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TRIANGULATING HOMOTOPY EQUIVALENCES

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INTRODUCTION

Let $f:(M,\partial M)\dashrightarrow(L,\partial L)$ be a homotopy equivalence of compact piecewise linear (PL) manifold pairs. When is f homotopic to a PL-homeomorphism ?

In general, homotopically equivalent M and L need not be PL-homeomorphic. For example, there is a fibre homotopically trivial S^4 bundle over S^4 with a non-zero first Pontryagin class; and there is a fibre homotopically trivial (PL) S^{10} bundle over S^{10} whose total space is not smoothable. Thus we obtain homotopy equivalences

$$f:E^8\longrightarrow S^4\times S^4$$

$$g:M^{20}\dashrightarrow S^{10}\times S^{10}$$

which are not homotopic to PL-homeomorphisms.

These examples (and two others) are in some sense generic in the simply connected, high dimensional case. Suppose dim $M\geqslant 6$ and $\pi_1 M=\pi_1\partial M=0$. Let

$$f:(M,\partial M)\longrightarrow(L,\partial L)$$

be a homotopy equivalence. Let P_i be the sequence of

Abelian groups,

$$P_i = \begin{cases} 0 & \text{if } i \text{ is odd} \\ Z_2 & \text{if } i \equiv 2 \pmod 4 \\ Z & \text{if } i \equiv 0 \pmod 4 \end{cases} .$$

THE MAIN THEOREM

THEOREM 1: There is an obstruction theory for the problem of deforming a homotopy equivalence $f:(L,\partial L)\to(M,\partial M)$ to a PL-homeomorphism over the skeletons of M. f is homotopic to a PL-homeomorphism iff a sequence of obstructions in

$$H^i(M,P_i) \quad , \; 0<i<\dim M \, ,$$

vanish .

Remark: The precise skeletal nature of the obstruction theory is described in (10).

THE OBSTRUCTIONS

The obstructions in dimensions $H^{4i}(M,Z)$ can be computed modulo torsion and we obtain

THEOREM 2: Let $f:(M,\partial M)\longrightarrow(L,\partial L)$ be as above and suppose that

1) $H^{4i+2}(M,Z_2) = 0 \qquad 4i+2 < \dim M$
2) $H^{4i}(M,Z)$ is free
3) f corresponds rational Pontryagin classes, $f^* p_i(L) = p_i(M)$.

Then f is homotopic to a PL-homeomorphism.

Corollary: Suppose M satisfies 1) and 2) of Theorem 2). Then the Hauptvermutung is true for M.

Proof: If f is a homeomorphism, 3) is satisfied by Novikov (6).

The obstructions in $H^{4i+2}(M;Z_2)$ are uniquely determined and can be computed in terms of geometric properties of the homtopy equivalence $f:(M,\partial M)\longrightarrow(L,\partial L)$ (Theorem 2/;3). These properties can be analyzed when f is a homeomorphism. This leads to stronger theorems about the Hauptverutung: Condition 1) may be dropped and Condition 2) may be weakened. Compare (11).

The Corollary implies the Hurewicz Conjecture (that homotopically equivalent closed manifolds are homeomorphic) is true if $H^{4i+2}(M,Z_2) = H^{4i}(M,Z) = 0$ for $0<4i+2$, $4i < \dim M$. Examples such as those above show that from the point of view of cohomology hypotheses, the corollary is essentially a best possible result.

THE PROOF:

Theorem 1 may be developed from two points of view, one geometric and one homotopy theoretical.

These discussions are given in (9) and will be published elsewhere.

The geometric proof arises from a procedure for supplying the hypothesis of the uniqueness Theorem in the Browder-Novikov Theory. (1) and (7).

The homotopy theoretic proof combines a general construction with PL-surgery on maps (1).

In the geometric situation the coefficient groups P_i appear as the cobordism groups of framed i-manifolds (with boundary PL-homeomorphic to the (i-1)-sphere) in Euclidean space. See (3).

HOMOTOPY INTERPRETATION

In the second point of view the P_i appear as the homotopy groups of a universal H-space F/PL. F/PL is the fibre of the homomorphism

$$B_{PL} \xrightarrow{J} B_F$$

which maps equivalence classes of PL-bundles into fibre homotopy equivalence classes of spherical fibre spaces (8).

To state Theorem 1 in this framework we make two definitions.

Definition 1: A PL-structure on M is a pair (L,g) where $g:(L,\partial L)\to(M,\partial M)$ is a homotopy equivalence.

Definition 2: Two PL-structures on M, (L,g) and (L',g'), are equivalent (or concordant) if there is a PL-homeomorphism $c:L\to L'$ so that $g'c$ is homotopic to g (as maps of pairs). Let PL(M) denote the set of equivalence classes.

Remark: The set PL(M) depends only on the homotopy type of the pair $(M,\partial M)$. It may be viewed as the set of homotopy equivalence classes of manifold structures on an underlying CW complex pair for $(M,\partial M)$.

Remark: Note that (L,g) is equivalent to (M, identity) iff g is homotopic to a PL-homeomorphism.

Let $M_0=M$ if $\partial M\neq 0$ and $M_0=M-pt$ if $\partial M=\emptyset$. Assume $\pi_1 M=\pi_1\partial M=0$ and dim $M\geq 6$.

