Triangulating Homotopy Equivalences

By

D. Sullivan

A DISSERTATION

Presented to the

Faculty of Princeton University

in Candidacy for the Degree

of Doctor of Philosophy

Recommended for Acceptance by the

Department of

Mathematics

December 1965

Abstract

In this thesis we consider the problem of deforming a homotopy equivalence

$$M \xrightarrow{g} L$$

between compact PL-manifolds M and L into a PL-homeomorphism. If M is simply connected and $\dim M \geq 6$, an obstruction theory is developed for such a deformation from two points of view - one geometric, the other homotopy theoretical.

The obstructions lie in

$$H^i(M;P_i) \quad ,$$

where

$$P_i = \begin{cases} 0 & i \text{ odd} \\ Z_2 & i \equiv 2 (\text{mod } 4) \\ Z & i \equiv 0 (\text{mod } 4) \end{cases} .$$

P_i arises in the geometric point of view as a cobordism theory and in the second point of view as the homotopy groups of a universal space F/PL.

Introduction

Consider a homotopy equivalence

$$f:(L,bL)\longrightarrow(M,bM)$$

between two PL n-manifolds. Is f homotopic (as a map of pairs) to a PL-homeomorphism?

Suppose $M=M'xI$ and $L=L'xI$ where L' and M' are closed and f is a homotopy between two PL-homeomorphisms c_0 and c_1,

$$c_i:L'xi\longrightarrow M'xi \qquad i=0,1 .$$

Can $f/Lx(0,1)$ be deformed to a pseudo-isotopy between c_0 and c_1?

The purpose of this thesis is to develop a framework for answering such questions.

The starting point is the observation that these problems are equivalent to a certain "cobordism" problem. We illustrate this for the problem of deforming a homotopy equivalence

$$f:L\longrightarrow M , \qquad bM=0$$

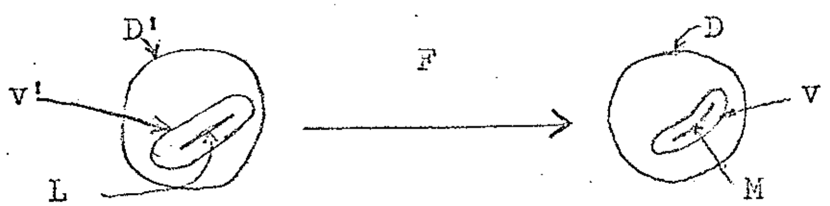
into a PL-homeomorphism .

Let v be the PL-normal disk bundle of $M\subset \operatorname{int} D$ ($D=D^{n+k}$, k large) and form $v'=f^*v$. Let $b(f)$ be a bundle map $v'\xrightarrow{b(f)}v$ covering f.

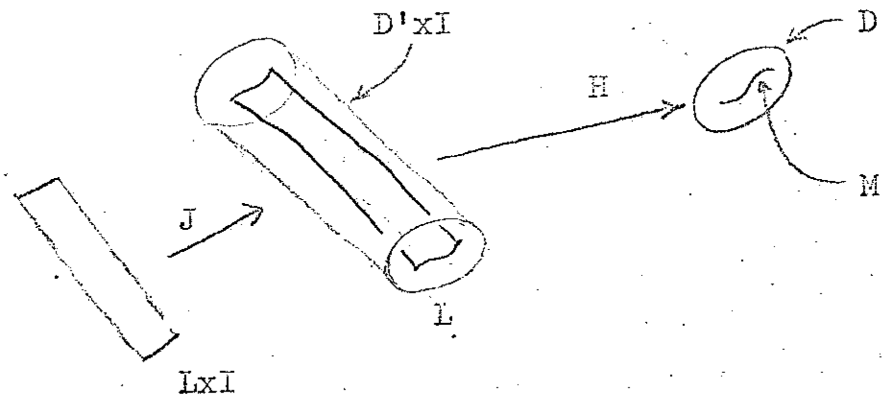
Suppose <u>we can build an $(n+k)$-disk D' containing v'</u> <u>in its interior</u> and <u>extend $b(f)$ to a homotopy equivalence</u>

$$(D', bD') \xrightarrow{F} (D, bD)$$

which carries $D'-v'$ into $D-v$. Then if $\dim M \geq 6$ and $\pi_1(M) = 0$ it follows that f is homotopic to a PL-homeomorphism (Theorem 1.16).



The basic idea here is to choose a homotopy H between F and a PL-homeomorphism C. Then change $H/D'x(0,1)$ so that H is t-regular to $M \subset \operatorname{int} D$ and $H^{-1}(M)$ is an h-cobordism between $H^{-1}(M) \cap D'x0 = L$ and $H^{-1}(M) \cap D'x1 = C^{-1}(M)$. Then we can straighten out (J) the h-cobordism to obtain the desired homotopy



In order to carry out these operations we need to work with manifolds with non-empty boundary.

This is essentially the uniqueness theorem in the PL Browder-Novidov theory.

Note that $b(f)$ defines a homotopy equivalence

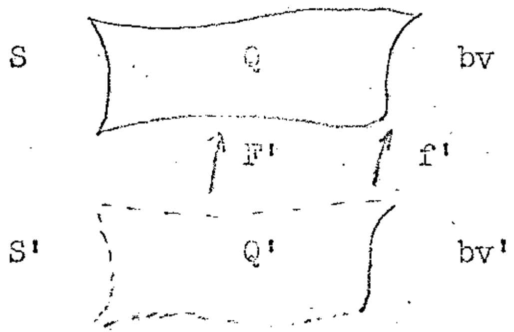
$$f':bv' \longrightarrow bv$$

and that $Q = D\text{-int } v$ defines a cobordism between bv and $S = S^{n+k-1}$.

It is easy to see then that the problem of constructing D' and F is equivalent to the problem of constructing a cobordism Q' between bv' and ~~the~~ some $(n+k-1)$- sphere S' and extending f' to a homotopy equivalence

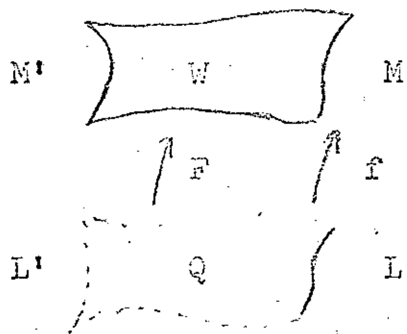
$$F':(Q',bv',S') \longrightarrow (Q,bv,S)\ .$$

(Q',F') is called a <u>cobordism construction for</u> $(Q;bv;f')$.



