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ON THE REGULAR NEIGHBORHOOD OF A TWO-SIDED SUBMANIFOLD

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§0. INTRODUCTION

THE PURPOSE of this paper is to prove the

THEOREM. *Suppose that* $M = M^{n-1}$ *is a closed two-sided p.l. submanifold of the p.l. manifold* $V = V^n$, *where* $M \subset \operatorname{Int} V$ *and* $n \neq 5$. *Let* $\mathcal{N}$ *be the regular neighborhood of M in V. Then there is a p.l. homeomorphism* $h : \mathcal{N} \to M \times [-1, 1]$.

From this theorem and its proof we get the

COROLLARY. *If M and V are as above and if* $\varepsilon > 0$ *then there is an ambient* ε*-isotopy* $H : V \times I \to V \times I$ *(in the topological category) such that* $H_0 = 1$ *and* $H_1(M)$ *is a p.l.-bicollared subpolyhedron of V which is p.l. homeomorphic to M.*

Remarks. (1) *Two-sided* means that every component of M has a connected neighborhood in V which it separates into exactly two components. Clearly we may assume (by considering each component separately) that M and V are connected and that M–V has exactly two components. We shall do this, letting V_0 and V_1 be the closures of these components and letting $\mathcal{N}_i$ denote the regular neighborhood of M in V_i.

(2) Unfortunately the theorem does not assert that M is p.l.-bicollared; i.e. that $h(M) = M \times 0$. (In the topological category this can be asserted, as we prove in Corollary 2.2.) It is well known that this stronger assertion would imply the combinatorial Schoenflies conjecture in all dimensions less than n. Conversely, since the combinatorial Schoenflies conjecture is known in dimensions ≤ 3, our theorem is true, even with this added assertion, as long as $n \leq 4$. Hence we shall assume, when we prove the theorem in §3, that $n > 5$.

(3) The theorem answers a question raised by Husch in [6]. Partial solutions have been given by Husch [6] and Duvall [4].

(4) If, instead of assuming that $\partial M = \varnothing$, we assume that $M \cap \partial V = \partial M$ and that $n \neq 5, 6$ then the same result holds. One simply uses the s-cobordism theorem for manifolds with boundary where we use it for closed manifolds.

(5) In Lemma 3.2 we construct an s-cobordism between M and a p.l. manifold M_0.

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Seen from a more general point of view we are "resolving the singularities" of $\mathcal{N}_0$ to get an s-cobordism. There are general obstruction theories to resolving the singularities of a homotopy or homology manifold (for star manifolds such as $\mathcal{N}_0$ the obstructions vanish) which will be presented in [12]. However the proof of the present theorem is not much shortened in the general setting, and we feel that the direct proof given here (in particular the material in §1) has independent interest.

We shall use standard notation and terminology unless otherwise indicated. In particular, terminology in the piecewise linear category is that of [13] with the following exceptions: A mapping is not assumed to be p.l. unless we say so explicitly. All polyhedra are locally compact and finite dimensional.

§1. MANIFOLD COMPLEXES

By a *manifold complex* we mean a pair (X, F) where X is a polyhedron and $F = \{F_\alpha\}$ is a locally finite family of compact subpolyhedra of X satisfying:

(1) Each F_α is a topological manifold
(2) $X = \cup F_\alpha$ (union over all elements of F)
(3) If $\alpha \neq \beta$ then $\mathring{F}_\alpha \cap \mathring{F}_\beta = \emptyset$ and $(F_\alpha \cap F_\beta) \in F$
(4) ∂F_α is the union of elements of F.

(X, F) will be called a *topological-ball complex* or a *PL-manifold complex*, respectively, if each F_α is a topological ball or a p.l. manifold. Finally a *C-complex* is a manifold complex in which each F_α is a contractible manifold with homotopy sphere boundary; and a *PLC-complex* is a C-complex in which each F_α is a p.l. manifold.

