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SMOOTHING HOMOTOPY EQUIVALENCES

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INTRODUCTION:

We consider the problem of deforming a homotopy equivalence $f:(L, \partial L) \to (M, \partial M)$ between smooth manifolds into a diffeomorphism. There is a theory for this problem analogous to that in (6).

We describe this theory-concentrating on points of difference between the two theories and on points omitted in (6).

THE OBSTRUCTION THEORY:

Definition 1: If $n > 5$, let $\mathcal{M}_n$ denote the category whose objects are smooth compact n-manifolds M such that $\pi_1 M = \pi_1 \partial M = 0$ and whose morphisms are embeddings $M_1^n \subset$ interior M_2^n such that $\pi_1(M_2 - M_1) = 0$. If $n \leqslant 5$ let $\mathcal{M}_n = \emptyset$.

Definition 2: If $M \in \mathcal{M}_n$, a k-skeleton of M is an embedding $M_k \subset M$ in $\mathcal{M}_n$ such that $\pi_i(M, M_k) = 0$ for $i \leqslant k$. A homotopy equivalence $f:(L, \partial L) \to (M, \partial M)$ is homotopic to a diffeomorphism over the k-skeleton of L if there is a k-skeleton $L_k \subset L$ and a map $g:(L, \partial L) \to (M, \partial M)$ such that

1) f is homotopic to g as maps of pairs

2) $g/L_k : L_k \to M$ is an embedding

3) $g(L - L_k) \subset M - g(L_k)$.

<u>Definition</u> 3: Let A_i denote the Abelian group of (almost) framed cobordism classes of almost parallelizable manifolds $M_i \subset R^{i+k}$, $k >> i$.

<u>Theorem</u> 1: Let $f:(L, \partial L) \to (M, \partial M)$ be a homotopy equivalence between manifolds in $\mathcal{M}_n$. Let $L_{k-1} \subset L_k \subset L_{k+1} \subset L$ be skeletons of L. Suppose that f/L_k is an embedding, $f(L - L_k) \subset M - f(L_k)$, and $k + 1 < n$. Then there is a homomorphism

$$H_{k+1}(L_{k+1}, L_k) \xrightarrow{C_{k+1}} A_{k+1}$$

with the following properties:

1) $C_{k+1} = 0$ iff f is homotopic to a diffeomorphism over the (k+1)-skeleton of M by a homotopy which is fixed on L_k and keeps $L - L_k$ in $M - f(L_k)$.

2) Under the identification of $H_{k+1}(L_{k+1}, L_k)$ with the (k+1)-chain group for $H_*(L)$, C_{k+1} becomes a cocycle. Let θ_{k+1} denote the cohomology class of C_{k+1}.

3) $\theta_{k+1} = 0$ in $H^{k+1}(L; A_{k+1})$ iff f is homotopic

to a diffeomorphism over the (k+1)-skeleton of L by a homotopy which is fixed on L_{k-1} and keeps $L - L_{k-1}$ in $M - f(L_{k-1})$.

Corollary: If $f:(L, \partial L) \to (M, \partial M)$ is a homotopy equivalence with L and M in $\mathcal{M}_n$ and $\partial L \neq 0$, then f is homotopic to a diffeomorphism iff a sequence of obstructions in

$$H^i(L; A_i) \qquad 0 < i < \dim M$$

vanish.

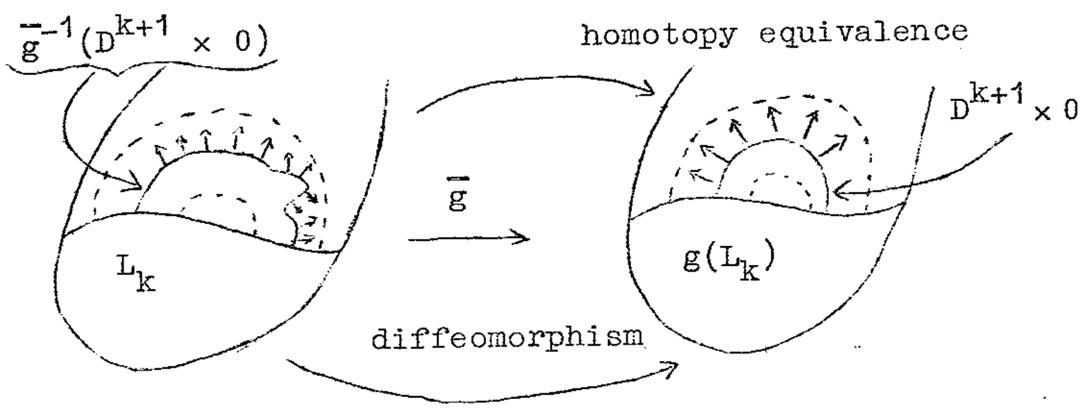
Remark: If a (k+1)-skeleton of M is obtained by attaching (k+1)-handles to $\partial g(L_k)$,