This indicates how the problem of deforming f into a PL-homeomorphism is reduced to the problem of making a certain <u>cobordism construction</u>.

Thus in section 1 of Part I we consider the general problem of making <u>cobordism constructions</u>,



for some triple $(W,M;f)$. We treat the case where (W,M) looks like the complement of the normal disk bundle of some (M,bM) in (D,bD). (Definition 1.4)

Under these conditions we derive an obstruction theory for the problem of making a cobordism construction for $(W,M;f)$. The obstructions lie in

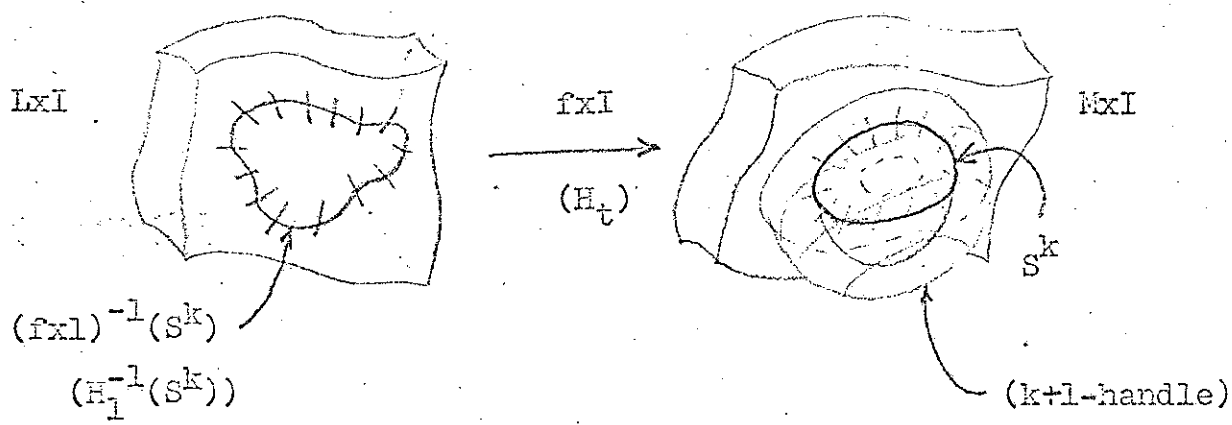
$$H^{i+1}(W,M;P_i)$$

where

$$P_i = \begin{cases} 0 & i \text{ is odd} \\ Z_2 & i \equiv 2 \pmod 4 \\ Z & i \equiv 0 \ (4) \end{cases} .$$

(See Theorem 1.12)

Geometrically, P_k is the set of framed cobordism classes of framed k-manifolds (M,bM) in $(D,bD)(D=D^{n+k}, n \text{ large})$ where $bM=S^{k-1}$(P_k becomes a group by forming connected sums along the boundary.).



We can illustrate how P_k enters into the theory by considering the case when $W = M \times I \cup (k+1\text{-handle})$. Suppose $S^k \subset M \times 1$ is the core sphere and $f \times 1$ is t-regular to S^k. Then $P = (f \times 1)^{-1}(S^k)$ is a framed submanifold of $L \times 1$. In our case P has high codimension and can be constructed so that $P\text{-int } D^k$ is a contractible submanifold of $L \times 1$. Thus P determines an element in P_k. This element is the precise obstruction to changing f by a homotopy H_t so that H_1 is a PL-homeomorphism on a neighborhood of $H_1^{-1}(S^k)$. (Lemma 1.5)

If H_1 has this property we can attach a $(k+1)$-handle to $(L \times I)$ (using $H_1^{-1}/(\text{nghd of } S^k)$) to form the desired cobordism of L. Then f extends to a homotopy equivalence of cobordisms using H_t on $L \times I$ and the identity on the handle.

This indicates how a cochain with values in P_k may be defined to measure the possibility extending a given partial construction over the $(k+1)$-handles of (W,M). (Theorem 1.7)

The fact that an obstruction theory arises from this procedure is due to the fact that there are (card P_{k+1}) different possible extensions of H_t over the $(k+1)$-handle on $L \times I$. (Theorem 1.18)

The main purpose of section 2 of Part I is to prove the following

Theorem: Let $f:(L,bL) \to (M,bM)$ be a homotopy equivalence. Suppose $\dim M \geqslant 6$, $\pi_1(M) = 0$, $\pi_1(bM) = 0$. Then f is homotopic to a PL-homeomorphism iffi a sequence of obstructions in

$$H^i(M;P_i)$$

vanish, $i < \dim M$.

This theorem is proved by relating cobordism constructions to the "normal invariants" of Browder and Novikov (1). This relationship (essentially described above) has other applications to problems where "normal invariants" enter.

It follows immediately from the existence of the obstruction theory and Theorem 1.14 of section 1 that <u>f is homotopic to a PL-homeomorphism</u> if

i) $H^{4i+2}(M,Z_2) = 0 \qquad 4i+2 < n$

ii) $H^{4i}(M,Z)$ is free

iii) f is a correspondence of rational Pontryagin classes. $(f^*p_i(M) = p_i(L))$.

Note that this theorem applies when f is a homeomorphism because of Novikov's result (20).

There are other theorems where more is known about f and many examples showing why conditions i) and ii) are "necessary". Some of these are reserved for Part II. Others are omitted.

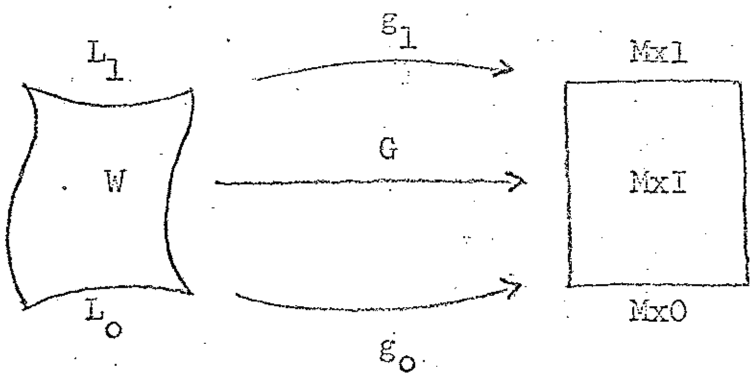
The discussion of the pseudo-isotopy problem is also reserved for Part II.