Two manifold complexes (X, F) and (Y, G) are *isomorphic* if there exists a bijection $\psi : F \to G$ such that both ψ and ψ^{-1} are incidence preserving. (It follows that ψ is dimension preserving and that $\overline{\psi(\partial A)} = \partial\overline{\psi(A)}$, where $\overline{Z}$ is the complex underlying Z.) In general we do not require that $\psi(A)$ be homeomorphic to A for every A in F or that X be homeomorphic to Y. However we do have:

LEMMA 1.1. *If (X, F) and (Y, G) are topological-ball complexes and if $\psi : F \to G$ is an isomorphism then there exists a homeomorphism $h : X \to Y$ such that $h(A) = \psi(A)$ for any A in F.*

The proof is an easy argument by induction up the skeletons. Having constructed $h \mid X^i$, one defines h over each $(i + 1)$-ball B as the cone on $h \mid \partial B$.

A more surprising result is

LEMMA 1.2. *If (X, F) and (Y, G) are PL-manifold complexes, where X is a p.l. n-manifold, and if $\psi : F \to G$ is an isomorphism then Y is a p.l. n-manifold. Moreover if ∂F is the subcomplex of F underlying ∂X then $\partial Y = |\psi(\partial F)|$.*

(The lemma tacitly assumes that ∂X is the underlying set of some subcomplex of F. This is an easy consequence of invariance of domain.)

Proof. Let us for the moment drop the assumption that X is a p.l. manifold. Suppose that $\psi : (X, F) \to (Y, G)$ is an isomorphism of arbitrary PL-manifold complexes and let $x \in X$ and $y \in Y$ be interior points of corresponding elements of F and G. Triangulate X and Y (by simplicial complexes of the same name) so that x and y become vertices and so that the elements of F and G underly subcomplexes of X and Y respectively. Take second deriveds X'' and Y''.

CLAIM. *In this general setting there exist a p.l. homeomorphism* $h : N(x, X'') \to N(y, Y'')$ *such that* $h(x) = y$ *and* $h(N(x, A'')) = N(y, \psi(A)'')$ *for all* $A \in F$. (Of course $N(z, Z) = \varnothing$ if $z \notin Z$.)

It is obvious that the claim implies Lemma 1.2.

We prove the claim by induction on n-dimension X. It is trivial when $n = 0$, so assume $n > 0$ and the claim is known for integers $<n$. Then $\psi|X^{n-1} : X^{n-1} \to Y^{n-1}$ is an isomorphism (here X^i denotes the i-skeleton of F, not of the simplicial complex which subdivides X), so there is a p.l. homeomorphism $h_0 : N(x, (X^{n-1})'') \to N(y, (Y^{n-1})'')$ as in the claim. Suppose that $x \in A^n \in F$ and $B^n = \psi(A^n)$.

If $x \in \mathring{A}^n$ then $y \in \mathring{B}^n$ and h_0 is vacuous. Since A^n and B^n are both p.l. manifolds there exists a p.l. homeomorphism $h : N(x, A'') \to N(y, B'')$ such that $h(x) = y$.

If $x \in \partial A^n$ then $N(x, (A^n)'') \cap (\partial A^n)'' = N(x, \partial(A^n)'')$, which is an $(n-1)$-face of A^n. Moreover, since ψ is an isomorphism and these are manifold-complexes we have:

$$\begin{aligned} h_0 N(x, (\partial A^n)'') &= h_0 N(x, \cup\{(A^{n-1})'' \mid A^{n-1} \lneqq A^n\}) \\ &= \cup\{h_0 N(x, (A^{n-1})'') \mid A^{n-1} \lneqq A^n\} \\ &= \cup\{N(y, (B^{n-1})'') \mid B^{n-1} \lneqq B^n\} \\ &= N(y, (\partial B^n)''). \end{aligned}$$

Thus h_0 takes a face of $N(x, A^{n''})$ to a face of $N(y, B^{n''})$ and so it may be extended to a p.l. homeomorphism $h : N(x, A^{n''}) \to N(y, B^{n''})$. Since the interiors of distinct n-manifolds of F (or of G) are disjoint, this gives a well-defined p.l. homeomorphism

$$h : N(x, X'') = \cup N(x, A'') \to N(y, Y'') = \cup N(y, B'')$$

where the unions are taken over all elements of F and G respectively. Q.E.D.