$$M_{k+1} = g(L_k) \cup_i D_i^{k+1} \times D_i^{n-k-1} \quad ,$$

then $C_{k+1}:H_{k+1}(L_{k+1}, L_k) \to A_{k+1}$ may be defined by the framed submanifolds

$$\bar{g}^{-1}(D_i^{k+1})$$

where $\bar{g}$ is a suitable (t-regular to $\cup_i D_i^{k+1} \times 0$) approximation to g such that $\bar{g}/L_k = g$.



FIGURE

<u>Remark</u>: If θ_i denotes the group of differentiable structures on S^i and P_i, $i = 1, 2, 3, \ldots$ denotes the sequence of Abelian groups, $0, Z_2, 0, Z, 0, Z_2, 0, Z, \ldots$, there is an exact sequence

$$\cdots \to P_{i+1} \xrightarrow{\partial} \theta_i \xrightarrow{i} A_i \xrightarrow{j} P_i \xrightarrow{\partial} \theta_{i-1} \cdots \quad .$$

See (10).

Using this sequence, (3), and (9) we compute A_i for $i \leqslant 19$ as follows:

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_i	0	Z_2	0	Z	0	Z_2	0	$Z \oplus Z_2$	$(Z_2)^2$	Z_6	0	Z	Z_3	$(Z_2)^2$

15	16	17	18	19
Z_2	$Z \oplus Z_2$	$(Z_2)^3$	$Z_8 \oplus Z_2$	Z_2

<u>HOMOTOPY INTERPRETATION</u>:

<u>Definition</u> 1: A <u>smooth structure</u> on M in $\mathcal{M}_n$ is a pair (L, g) where L is in $\mathcal{M}_n$ and $g:(L, \partial L) \to (M, \partial M)$ is a homotopy equivalence.

<u>Definition</u> 2: A smooth structure $g:(L, \partial L) \to (M, \partial M)$ <u>restricts to a smooth structure</u> on $M' \subseteq M$ if $M' \subseteq M$ and $L' = g^{-1}(M') \subseteq L$ are morphisms of $\mathcal{M}_n$ and $(L', g/L')$ is a smooth structure on M'.

<u>Definition</u> 3: $(\mathcal{S}(M))$ Two smooth structures, (L, g) and (L', g') on M, <u>are equivalent</u> (or <u>concordant</u>)

if there is a diffeomorphism $d: L \to L'$ so that $g' \cdot d$ is homotopic to g. Denote the set of equivalence classes by $\mathcal{S}(M)$.

Remark: The preferred element in $\mathcal{S}(M)$ is the concordance class of (M, identity). (L, g) is concordant to (M, identity) iff g is homotopic to a diffeomorphism.

$\mathcal{S}(M)$ may be regarded as the set of homotopy equivalence classes of smooth manifold structures on an underlying CW pair for $(M, \partial M)$.

$\mathcal{S}$ is a functor:

Theorem (Browder) Let $M' \subset M$ be a morphism in $\mathcal{M}_n$ and let (L, g) be a smooth structure on M. Then g is homotopic to $\bar{g}: (L, \partial L) \to (M, \partial M)$ so that $(L, \bar{g})$ restricts to a smooth structure (L', g') on M'. The concordance class of (L', g') depends only on the concordance class of (L, g).

Proof: This is the codimension one embedding theorem of Browder (1) without the π_2 hypotheses. This form follows from properties of θ_M in Theorem 2.

Corollary: The assignment $M \to \mathcal{S}(M)$ extends to a contravariant functor from $\mathcal{M}_n$ to the category of based sets.

A PRODUCT OPERATION IN $\mathcal{S}(M)$:

We use the restriction homomorphism to define a binary operation in $\mathcal{S}(M)$. Let $\tau \xrightarrow[\pi]{} M$ denote the tangent n-disk bundle of M. Then there is an embedding $i:\tau \to M\times M$ representing τ as a tubular neighbourhood of the diagonal in $M \times M$.

Definition: Suppose either $\partial M \neq 0$ or $\dim M \neq 4i$. Define a product

$$\mathcal{S}(M) \times \mathcal{S}(M) \xrightarrow{\mu} \mathcal{S}(M)$$

by the composition

$$\mathcal{S}(M) \times \mathcal{S}(M) \xrightarrow{e} \mathcal{S}(M\times M) \xrightarrow{i^*} \mathcal{S}(\tau) \xleftarrow{\pi^*} \mathcal{S}(M) .$$

e is defined on representatives by $(L, g) \times (L', g') \xrightarrow{e} (L \times L', g \times g')$. i^* is the map corresponding to the inclusion $i:\tau \subset M \times M$. π^* is defined on representatives by $(L, g) \xrightarrow{\pi^*} (\pi^*\tau, \pi^*g)$.