The existence of the above obstruction theory suggests that the problem of deforming a homotopy equivalence $f: L \to M$ into a PL-homeomorphism should be "classified" by a map

C_f of M into some universal space X. We should have $\pi_1(X) = P_i$ and $C_f \simeq$ pt map iffi f is homotopic to a PL-homeomorphism.

This suggestion is correct, and Part II is devoted to developing this point of view.

A PL-structure on M is a pair (L,g) consisting of a PL n-manifold L and a homotopy equivalence $g:(L,bL) \longrightarrow (M,bM)$. Two PL-structure (L_0,g_0) and (L_1,g_1) are concordant if there exists an h-cobordism W between L_0 and L_1 and a homotopy equivalence $G:W \longrightarrow M\times I$ which extends $g_0 \cup g_1$, and preserves appropriate subspaces.



(see Definition 5).

Let $\mathcal{M}_n$ denote the category whose objects are

$$\begin{cases} \text{PL } n\text{-manifolds } (M,bM) \text{ such that} \\ \pi_1(M) = \pi_1(bM) = 0, \quad bM \neq 0 & : \quad n \geq 6 \\ \{\emptyset\} & \quad\;\; n < 6 \end{cases}$$

whose morphism are

embeddings $M_1 \subset \operatorname{int} M_2$ such that $\pi_1(M_2 - M_1) = 0$.

Let $\overline{\mathcal{M}}_n$ denote the category obtained by dropping the condition $bM \neq 0$.

In chapter 1 we show that the correspondence

$$M \longrightarrow PL(M)$$

defines a contravariant functor

$$\overline{\mathcal{M}}_n \xrightarrow{PL} S$$

from $\overline{\mathcal{M}}_n$ to the category of pointed sets and base point preserving functions. The key techinical lemma here is provided by Browder (2). This enables one to change a PL-structure

$$L \xrightarrow{g} M_2$$

on M_2 by a concordance so that it "induces" a PL-structure on

$$M_1 \overset{i}{\subset} \operatorname{int} M_2 .$$

Thus we can define an induced map

$$PL(M_2) \xrightarrow{i^*} PL(M_1) \quad .$$

We would now like to "classify" elements in PL(M) by defining some natural transformation

$$PL \xrightarrow{C} [\ \ ,X]$$

where X is our universal space. This is hard to do on the level of representatives so we have to employ an intermediate functor B_k.

B_k is defined on $\mathcal{C}$ the category of countable, connected, locally finite, simplicial complexes and PL-maps.

To each K in $\mathcal{C}$ we assign the set of equivalence classes of diagrams

$$\begin{array}{ccc} E & \xrightarrow{t} & D^k \\ {\scriptstyle p}\downarrow & & \\ K & & \end{array}$$

where $E \xrightarrow{p} K$ is a PL-k-disk bundle and $E \xrightarrow{t} D^k$ is a fibre homotopy (F-) trivialization. Two diagrams are equivalent if there is a diagram over KxI which restricts to the appropriate diagram on KxbI. Such a diagram is called an $F/PL)_k$-bundle.

The induced bundle construction makes B_k into a contravariant functor on $\mathcal{C}$.

In chapter 2 we define a natural transformation

$$PL \xrightarrow{C} B_k$$

of functors on $\mathcal{M}_n$. (C is actually defined for any M.)

Let a in PL(M) be represented by $(L,bL) \xrightarrow{g} (M,bM)$. Let g' be a homotopy inverse for g and approximate g' by an embedding.

$$(M,bM) \xrightarrow{i} (L,bL)\text{x int } D^k, \text{ k large } .$$

Choose i and a closed tubular neighborhood

$$E \xrightarrow{j} L\text{x int } D^k$$

of $i(M)$ so that

$$j\,(\text{open disk bundle of } E) \supset L\times 0.$$

(Lemma 3). Then choose $r: D^k \to D^k$ so that

$$E \xrightarrow{j} L\times D^k \xrightarrow{p_2} D^k \xrightarrow{r} D^k$$

defines an F-trivialization of E(Lemma 4). The <u>pair</u> <u>(E, rp_2j) is called a classifying bundle for the PL-structure</u> (L,g) <u>and the correspondence</u>

$$(\text{PL-structure}) \longrightarrow (\text{classifying bundle})$$

<u>defines a natural transformation</u>

$$PL \xrightarrow{C} B_k\,,\quad k \text{ large}$$

of functors on $\mathcal{M}_n$(Theorem 6).

If M is embedded in bW and $a = (L,g)$ is a PL-structure on M we say that a <u>extends to a PL-structure on</u> W if there is a PL-structure (Q,G) on W so that

i) L is embedded in bQ and $G/L = g$

ii) $(Q,G)/L'$ defines a PL-structure on $M' = bW\text{-int } M$ $(L' = bQ\text{-int } L)$)

Most extension problems for PL-structures reduce to one of this type. (e.g. the cobordism construction problem of part I)

<u>Two extensions of</u> a <u>are said to be equivalent (rel M) if the product concordance of a extends to a concordance between them. The set of</u>

equivalence classes is denoted by $PL(W,M;a)$.

Two $F/PL)_k$-bundles on W which restrict to the same $F/PL)_k$-bundle v on M are said to be equivalent (rel M) if the product equivalence on M extends to an equivalence between them. The set of equivalence classes is denoted by $B_k(W,M;v)$.

Note that there are natural projections
$PL(W,M;a) \xrightarrow{p} PL(M)$ and $B_k(W,M;v) \xrightarrow{p} B_k(W)$.

Suppose that $bW\text{-int } M \neq 0$, $\pi_1(bW\text{-int } M) = \pi_1(W) = 0$, and $\dim W \geqslant 6$. Chapter 3 is denoted to the proof of the

Theorem 9: Let (W,M) be as above. Let $a = (L,g)$ be a PL-structure on M with k-dimensional classifying bundle $v = (E,p,t)$, k large. Then there exists a one-to-one correspondence

$$PL(W,M;a) \xrightarrow{C_a} B_k(W,M;v)$$

so that $pC_a = Cp$ where

$$PL(W) \xrightarrow{C} B_k(W)$$

is defined above.

Corollary: If $k \gg n$, $PL \xrightarrow{C} B_k$ is a natural equivalence of contravariant functors on $\mathcal{M}_n$. (not $\bar{\mathcal{M}}_n$)

Theorem 9 is stated in a relative form so that the pseudo-isotopy problem can be treated more completely.