Definition. Let (X, F) be a C-complex. Let A^n be a principal element of F and let B^{n-1} be a free face of B^n (i.e. A^n is incident with no larger manifold and B^{n-1} is incident only with A^n). Let $Y = X - \mathring{A}^n - \mathring{B}^{n-1}$ and let $G = F - \{A^n, B^{n-1}\}$. Then we say that there is an *elementary pseudo-collapse* from (X, F) to (Y, G). More generally, if $(Y, G) < (X, F)$ we say that (X, F) *pseudo-collapses* to (Y, G) if there is a finite sequence of elementary pseudo-collapses from (X, F) to (Y, G).

LEMMA 1.3. *Suppose that* (X, F) *and* (Y, G) *are C-complexes isomorphic under the isomorphism* ψ. *Suppose that* (X, F) *pseudo-collapses to* (X_0, F_0) *and that* $\psi(X_0, F_0) = (Y_0, G_0)$. *Then*

(1) (Y, G) *pseudo-collapses to* (Y_0, G_0)
(2) Y *deformation retracts to* Y_0
(3) *The Whitehead torsion* $\tau(Y, Y_0) = 0 \in Wh\, \pi_1 Y$.

Proof. (1) is obvious. (2) and (3) will follow from induction and Lemma 7.4 of [8] once known for the case of an elementary pseudo-collapse. Thus assume that $Y = Y_0 \cup A^n$ and $A^n \cap Y_0 = \mathrm{Cl}(\partial A^n - B^{n-1})$. Then $A^n \cap Y_0$ is contractible, since ∂A^n and ∂B^{n-1} are homotopy spheres and B^{n-1} is contractible. (Van Kampens theorem applies because everything is polyhedral.) Hence the inclusion $i : A^n \cap Y_0 \to A^n$ is a homotopy equivalence, and Y deformation retracts to Y_0. Finally $(Y - Y_0) = (A^n - Y_0)$ is homeomorphic to $\mathring{A}^n \cup (\mathring{B}^{n-1} \times [0, 1])$ with the identifications $b = (b, 0)$ if $b \in \mathring{B}^{n-1}$. Therefore $Y - Y_0$ is contractible. So $\tau(Y, Y_0) = 0$, by Lemma 7.2 of [9]. Q.E.D.

LEMMA 1.4. *Suppose that (M, F) and (Y, G) are PLC-complexes where M is a closed p.l. manifold. Let $(M \times I, F \times I)$ be the natural product PLC-complex and assume that there is an isomorphism $\psi : F \times I \to G$. Let F_i be the subcomplex underlying $M \times \{i\}$ and let $Y_i = |\psi(F_i)|$ $(i = 0, 1)$. Then Y is a p.l. s-cobordism from Y_0 to Y_1.*

Proof. By (1.2), Y is a p.l. manifold with boundary $Y_0 \cup Y_1$. Because $(M \times I) \searrow (M \times \{i\})$, it follows from (1.3) that Y deformation retracts to Y_i and that $\tau(Y, Y_i) = 0$ $(i = 0, 1)$. Hence Y is an s-cobordism.

§2. THE CELL STRUCTURE ON THE REGULAR NEIGHBORHOOD OF M

In this section we assume that M^{n-1} is a two-sided p.l. submanifold of Int V^n, where $\partial M = \varnothing$. We make no compactness assumptions and put no restrictions on n. Remark 1 of the Introduction applies and we let V_0, V_1 be as in that remark. Triangulate V so that M is a full subcomplex of V. Set $\mathcal{N}_i = N(M', V_i')$ and $M_i = \dot{N}(M', V_i')$, $(i = 0, 1)$.