π^* is an injection which contains the image of i^*e if $\partial M \neq 0$ or $\dim M \neq 4i$. (These facts are proved in (8) where $\mathcal{S}(M)$ and $\mathcal{S}$(bundle over M) are compared.)

THE CLASSIFYING SPACE FOR $\mathcal{S}(M)$.

Let $F/O \xrightarrow{f} B_O$ be the fibre of the homomorphism

$$B_O \xrightarrow{J'} B_F \quad ,$$

which maps equivalence classes of stable vector bundles to equivalence classes of stable spherical fibre spaces.

Theorem 2: a) For each M in $\mathcal{M}_n$ there is a homomorphism

$$\mathcal{S}(M) \xrightarrow{\theta_M} [M, F/O].$$

b) The collection $\{\theta_M\}$ comprise a natural transformation of functors on $\mathcal{M}_n$,

$$\mathcal{S} \xrightarrow{\theta} [\quad, F/O].$$

c) θ_M is an isomorphism if $\partial M \neq 0$.

Remark: In a) replace the word "homomorphism" by "function" if $\partial M = 0$ and dim. $M = 4i$ since μ is not defined in this case. A product operation is always defined in the piecewise linear analogue of $\mathcal{S}(M)$, PL(M), because $PL(M) = PL(M_o)$.

If $\partial M = \emptyset$, θ need not be surjective nor injective. We can describe the situation however.

Definition: Define an action (connected sum)

$$\theta_n \times \mathcal{S}(M) \overset{\#}{\to} \mathcal{S}(M)$$

on representatives by

$$(\sigma, (L, g)) \to (\sigma \# L, \text{pt map} \# g) \equiv \sigma \# (L, g)$$

Definition: Define functions

$$[M^n, F/O] \overset{K}{\to} P_n$$

on representatives $M^n \overset{f}{\to} F/O$ by

$$K(M^n, f) = \begin{cases} 0 & n \text{ odd} \\ (W(M) \cup f^*(U))\,[M]_2 & n \equiv 2 \pmod 4 \\ (L(M) \cup f^*\mathcal{K})\,[M] & n \equiv 0 \pmod 4 \end{cases}$$

Here $U \in H^{4*+2}(F/0, Z_2)$ is defined in (8), $W(M)$ is the total Stiefel Whitney class of M, and $[M]_2$ is the generator of $H_n(M, Z_2)$). $\mathcal{L} = 1/8j^*(L - 1)$ in $H^{4*}(F/0; Q)$, where $j:F/0 \to B_0$ and L is the Hirzebruch L-genus in $H^{4*}(B_0, Z_2)$, $L(M)$ is the total Hirzebruch class, and $[M]$ is a generator of $H_n(M; Q)$.

<u>Remark</u>: If $n \equiv 2 \pmod 4$, the expression for $K(M^n, f)$ represents a <u>formula for the Kervaire Invariant of an F/0-bundle</u>. See (8). In this case K is a homomorphism.

If $n \equiv 0 \pmod 4$, then $K(M^n, f)$ is actually an integer (3). K is not a homomorphism in this case (it is mod 2, however).

<u>Theorem 3</u>: Let M^n belong to $\mathcal{M}_n$ and consider

$$\mathcal{S}(M) \xrightarrow{\theta_M} [M, F/0].$$

a) $\theta_M(L, g) = \theta_M(L', g')$ iff there is a σ in $\theta_n \partial\pi$ such that $(L, g)\#\sigma = (L', g')$.

b) If $\alpha \in [M, F/0]$, then $\alpha = \theta_M(L, g)$ iff $K\alpha = 0$ in P_n.

<u>Corollary</u>: The sequence

$$\theta_n(\partial\pi) \xrightarrow{\#(M,id)} \mathcal{S}(M) \xrightarrow{\theta_M} [M, F/0] \xrightarrow{K} P_n$$

is exact.

<u>PROPERTIES OF θ_M</u>:

Let $g:(L, \partial L) \to (M, \partial M)$ be a homotopy equivalence and denote $\theta_M(L, g)$ by $\theta g: M \to F/0$. Note $\theta g \simeq$ pt. map iff g is homotopic to a diffeomorphism (mod $\theta_n \partial\pi$)

In (6) we described a map $\zeta g: M_0 \to F/PL$ which is homotopic to zero iff g is homotopic to a PL-homeomorphism. ($M_0 = M$ if $\partial M \neq \emptyset$ and $M_0 = M - pt$ if $\partial M = \emptyset$.)