If M is closed let $M_o = M\text{-int } D^n$. (If $bM \neq 0$, let $M_o = M$.)

Then $M_o \subset M$ defines a map

$$PL(M) \xrightarrow{r} PL(M_o) \quad ,$$

which is an isomorphism (Lemma 21). Thus

$$\bar{\mathcal{M}}_n \xrightarrow{PL} S$$

is determined by $PL/\mathcal{M}_n$.

In chapter 4 we construct (using Brown's theory(4)) a space F/PL which classifies stable equivalence classes of $F/PL)_k$-bundles over finite simplicial complexes. Let (,F/PL) denote the functor which assigns to finite X the set of free homotopy classes of maps $X \to F/PL$. Then Theorem 9 implies that there is a natural equivalence

$$PL \xrightarrow{C} (\ \ ,F/PL)$$

of contravariant funtors on $\mathcal{M}_n$ (theorem 25).

In fact, it follows from results in chapter 4 that $PL(W,M;a)$ is in one-to-one correspondence with

$$(W \cup (\text{cone } M),\ F/PL) \quad .$$

(Theorem 12, 14).

Thus the problem of deforming the homotopy equivalence

$$(L,bL) \xrightarrow{f} (M,bM)$$

to a PL-homeomorphism in classified by a map

$$M_o \xrightarrow{C_f} F/PL \quad . \quad \text{(Theorem 26)}$$

The problem of deforming a homotopy

$$L \times I \xrightarrow{f} M \times I$$

between two PL-homeomorphisms to a pseudo-isotopy is classified by a map

$$(\text{susp. } M_o) \xrightarrow{C_f} F/PL \qquad \text{(Chapter 6)}.$$

We can use the fact that

$$PL \xrightarrow{C} (\quad , F/PL)$$

is natural to develope an obstruction theory for deforming

$$(L, bL) \xrightarrow{f} (M, bM)$$

to a PL-homeomorphism "over the skeletons of M" (Definition 22). One skeletal version of this obstruction theory is decribed in chapter 6. (Compare section I, Part I.)

The obstructions lie in

$$H^i(M_o, \pi_i(F/PL)) \quad ,$$

and are just the obstructions to deforming C_f to the point map. We prove in chapter 4 that $\pi_i(F/PL) = P_i$.

Now there is a map

$$F/PL \xrightarrow{b} B_{PL}$$

which is essentially the fibre of $B_{PL} \xrightarrow{J} B_F$.

(Theorem 14) It follows from the construction of C_f that the composition

$$M_o \xrightarrow{C_f} F/PL \xrightarrow{b} B_{PL}$$

classifies the normal bundle of $(M_o, bM_o) \xrightarrow{i_o} (L_o, bL_o) \times D^k$, where i_o is an embedding so that

$$M_o \xrightarrow{i_o} L_o x D^k \xrightarrow{(f/L_o)\ p_1} M_o$$

is homotopic to the identity (as a map of pairs). Thus by making tangential hypothesis on f we can prove triviality theorems for the obstructions. See the corollaries to Theorem 26.

Using facts about $(X, F/PL)$ we can give an upper bound on the number of PL-homeomorphism classes of manifolds within a given tangential equivalence class. For example for CP^4 this upper bound is 4. In the examples in chapter 6 the number is shown to be at least 2.

This and other estimates are described in the Corollaries to Theorem 26.

A provocative fact about the obstructions is described in Theorem 27 of chapter 6. This Theorem relates the <u>homomorphisms determined by the obstructions to a sequence of Bundle Problems</u>.

A <u>Bundle Problem</u> is the problem of showing that a PL-structure on a k-disk bundle E over M "comes from" a PL-structure on M; i.e. $W \xrightarrow{G} E$ is concordant to $g^*E \xrightarrow{b(g)} E$ for some PL-structure $L \xrightarrow{g} M$.

The Bundle Problem per se is discussed in chapter 5. The situation is very nice when M is in $\mathcal{M}_n$ or when M in in $\overline{\mathcal{M}}_n$, $k \geqslant 3$, and n is odd. (Theorem 23, Corollaries 1 and 2). The case M in $\overline{\mathcal{M}}_n$, $k \geqslant 3$, and n even is discussed in Theorem 24. Here it is shown that (W,G) in $PL(E)$ "comes from" $PL(M)$ iffi the "index" or "Kervaire Invariant" of a certain $F/PL)_k$-bundle vanishes. A "formula" for the "Kervaire Invariant" of an $F/PL)_k$-bundle is also described.

The formula for the "Kervaire Invariant" of an $F/PL)_k$-bundle over M arises from a homomorphism

$$\Omega_n^{PL}(F/PL) \xrightarrow{K} P_n$$

which is defined geometrically. (Theorems 16,17.)

This homomorphism is used to study homotopy properties of F/PL. The <u>main point is Theorem 18 which asserts that</u>

$$\pi_n(F/PL) \longrightarrow \Omega_n^{PL}(F/PL) \xrightarrow{K} P_n$$

<u>is an isomorphism if</u> $n \neq 4$. As a corollary we show for for example that <u>the k-invariants of</u> F/PL (<u>reduced mod 2</u>) <u>are zero</u>. This fact enables us to compute the Z_2-cohomology algebra of F/PL. (See Corollaries to Theorem 20.)

We remark that there is an analogous smooth theory which attacks the problem of deforming $f:(L,bL) \longrightarrow (M,bM)$ into a diffeomorphism. If $bM \neq 0$, f <u>may be deformed into a diffeomorphism iffi a sequence of obstructions in</u>

$$H^i(M, \pi_i(F/O))$$

vanish.

These two theories and that of Hirsch (8) are related by the fibration

$$PL/O \to F/O \to F/PL \quad .$$

This smooth theory is omitted for reasons of space.

At this point I would like to acknowledge several debts. I have had numerous useful conversations with Professor Milnor and Professor Steenrod.

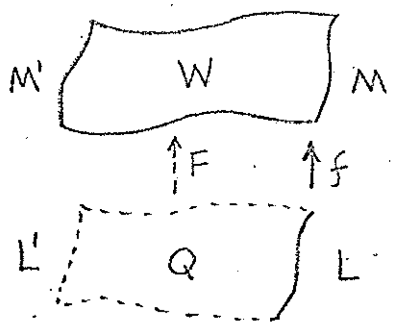
Several graduate students have offered much encouragement. I would especially thank George Cooke for asking a particularly appropriate question at the very beginning of this work.