We introduce some notation. If A is a simplex of the simplicial complex K then

$$\dot{D}(A, K) = \{\hat{A}_1 \ldots \hat{A}_q \mid A \lneqq A_i\} < K'$$

$$D(A, K) = \hat{A}\, \dot{D}(A, K) \text{ (the \textit{dual cell} to } A \text{ in } K).$$

It is well known that $\dot{D}(A, K)$ is simplicially isomorphic to $Lk(A, K)'$. Also, if L is a subcomplex of K, let us denote

$$C(L, K) = \{A < K \mid A \cap L = \varnothing\}$$

Now suppose that A denotes a variable simplex of M. We set

$$C_A = C(\dot{D}(A, M), \dot{D}(A, V_0))$$

$$F = \{D(A, M) \mid A < M\}$$

$$G = \{C_A \mid A < M\}$$

$$F_0 = F \cup G \cup \{D(A, V_0) \mid A < M\}.$$

$F \times I$ and $F \times [-1, 1]$ denote the natural product complexes gotten by viewing I as a 1-simplex and $[-1, 1]$ as a 1-complex with vertices $-1, 0, 1$. Define $\psi : F \times I \to F_0$ by

$$D(A, M) \times 0 \to D(A, M)$$

$$D(A, M) \times 1 \to C_A$$

$$D(A, M) \times I \to D(A, V_0)$$

The point of this section is to prove

THEOREM 2.1. *Under the above conditions we have*

(a) $(\mathcal{N}_0, F_0)$ *is a topological-ball complex with subcomplexes* (M, F) *and* (M_0, G)

(b) (M, F) *and* (M_0, G) *are PL-manifold complexes*

(c) ψ *is an isomorphism from* $(M \times I, F \times I)$ *to* $(\mathcal{N}_0, F_0)$.

Since Lemma 1.1 allows us to trade isomorphisms of topological-ball complexes for homeomorphisms, we may use the symmetry of $\mathcal{N}_0$ and $\mathcal{N}_1$ to get

COROLLARY 2.2. *If* M^{n-1} *is a p.l. manifold without boundary, piecewise-linearly embedded as a two-sided submanifold of* $\mathring{V}^n$ *and if* $\mathcal{N}$ *is the regular neighborhood of* M *in* V *then there is a homeomorphism* $h : (\mathcal{N}, M) \to (M \times [-1, 1], M \times 0)$.

Remark. This corollary is a sharpening of [2, Theorem 6] and [10, Lemmas 10, 11]. Actually we are not so much interested in Theorem (2.1) for the Corollary as for the opportunity it will afford us to recognize s-cobordisms. However, since this work was done the Hauptvermutung has been proved by Kirby and Siebenmann and one might wish to combine (2.2) with the Hauptvermutung to prove our theorem. To do this one would have to show that their obstruction vanishes. Essentially this is what Lemma 3.2 accomplishes.

Proof of Theorem 2.1. We proceed by induction on n, the proof being trivial when $n = 1$. Assume that $n > 1$ and that both (2.1) and (2.2) are known for integers less than n. We shall show that each element of F_0 is a topological ball and that each element of $F \cup G$ is also a p.l. manifold. We then leave it to the reader to check that the incidence relations are such that these are manifold-complexes and ψ is an isomorphism. (Compare the proof of Lemma 4 of [3].)

Suppose that $A < M$. Since $\dot{D}(A, M) \cong Lk(A, M)'$ (where "$\cong$" denotes simplicial isomorphism), $D(A, M)$ is a p.l. ball. Hence each element of F is a p.l. ball. Under the isomorphism $\varphi : \dot{D}(A, V_0) \to Lk(A, V_0)'$ we have

$$\begin{aligned}\varphi(C_A) &= C(Lk(A, M)', Lk(A, V_0)') \\ &= C(Lk(A, V_1)', Lk(A, V)').\end{aligned}$$

Since M is full in V and separates V, V_1 is full in V. Thus $\varphi(C_A)$ is the closure of the complement of a regular neighborhood in the p.l. sphere $Lk(A, V)$. Thus $\varphi(C_A)$ is a p.l. manifold. Hence so is C_A. So each element of G is a p.l. manifold.

