P^2) \times (\text{euclidean space})$$

is not an analytic singularity.

Stiefel Whitney homology classes of analytic spaces.

One can now make a purely combinatorial discussion in the class of triangulated spaces V satisfying the local Euler characteristic condition of corollary 2. Following Whitney (1940) consider the mod 2 chains s_i defined as the sum of all the i-simplices in a first barycentric subdivision of V, $i = 0, 1, \ldots, \dim V$.

It is a pleasant combinatorial exercise to verify that these chains are cycles mod 2. In fact if $\sigma = (\tau_1 < \tau_2 < \ldots < \tau_i)$ is an $(i-1)$ - simplex of the subdivision, then the coefficient of σ in ∂s_i is just the mod 2 Euler characteristic of a deleted conical neighbourhood of the barycentre of τ_i, so the statement is equivalent to the local Euler characteristic condition. We obtain the Stiefel homology classes

$$s_i \in H_i\ (V\ ;\ Z/2) \qquad\qquad i = 0, 1, \ldots, \dim V\ ,$$

Now suppose that $f : V \to W$ is a semi-triangulable map, i.e. that the mapping cylinder of f is triangulable.* If the Euler characteristic of each fibre $f^{-1}(pt)$ is odd, then

$$f_*(s_i V) = s_i W\ .$$

For if M_f denotes the mapping cylinder of f with $W \times I$ added to the end, the condition on $f^{-1}(pt)$ implies that M_f has even local Euler characteristics in its its interior. The boundary formula above now implies

$$\partial\, s_{i+1}(M_f) = s_i(V) + s_i(W)\ .$$

<u>Example</u> In case V is a manifold, the Stiefel classes $s_i(V)$ are dual to the (Stiefel) Whitney cohomology classes $w_{n-i}(V)$ (Whitney 1940, Cheeger 1968).

In case $f : V \to W$ is obtained by blowing up a submanifold of odd codimension, then

$$f_*(s_i V) = s_i W\ .$$

<u>Historical Note</u> J. Cheeger and J. Simons were seeking (1967-68) a combinatorial formula for the rational Pontryagin classes of a manifold (solid angle, incidence, etc.) This problem is still unsolved, difficult, and extremely provocative. What resulted however was Cheeger's rediscovery of combinatorial Stiefel-Whitney classes for manifolds (the remarks of Whitney on the subject in 1940 are little known - there are no proofs) and the Chern-Simons work on $4\ell - 1$ forms in the principal frame bundle of a Riemannian manifold.

Cheeger proceeded by constructing specific k-fields with singularities on a triangulated manifold. He then identified the obstruction co-chain for removing the singularities to the dual of s_{k-1}. The identification was a formidable calculation involving the evaluation of a complicated integral and certain solid angle formulae.

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* Proper algebraic maps are semi-triangulable using Lojasiewicz theory. So are stratified maps and it seems reasonable that proper analytic maps should be.

We wondered if there was a direct argument that these simply defined Stiefel chains were mod 2 cycles. The "even local Euler characteristic" condition was worked out with E.Akin at Berkeley.

Then it seemed natural to ask whether other classes of spaces besides manifolds satisfied this Euler condition and had natural Stiefel homology classes.

It was fairly clear that complex varieties of complex dimension one and two had vanishing local Euler characteristic - essentially Corollary 1 for a small number of strata. I asked P. De ligne if he could give an example of a complex algebraic variety not satisfying corollary 1. To my surprise he almost immediately replied with a convincing argument that no such example existed using Hironaka's local resolution of singularities. He has since outlined a general conjectural theory of Chern classes for singular varieties based on ideas of Grothendieck and this resolution idea. Progress on the theory now depends on the following conjecture.

If $V \xrightarrow{f} W$ is an algebraic map between complete non-singular complex algebraic varieties (V and W not necessarily connected), so that the Euler characteristic of each point inverse $f^{-1}(p)$ is one, then

$$f_* \text{ (Chern class of } V) = \text{Chern class of } W .$$

De ligne's verification for complex varieties inspired me to finish the naïve (but complicated) geometrical discussion required for Corollary 1.

Finally one should point out that the generality of corollary 1 shows the result about complex varieties follows from "dimensional considerations". The mod 2 result about real varieties is however <u>geometrically</u> surprising. It depends on the existence of the associated complex variety. This was also true in the "mod 2 fundamental cycle" result of Borel and Haefliger. The sequence of cycles $s_0, s_1, \ldots, s_n$ $n = \dim V$ (or the local Euler condition) provides a generalization of their result - s_n being the fundamental class. One now wonders at the significance of the lower Stiefel homology classes of these analytic spaces - of course s_o is just the mod 2 Euler characteristic of V .