Finally, I express my warmest thanks to my thesis advisor Professor Browder for patiently explaining his recent research, for suggesting the viewpoint of part II and for providing a friendly, informal and inspiring atmosphere for this research.

PART I

The PL Obstruction Theory

Let $f: L \longrightarrow M$ be a homotopy equivalence of PL n - manifolds. Let $(W; M, M')$ be a cobordism between M and another PL n - manifold M':



We wish to consider the problem of constructing a cobordism $(Q; L, L')$ such that f extends to

$$F: (Q; L, L') \longrightarrow (W; M, M'),$$

a homotopy equivalence of triples.

If certain hypotheses are satisfied by the pair (W, M), we will show that an obstruction theory for this problem exists. The obstructions lie in

$$H^{k+1}(W,M ; P_k),$$

where

$$P_k = \begin{cases} 0 & k \text{ odd} \\ Z_2 & k \equiv 2 \pmod 4 \\ Z & k \equiv 0 \pmod 4) . \end{cases}$$

Section I

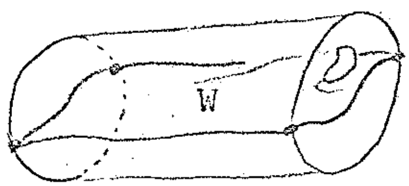
First we define P_k geometrically. Let (M^k, bM^k) be an oriented smooth parallelizable k-manifold embedded in (D^{k+r}, bD^{k+r}) with boundary in the boundary and interior in the interior. Suppose that

1) r is greater than k
2) bM^k is PL - homeomorphic to the standard PL $(k-1)$ sphere
3) the normal bundle of M has a given product structure F.

The pair (M,F) will be called a <u>framed almost-closed</u> (f.a.c.) <u>k - manifold</u> in D^{k+r}.

<u>Definition 1.1</u> Two framed almost - closed k-manifolds (M_o, F_o) and (M_1, F_1) are <u>framed cobordant</u> if there is a framed $(k+1)$-manifold (W,G) in $D^{k+r} \times I$ such that

1) $(W,G) \cap (D^{k+r} \times i) = (M_i, F_i)$, $i = 0,1$.
2) $W \cap (bD^{k+r} \times I)$ is an h-cobordism between bM_o and bM_1.

M_o  M_1

This defines an equivalence relation on the set of framed almost-closed k-manifolds.

Definition 1.2 Let P_k denote the set of framed cobordism classes of framed almost-closed k-manifolds in D^{k+r}, $r>k$.

Two f.a.c. k-manifolds (M_1,F_1) in D_1^{k+r} and (M_2,F_2) in D_2^{k+r} may be added by identifying a framed (k-1)-cell in bM_1 to a framed (k-1)-cell bM_2. Thus D_1^{k+r} is attached to D_2^{k+r} along (k+r-1)-cells in their boundaries. If orientations are respected, an oriented f.a.c. k-manifold in $D_3^{k+r} = D_1^{k+r} \cup D_2^{k+r}$ results. This operation, called "connected sum along the boundary," induces an Abelian group structure on P_k.

Lemma 1.1 With the operation "connected sum along the boundary,"

$$P_k \text{ is isomorphic to } \begin{cases} 0 & \text{if } k \text{ is odd} \\ Z_2 & \text{if } k \equiv 2 \pmod 4 \\ Z & \text{if } k \equiv 0 \pmod 4 \end{cases}$$

<u>Lemma 1.2</u> P_k is generated by

a) $(S^{k/2} \times S^{k/2} - \text{int } D^k)$ with a <u>certain</u> framing if $k=2,6,\text{or }14$.

b) the Kervaire manifold obtained by "plumbing together" two copies of the tangent disk bundle of $S^{k/2}$ with any framing if $k \equiv 2 (\text{mod } 4)$ $\neq 2,6,\text{or } 14$.

c) a certain $(k/2 - 1)$-connected manifold with index a_k and any framing if $k \equiv 0 (\text{mod } 4)$

$(a_4 = 16,\ a_{4r} = 8$ for r greater than 1.)

d) any k-disk and any framing if k is odd.

<u>Lemma 1.3</u> If (M^k, F) represents the zero element in P_k, then there is a framed cobordism (W,G) between (D^k, F_1) and (M^k, F) where W is the trace of surgeries on M of dimension $\leq k/2$, $k \neq 3,4$ $(\leq k-1, k=3,4)$.

These lemmas are proved in (11) except for $k = 3$ and $k = 4$. For these cases see the appendix to Section I.

Now we will discuss transverse regularity in the PL-category. The theory in this case is analogous to the smooth theory if the pertinent submanifolds have PL normal microbundles (34).

Definition 1.3 Let $f: M \to M'$ be a map of PL manifolds. Let N' be a submanifold of M' with a normal microbundle. Then f is t-regular on M if $N = f^{-1}(N')$ is a submanifold of M with a normal microbundle, and so that f restricted to a neighborhood of N is a bundle map of normal microbundles.

Theorem 1.4 (Mazur, Williamson) Let $f: M \to M'$ be a map of compact PL-manifolds. Suppose N' is a closed submanifold of M' with a normal microbundle. Then f can be approximated in the C^0 topology by a mapping g t-regular on N'. Furthermore $N = g^{-1}(N')$ and N' have PL normal bundles, and g restricted to a suitable neighborhood of N is a map of PL normal bundles.

Proof: The approximation part is proved by Williamson in (34). By Mazur the normal microbundle of N' contains a unique PL bundle E'. Then $E = g^*E'$ is a PL normal bundle for N. The last statement follows from the t-regularity of g and the construction of g^*E'.

There is a relative version of Theorem 1.4 (to the effect that f need not be changed on closed subsets where it is already t-regular) and a form for submanifolds with boundary.

Now we relate t-regularity and the coefficient groups P_k. The lemma will be used to define the obstructions in the cobordism problem.

Lemma 1.5 Let $f\colon (M, bM) \longrightarrow (M', bM')$ be a map of oriented PL n-manifolds with degree 1. Suppose that M and M' are $(k-1)$ connected, that $2k+1 < n$, and $k \geqslant 1$. Then f determines a homomorphism

$$c_f\colon \pi_k(M') \longrightarrow P_k$$

with the following two properties.

1) If f is homotopic to g (as a map of pairs), then $c_f = c_g$.