Now $Lk(A, M)$ is a codimension-one p.l. sphere in $Lk(A, V)$, so by induction hypothesis $N(Lk(A, M)', Lk(A, V_0)')$ is homeomorphic to $S^i \times I$, where $\dim A = n - i - 2$ $(0 \leq i \leq n - 2)$. [If $\dim A = n - 1$, it is obvious that C_A, $D(A, V_0)$ and $D(A, M)$ are topological balls, so we don't consider $i = -1$.] Pulling back by φ^{-1} this shows that $\dot{N}(\dot{D}(A, M), \dot{D}(A, V_0))$ is a sphere. As the boundary of the regular neighborhood $N(\dot{D}(A, M), \dot{D}(A, V_0))$, this sphere is bicollared in $\dot{D}(A, V_0)$, so by the topological Schoenflies theorem [1] C_A is a topological ball. Finally

$$\begin{aligned}D(A, V_0) &= \hat{A} * [C_A \cup N(\dot{D}(A, M), \dot{D}(A, V_0))] \\ &= \hat{A} * [\text{ball} \cup \text{collar}] \\ &= \text{topological ball}.\end{aligned}$$

Thus every element of F_0 is a topological ball. Q.E.D.

§3. PROOF OF THE MAIN THEOREM

We assume that M, V and $\mathcal{N}$ are as in the theorem announced in the Introduction. By Remark 1 of the Introduction we assume that M and V are connected and that V_0 and V_1 are the closures of the components of $V-M$. By Remark 2 we assume that $n > 5$. We triangulate the situation so that M is a full subcomplex of V. Let $\mathcal{N}_i = N(M', V_i')$ and let $M_i = \dot{N}(M', V_i')$. Thus the results of §2 apply.

LEMMA 3.1. *The regular neighborhood $\mathcal{N}$ is p.l. homeomorphic to $M_0 \times [-1, 1]$.*

Proof. Since, by (2.2), $\mathcal{N}$ is homeomorphic to $M_0 \times [-1, 1]$, $\mathcal{N}$ is an h-cobordism between M_0 and M_1. Further $\mathcal{N} \searrow \mathcal{N}_0$ (since $\mathcal{N} \searrow M$ and M separates $\mathcal{N}$) and $\mathcal{N}_0$ pseudo-collapses to M_0 by (1.3) and (2.1). Hence $\mathcal{N}$ pseudo-collapses to M_0, so by (1.3) again, $\tau(\mathcal{N}, M_0) = 0$. Similarly $\tau(\mathcal{N}, M_1) = 0$. Hence $\mathcal{N}$ is an s-cobordism; so by the s-cobordism theorem [11], $\mathcal{N}$ is p.l. homeomorphic to $M_0 \times [-1, 1]$. Q.E.D.

Let F, G, F_0 and ψ be as in §2. By (2.1) each element of $F \cup G$ is a p.l. manifold and a topological ball. Thus $(M \cup M_0, F \cup G)$ is a PLC-complex. Let $\alpha_0 = \psi \mid F \times \{0, 1\} : F \times \{0, 1\} \to F \cup G$.

LEMMA 3.2. *There is a PLC complex (W, H) which has $(M \cup M_0, F \cup G)$ as a subcomplex such that the isomorphism $\alpha_0 : F \times \{0, 1\} \to F \cup G$ extends to an isomorphism $\alpha : F \times I \to H$.*

Proof. As pointed out in the proof of (2.1), each i-dimensional dual cell $D(A^{n-i}, V_0)$ is the cone on a topological $(i-1)$-ball. Hence, by the Hauptvermutung in low dimensions [10], each $D(A^{n-i}, V_0)$ is a p.l. ball for $i \leq 4$. Let us set

$$\alpha_4 = \psi \mid (F \times \{0, 1\}) \cup (F^3 \times I)$$
$$W_4 = M \cup M_0 \cup \cup \{D(A^{n-i}, V_0) \mid A^{n-i} < M, i \leq 4\}$$
$$H_4 = F \cup G \cup \{D(A^{n-i}, V_0) \mid A^{n-i} < M, i \leq 4\}.$$

Then α_4 is an isomorphism of PLC complexes which extends α_0.