THEOREM 3: There is a bijective correspondence

$$PL(M)\xrightarrow{\zeta}[M_0,F/PL]$$

between PL(M) and the set of homotopy classes of maps, $M_0\to F/PL$. ζ corresponds the equivalence class of (M, id) to the class of the point map in $[M_0,F/PL]$.

Remark: If $g;(L,\partial L)\to(M,\partial M)$ is a homotopy equivalence, denote $\zeta(L,g)$ by $\zeta g;M_0\to F/PL$. Since ζ is injective $\zeta g\simeq$ pt. map iff g is homotopic to a PL-homeomorphism.

<u>Corollary</u>: Every homotopy equivalence $g:(L,\partial L)\to(M,\partial M)$ is homotopic to PL-homeomorphism iff $[M_0, F/PL]$ contains only one element.

<u>PROPERTIES OF</u> ζ.

I. We give further properties of the correspondence ζ. It is possible to define a group operation in PL(M) geometrically and to "restrict" PL-structures on M to PL-structures on M', where dim M= dim M' and M is embedded in the interior of M with simply connected complement. Thus the assignment

$$M \longrightarrow PL(M)$$

extends to a contravariant functor on a category of simply connected n-manifolds and "nice" embeddings to the category of Abelian groups. <u>The correspondence</u>

$$PL(M) \xrightarrow{\zeta} [M_0, F/PL]$$

<u>is a natural equivalence of functors on this category</u>.

The group structure on PL(M) and the restriction homomorphism

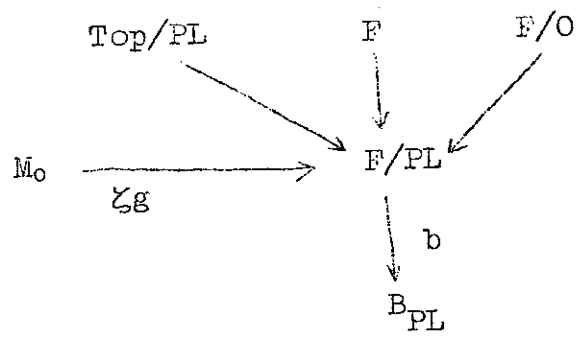
$$PL(M) \longrightarrow PL(M')$$

are described in (10).

The obstruction theory of Theorem 1 follows from the naturality of ζ and the computation

$$\pi_i(F/PL) = P_i \quad .$$

II. Consider the diagram



$\zeta(L,g)=\zeta g$ has the following properties:

1) If g/L_o is a PL-tangential equivalence, then ζg lifts to F. (In general $b \cdot \zeta g$ measures the precise deviation of g from a tangential equivalence.)

2) If M and L are smooth, then ζg lifts to F/O. A certain (canonically defined) lifting is homotopic to zero iff g is homotopic to a diffeomorphism between $L\#\theta$ and M where θ is in $\theta_n(\partial\pi)$. Compare (10).

3) If g is a homeomorphism, then ζg lifts to Top/PL. (In fact, this is true if g is topologically h-cobordant to a homeomorphism.)

From these properties of ζ we obtain

THEOREM 4: Let $\pi_1 M = \pi_1 \partial M = 0$ and suppose $\dim M \geq 6$. Recall $M_0 = M$ if $\partial M \neq 0$ and $M_0 = M - pt$ if $\partial M = 0$. Then

a) Every tangential equivalence $f:(L,\partial L) \longrightarrow (M, \partial M)$ is homotopic to a PL-homeomorphism iff

$$[M_0, F] \longrightarrow [M_0, F/PL] \quad \text{is zero.}$$

b) Every homotopy equivalence $f:(L,\partial L) \longrightarrow (M,\partial M)$ with M and L smooth is homotopic to a PL-homeomorphism if

$$[M_0, F/O] \longrightarrow [M_0, F/PL] \quad \text{is zero.}$$

The converse holds if $\partial M \neq 0$ and M is smoothable.

c) Every homeomorphism $f:(L,\partial L) \longrightarrow (M,\partial M)$ is homotopic to a PL-homeomorphism if

$$[M_0, Top/PL] \longrightarrow [M_0, F/PL] \quad \text{is zero.}$$

Theorem 3 can be used to construct examples of various kinds by constructing appropriate maps into F/PL.

<u>Example</u>: 1) Using the composition

$$CP_0^4 = CP^4 - pt \cong CP^3 \xrightarrow[\deg 1]{} S^6 \xrightarrow[\text{gen } \pi_6]{} F/PL$$

we construct a smooth 8-manifold which is tangentially equivalent to CP^4 but which is not (topologically) homeomorphic to CP^4. Compare (4).

2) Using

$$CP^2 \times S^8 \xrightarrow[p_2]{} S^8 \xrightarrow[\text{gen } \pi_8]{} F/PL$$

we construct a PL 12-manifold M^{12} which is homotopically equivalent to $CP^2 \times S^8$ but which is not cobordant (mod 2) to $CP^2 \times S^8$.

Remark: Theorem 3 may be stated in a relative form that is useful for studying weak-isotopy classes of PL-homeomorphisms $c: M \to M$. The problem of deforming a homotopy between two PL-homeomorphisms into a weak-isotopy between them is classified by a map of the suspension of M_0 into F/PL. Compare (9).

There is an analogous theory for constructing diffeomorphisms from homotopy equivalences. Compare (10). The classifying space is F/O. These two theories and the smoothing theory of (5) and (2) are compatibly related by the fibration

$$PL/O \longrightarrow F/O \longrightarrow F/PL \quad .$$

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