2) Let $S^k_1, \ldots, S^k_r$ be disjoint k-spheres in M' representing elements $(S^k_1), \ldots, (S^k_r)$ in $\pi_k(M')$. Then f is homotopic to g such that g is t-regular to each S^k_i and so that each $g^{-1}(S^k_i)$ is PL-homeomorphic to S^k iff $c_f(S^k_1) = \ldots = c_f(S^k_r) = 0$.

Proof: c_f is defined as follows: let x be in $\pi_k(M')$. Embed S^k in M' to represent x. Make f t-regular to S^k and consider $f^{-1}(S^k)$-int D^k.

We can assume that this is a framed submanifold of M which is contractible in M. We may assume also by general position that it is contained in an n-cell of M. Thus $f^{-1}(S^k)$ determines an element y in P_k. Let

$$c_f(x) = y .$$

y is the precise obstruction to changing f by a homotopy so that $f^{-1}(S^k)$ becomes a PL k-sphere.

In the appendix to section I we will give a detailed proof that c_f is a well defined homomorphism from $\pi_k(M')$ to P_k and satisfies 1) and 2).

Now we describe the cobordism problem more precisely and develope the accompanying obstruction theory.

Definition 1.4 Let (M^n, bM^n) be a (smooth) PL-manifold pair. A (smooth) admissable cobordism of (M, bM) is a (smooth) PL (n+1)-manifold pair (W, bW) satisfying:

1) $n \geq 5$

2) $\pi_1(M) = \pi_1(W) = 0$

3) $bW = M \cup M'$

$bM = M \cap M' = bM'$

4) there is an integer $s \leq n$ so that

$$H_i(W,M) \xrightarrow{b} H_{i-1}(M)$$

is an isomorphism for $2i+1 < s$ and so that $H_i(W,M)=0$ for $2i+1 \geq s$.

Let $j: \bigcup_{i=1}^{r} S_i^k \times D_i^{n-k} \longrightarrow (\text{int } M) \times 1$ be a (*smooth*) embedding. Let $A = j\,(\bigcup_{i=1}^{r} S_i^k \times D_i^{n-k})$. A (*smooth*) *(k+1)-elementary cobordism of (M, bM)* is a (smooth) admissable cobordism of (M, bM) where W is (diffeomorphic) PL-homeomorphic to

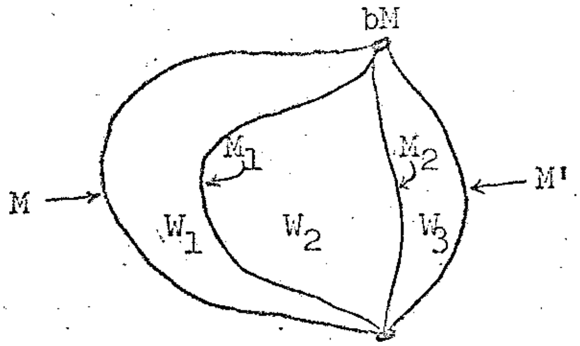
$$M \times I \cup_A (\bigcup_{i=1}^{r} D_i^{k+1} \times D_i^{n-k}) .$$

An admissable cobordism of (M, bM) by W will be denoted by $(W; M, M')$.

Note that admissable cobordisms, $(W_1; M_o, M_1)$ and $(W_2; M_1, M_2)$, may be "composed" to obtain a third admissable cobordism, $(W_1 + W_2; M_o, M_2)$ by setting

$$W_1 + W_2 = W_1 \cup_{M_1} W_2 .$$

The following decomposition lemma discusses smooth admissable cobordisms and implies inparticular that they are closed under composition.



$(W_1 + W_2 + W_3; M, M')$

<u>Lemma 1.6</u> Let $(W; M,M')$ be a smooth admissable cobordism of (M,bM). Let r be such that $H_i(W,M)=0$ for $i>r$. Then properties 1), 2), and 3) of <u>Definition 1.5</u> imply there exist smooth k-elementary cobordisms

$$(W_k; M_{k+1}, M_k) \qquad k=1,2,\ldots, r$$

with $M=M_o$ and $M'=M_r$ so that

$$(W; M,M')=(W_1+W_2+\ldots+W_r; M_o,M_r).$$

The decomposition satisfies

a) the inclusion i_o: $M \longrightarrow W_1+\ldots+W_k$ induces an isomorphism between $H_i(M)$ and $H_i(W_1+\ldots+W_k)$ for $i>k$,

b) the inclusion i_k: $M_k \longrightarrow W_1+\ldots+W_k$ induces an isomorphism between $H_i(M_k)$ and $H_i(W_1+\ldots+W_k)$ for $i<n-k$,

c) property 4) implies M_k is $(k-1)$ connected when r is minimal.

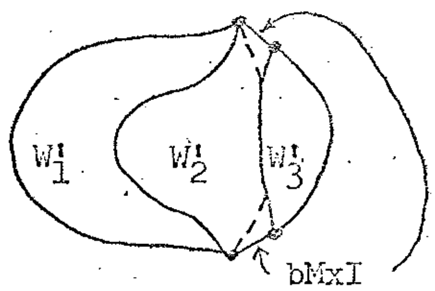
<u>Proof:</u> Write $bW = M'' \cup bM \times I \cup M$ by choosing a collar neighborhood of bM in M' and setting $M'' = M' - (bM \times [0,1))$. Define $f\colon bW \longrightarrow R$ by

$$f(x)=\begin{cases} \frac{1}{2} & x \text{ in } M \\ ry+\frac{1}{2} & x=(m,y) \text{ in } bM \times I \\ r+\frac{1}{2} & x \text{ in } M''. \end{cases}$$

Extend f to a nice Morse function g (i.e. g has finitely many critical points in int W and g of (critical point of index i) equals i) with critical points of index $\leq r$ and set

$$W_k' = g^{-1}(k-\tfrac{1}{2},\ k+\tfrac{1}{2}) \quad k=1,\ldots, r .$$

Such a g exists by (21) .



Now the W_k' may be modified slightly in a neighborhood of $bM \times I$ (using a local product structure) to obtain the desired decomposition

$$(W;\ M,M') = (W_1+\ldots+W_r;\ M,M') .$$

Now a) and b) follow from the fact that $W_1+\ldots+W_k$ has the homotopy type of M with (1 thru k)-cells attached or M_k with (n+1-k thru n)- cells attached.