Now let $A^4 \in F$; i.e. A^4 is one of the dual 4-balls in M. Let $\Sigma = \alpha_4(\partial(A^4 \times I)) = A^4 \cup \alpha_4(\partial A^4 \times I) \cup B^4$ where B^4 is a contractible p.l. manifold and, by (1.4) $\alpha_4(\partial A^4 \times I)$ is an s-cobordism from ∂A^4 to ∂B^4. Clearly Σ is a p.l. homotopy 4-sphere. We claim that Σ bounds a compact contractible 5-manifold Q^5. For Σ can be compatibly smoothed by [5] and the resulting smooth homotopy 4-sphere bounds a smooth compact contractible manifold since, according to [7], $\theta_4 = 0$. Let Q^5 be the p.l. manifold underlying this smooth manifold.

For each $A^4 \in F$, we attach the corresponding Q^5 to W_4 by a p.l. homeomorphism $\partial Q^5 \to \alpha_4(\partial(A^4 \times I))$. This yields W_5 and we let $H_5 = H_4 \cup \{Q^5 \mid A^4 \in F\}$. Clearly α_4 extends to an isomorphism of PLC complexes,

$$\alpha_5 : (F \times \{0, 1\}) \cup (F^4 \times I) \to H_5.$$

Having constructed α_i and (W_i, H_i) for $i \geq 5$ then it follows, as before, that $\alpha_i(\partial(A^i \times I))$ bounds a p.l. homotpy i-sphere. But this must be a real p.l. sphere by the Poincarè conjecture for $i \geq 5$. Thus $\alpha_i(\partial(A^i \times I))$ certainly bounds a compact contractible p.l. manifold and the process continues. We set $\alpha = \alpha_n$ and $(W, H) = (W_n, H_n)$.

Q.E.D.

Proof of the main theorem. By (3.1.) $\mathcal{N}$ is p.l. homeomorphic to $M_0 \times [-1, 1]$. By (3.2) and (1.4) there exists an s-cobordism from M to M_0. The s-cobordism theorem [12] applies because $n > 5$. Thus M is p.l. equivalent to M_0. Therefore $\mathcal{N}$ is p.l. equivalent to $M \times [-1, 1]$. Q.E.D.

§4. PROOF OF THE COROLLARY

We use the notation of §2. Assume that V is triangulated so that M is a full subcomplex and the star of every simplex of M has diameter less than $\varepsilon/3$. Thus every dual cell has diameter less than $\varepsilon/3$. By (2.1), $\mathcal{N}_0 = N(M', V_0')$ has the topological-ball structure of $M \times I$, where each ball stretching from M to M_0 (i.e. each $D(A, V)$ stretching from $D(A, M)$ to C_A) has diameter less than $\varepsilon/3$. Since $\partial\mathcal{N}_0$ is topologically bicollared in V, we may choose collars of M and M_0 in $\mathrm{Cl}(V - \mathcal{N}_0)$ so small that we get a neighborhood W of $\mathcal{N}_0$ covered by a topological ball-complex isomorphic to $M \times [-1, 2]$ in which $\mathcal{N}_0$ corresponds to $M \times [0, 1]$ and in which each ball has diameter less than $\varepsilon/3$. Then there is an ambient isotopy of W, fixed on ∂W and respecting the blocks $D(A, M) \times [-1, 2]$ which takes each $D(A, M) \times 0$ onto $D(A, M) \times 1$. Clearly this is an ε-isotopy which extends to an ambient ε-isotopy H of V. But $H_1(M) = M_0$. Since M_0 is p.l. equivalent to M by the main theorem, and since $M_0 = \dot{N}(M', V_0')$ is p.l. bicollared, the corollary is proven.

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