If g is a PL-homeomorphism, then $g: L \to M$ defines a smoothing of the underlying PL-manifold of M. This smoothing is classified by a map $\alpha g: M \to PL/O$. (2) and (4).

Consider the fibration

$$PL/O \xrightarrow{i} F/O \xrightarrow{j} F/PL .$$

Theorem 4: a) Let k denote the inclusion $M_0 \subseteq M$. Then

$$\begin{array}{ccc} M & \xrightarrow{\theta g} & F/O \\ \uparrow k & & \downarrow j \\ M_0 & \xrightarrow[\zeta g]{} & F/PL \end{array}$$

is homotopy commutative.

b) If g is a PL-homeomorphism, then αg is defined and

$$\begin{array}{ccc} & & PL/O \\ & \overset{\alpha g}{\nearrow} & \downarrow i \\ M & \xrightarrow[\theta g]{} & F/O \end{array}$$

is homotopy commutative.

Corollary: a) g is homotopic to a PL-homeomorphism iff θg lifts to PL/O.

b) If $\partial M \neq 0$, the smoothings of M corresponding (under α) to $\ker([M, PL/O] \xrightarrow{i^*} [M, F/O])$ are determined by manifolds diffeomorphic to M. In fact the smoothing map $g: L \to M$ is homotopic to a diffeomorphism.

<u>Theorem 5</u>: Consider the diagram

$$\begin{array}{ccccc} & & F & & Top/O \\ & & & \searrow \quad \swarrow & \\ M & \xrightarrow[\theta g]{} & & F/O & \end{array}$$

a) θg lifts to F iff g is a tangential equivalence.

b) θg lifts to Top/O if g is a homeomorphism.

<u>Remark</u>: a) explains the existence of the obstructions (in $H^i(M, \pi_i F)$) defined by Novikov in (5). The vanishing of these was a sufficient (but not necessary) condition to deform a tangential equivalence to a diffeomorphism (mod $\theta_n \partial\pi$). In effect, Novikov has chosen an arbitrary lifting of θg to F to define his obstructions.

We relate α and θ in a simple example.

$$\begin{array}{ccc} & \overset{\alpha g}{\nearrow} & PL/O \\ & & \downarrow i \\ M & \xrightarrow[\theta g]{} & F/O \end{array}$$

<u>Example</u>: Let $c: \sigma \to S^n$ be a smoothing of S^n $n \geq 5$. Then $c \times \text{identity} = s$

$$s:\sigma \times D^k \to S^n \times D^k$$

defines a smoothing of $S^n \times D^k$. This smoothing is classified by $\alpha s: S^n \times D^k \to PL/O$, where αs is given by the composition

$$S^n \times D^k \xrightarrow[p_1]{} S^n \xrightarrow[\sigma]{} PL/O.$$

(We identify θ_n and $\pi_n PL/O$.)

The "smooth structure" in $\mathcal{S}(S^n \times D^k)$ determined by s is classified $(k \geq 3)$ (according to Theorem 4 part b) by the composition θs,

$$S^n \times D^k \xrightarrow[\alpha s]{} PL/O \xrightarrow[i]{} F/O, \qquad \theta s = i \circ \alpha s .$$

Thus we obtain

1. s is (PL) weakly isotopic to a diffeomorphism iff $\alpha s \simeq 0$, i.e. iff $\sigma = 0$ in θ_n. (2) and (4).

2. s is homotopic to a diffeomorphism iff $\theta s \simeq i\alpha s \simeq 0$, i.e. iff σ is in the subgroup $\theta_n \partial\pi \subseteq \theta_n$. (Since $\ker (\pi_n PL/O \xrightarrow{i_*} \pi_n F/O) = \theta_n \partial\pi$.)

3. If $k > n$, then $\sigma \times D^k$ and $S^n \times D^k$ are diffeomorphic. So if $\theta_n \partial\pi \neq \theta_n$, there is a PL-homeomorphism $s: S^n \times D^k \to S^n \times D^k$ which is not homotopic to a diffeomorphism. For example this happens if $n = 8$.

4. If $\theta_n = \theta_n \partial\pi$, all smoothings of $S^n \times D^k$ are determined by diffeomorphic manifolds $(k \geq 3)$. For

example if n = 7 or n = 11 this holds. If n = 15, there are no more than two (diffeomorphism classes of) manifolds PL-homeomorphic to $S^n \times D^k$. (There are respectively 28, 992, and 16, 256 smoothings of $S^n \times D^k$ for n = 7, 11, and 15.)

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