To prove c) note that the inclusion

$$j:\ W_1+\ldots+W_k \longrightarrow W$$

is a homotopy equivalence of the respective k-skeletons. However, 2) and 4) imply that W is $(k-1)$ - connected for each k such that $2k+1 < s$. Thus $W_1+\ldots+W_k$ is

$(k-1)$- connected if $(2k+1) \leq s$. Therefore, by the proof of b) M_k is $(k-1)$-connected if $(2k+1) < s$ and $k < n-k$. If we take r such that $2r+1 < s$ then $k \leq r$ implies $2k+1 < s$ and $k < n-k$. This completes the proof of c) and the lemma.

Lemma 1.6 shows that an admissable cobordism of essentially the trace of surgeries on the interior of M which kill the low dimensional homology of M.

The Lemma could have been stated in a much sharper form. The pairs

$$(W_1+\ldots+W_k, M) \qquad k = 1,\ldots,r$$

filter the pair (W,M). This filtration of (W,M) gives us a chain complex whose homology is $H_*(W,M)$. Furthermore this chain complex has an explicit geometric description in terms of the handlebodies W_k.

Conversely, if we are given any chain complex whose homology is $H_*(W,M)$, we can write $(W; M,M')$ as a sequence of elementary cobordisms whose geometric (or filtered) complex is chain isomorphic to the given complex.

We will use some of this information in the obstruction theory to follow. The facts used will be discussed in the appendix to Section I.

Note that the filtration of (W,M) by the pairs $(W_1+\ldots+W_k, M)$ is, homotopically speaking, just a skeletal decomposition of the C.W. pair (W,M).

<u>Definition 1.5</u> Let $f:(L,bL) \longrightarrow (M,bM)$ be a homotopy equivalence of PL manifold pairs. Let $(W; M,M')$ be an admissable cobordism of (M,bM). <u>A cobordism construction for the triple</u> $(W,M;\ f: L \rightarrow M)$ is an admissable cobordism $(Q; L,L')$ of (L,bL) and a homotopy equivalence of pairs

$$F: (Q,bQ) \longrightarrow (W,bW)$$

such that

$$F/L \text{ induces } f: (L,bL) \longrightarrow (M,bM)$$

and

$$F/L' \text{ induces } f': (L',bL') \rightarrow (M',bM') ,$$

where f' is a homotopy equivalence of pairs.
To be brief we whall write <u>(Q,F) is a construction for $(W,M;\ f)$</u>.

<u>Definition 1.6</u> Suppose that $(W; M,M')$ is a smooth admissable cobordism of (M,bM). Let $(W_k; M_k,M_{k-1})$, $k=1, \ldots, r$, define a skeletal decomposition of $(W;M,M')$, i.e. $(W;M,M') = (W_1+\ldots+W_r;\ M_o,M_r)$.
Let $f: (L,bL) \rightarrow (M,bM)$ be a homotopy equivalence.
<u>A cobordism construction for $(W,M;\ f)$ over the k-skeleton of (W,M)</u> is a cobordism construction for $(W_1+\ldots+W_k,M;\ f)$.

Let (Q_k,F_k) be such a construction. We say that <u>(Q_k,F_k) extends to a construction over the $(k+1)$-skeleton of (W,M)</u> if there is a construction for $(W_{k+1},M_k;\ g)$, where $g = F_k/(bQ_k - \text{int } L)$.

If (R,G) is a construction for $(W_k,M_{k-1};\ g)$, then by setting $Q_{k+1} = R \cup_{bQ_k-1} Q_k$ and $F_{k+1} = F_k \cup_g G$ we get a construction for $(W,M;\ f)$ over the $(k+1)$-skeleton of (W,M) which naturally extends (Q_k,F_k).

Note that the partial constructions of Definition 1.6 are defined with respect to a specific skeletal decomposition of $(W;\ M,M')$. We fix now a decomposition with minimal r and discuss the stepwise extension of a construction for $(W,M;\ f)$ over the skeletons of (W,M).

Let (Q_{k-1},F_{k-1}) be a construction for $(W,M;\ f)$ over the $(k-1)$-skeleton of (W,M). Let $L_{k-1} = bQ_{k-1} - \operatorname{int} L$. Then according to <u>Definition 1.5</u> F_{k-1} induces $g\colon (L_{k-1},bL_{k-1}) \longrightarrow (M_{k-1},bM_{k-1})$ a homotopy equivalence of PL manifold pairs. By Lemma 1.6 c) M_{k-1} is $(k-2)$-connected since the fixed decomposition has minimal r. Thus Lemma 1.5 applies to g.

<u>Definition 1.7</u> (<u>The obstruction cochain</u>) Let $c_k = c_k(Q_{k-1},F_{k-1})$ in $\operatorname{Hom}(H_k(W_k,M_{k-1}),\ P_{k-1})$ be defined by the composition

$$H_k(W_k,M_{k-1}) \xrightarrow{b} H_{k-1}(M_{k-1}) \underset{\cong}{\xleftarrow{h}} \pi_{k-1}(M_{k-1}) \xrightarrow{c_g} P_{k-1},$$

where c_g is defined by Lemma 1.5, and h is the Hurewicz Isomorphism.

<u>Theorem 1.7</u> Let (Q_{k-1}, F_{k-1}) be a cobordism construction for $(W,M;\ f:L \longrightarrow M)$ over the $(k-1)$-skeleton of (W,M). Then (Q_{k-1}, F_{k-1}) extends to a construction over the k-skeleton of (W,M) iffi ,

$$c_k(Q_{k-1}, F_{k-1}) = 0 .$$

<u>Proof:</u> Let $L_{k-1} = bQ_{k-1} - \overset{\circ}{L}$ and set $g = F_{k-1}/L_{k-1}$. We want to show that there is a construction for $(W_k, M_{k-1};\ g:L_{k-1} \longrightarrow M_{k-1})$ iffi $c_k(Q_{k-1}, F_{k-1}) = 0$. Now $c_k(Q_{k-1}, F_{k-1})$ is zero iffi $c_g = 0$. In fact, in the composition

$$\begin{tikzcd} H_k(W_k, M_{k-1}) \arrow[r, "b", "\cong"'] \arrow[rrr, bend right, "c_k"'] & H_{k-1}(M_{k-1}) & \pi_{k-1}(M_{k-1}) \arrow[l, "h"', "\cong"] \arrow[r, "c_g"] & P_{k-1} \end{tikzcd}$$

b and h are isomorphisms .

Suppose first that $c_g = 0$. Write

$$W_k = M_{k-1} \times I \cup_A \left(\bigcup_{i=1}^{r} D_i^k \times D_i^{n+1-k} \right),$$

where $A = j\left(\bigcup_{i=1}^{r} S_i^{k-1} \times D_i^{n+1-k}\right)$ in $\operatorname{int}(M_{k-1} \times 1)$.

Let $N_{k-1} = j\left(\bigcup_{i=1}^{r} S_i^{k-1} \times p_i\right)$ where p_i is a point in D_i^{n+1-k}. By Lemma 1.5 $c_g = 0$ implies there exists

$$H: (L_{k-1}, bL_{k-1}) \times I \longrightarrow (M_{k-1}, bM_{k-1})$$

so that $H/(L_{k-1}, bL_{k-1}) \times 0 = g$ and

so that $\bar{g} = H/(L_{k-1}, bL_{k-1}) \times 1$ is t-regular to N_{k-1} with $\bar{g}^{-1}(N_{k-1}) = R_{k-1}$ PL-homeomorphic to N_{k-1}. Since R_{k-1} is a disjoint union of $(k-1)$-spheres, $\bar{g}/R_{k-1} : R_{k-1} \to N_{k-1}$ is homotopic to a PL-homeomorphism. Thus by using the PL-covering homotopy theorem (19) and the homotopy extension theorem we can assume that $\bar{g}/R_{k-1}$ is a PL-homeomorphism..

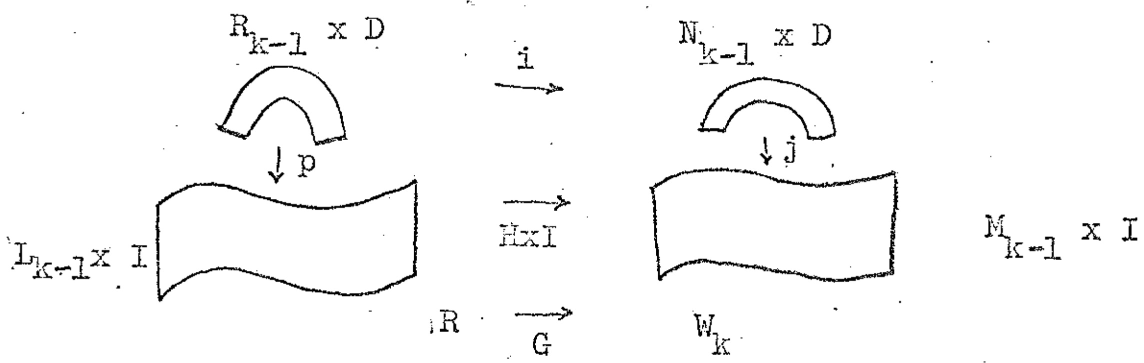
Now R_{k-1} has a product neighborhood

$$p:\left(\bigcup_{i=1}^{r} S_i^{k-1}\right) \times D^{n+k-1} \longrightarrow (\operatorname{int} L_{k-1}) \times 1$$

so that $\bar{g}\cdot p = j$. Let $B = p\left(\left(\bigcup_{1}^{r} S_i^{k-1}\right) \times D^{n+k-1}\right)$ and form $R = L_{k-1} \times I \cup_B \left(\left(\bigcup_{1}^{r} D_i^{k}\right) \times D^{n+k-1}\right)$. Define $G:(R, bR) \longrightarrow (W_k, bW_k)$ by $G = H \times I \cup_g i$ where

$$i: \left(\bigcup_{i=1}^{r} D_i^k\right) \times D^{n+k-1} \longrightarrow \bigcup_{i=1}^{r} (D_i^k \times D_i^{n+k-1})$$

is the obvious equivalence.



We claim that (R,G) is a cobordism construction for $(W_k, M_{k-1}; g)$. Let $L_k = bR - \mathring{L}_{k-1}$. Then G induces

$$g': (L_k, bL_k) \longrightarrow (M_k, bM_k).$$

We need to show that G and g' are homotopy equivalences of pairs.

We proceed in steps:

1) g'/bL_k is a h.e. by construction.

2) It follows from a standard lemma in homotopy theory that

$$G: R \to W_k$$

is a h.e. (See Lemma H in the appendix to section I.)

3) $G: (R, bR) \to (W_k, bW_k)$ has degree one. This follows from the commutative diagram 3).

4) $G^*: H^* (W_k, M_{k-1}) \to H^*(R, L_{k-1})$ is an isomorphism. This follows from diagram 2), diagram 4) and the five lemma.

5) $G_*: H_*(W_k, M_k) \longrightarrow H_*(R, L_k)$ is an isomorphism. This follows from 3), 4) and diagram 5).

6) $g_*': H_*(L_k) \longrightarrow H_*(M_k)$ is an isomorphism. This follows from 5) and the 5-lemma by an argument like that of 4).

7) Therefore g' is a h.e. when L_k and M_k are simply connected by the Whitehead Theorem.

8) $(G/bR)_*: H_*(bR) \to H_*(bW_k)$ is an isomorphism. This follows from 6) and diagram 8).

9) Therefore G/bR is a h.e. when bR is simply connected.

This takes care of all cases except possibly when $k = 1$ or $k = 2$. For these cases see the Appendix to section I.

Thus $c_k(Q_{k-1},F_{k-1}) = 0$ implies that there is a construction for $(W_k,M_{k-1};\ g)$.

3)

$$\begin{tikzcd} H_{n+1}(R,bR) \arrow[r, "\cong"] \arrow[d, "G_*^{n+1}"] & H_{n+1}(R,bR \cup L_{k-1} \times I) \arrow[d] & H_{n+1}(R_{k-1}\times D, b(R_{k-1}\times D)) \arrow[l, "\cong"', "exc"] \arrow[d, "i_*", "\cong"'] \\ H_{n+1}(W_k,bW_k) \arrow[r, "\cong"] & H_{n+1}(W_k,bW_k \cup M_{k-1}\times I) & H_{n+1}(N_{k-1}\times D, b(N_{k-1}\times D)) \arrow[l, "\cong"', "exc"] \end{tikzcd}$$

4)

$$\begin{tikzcd} \cdots & H^i(W_k) \arrow[l] \arrow[d, "\cong"] & H^i(W_k,M_{k-1}) \arrow[l] \arrow[d, "G^*"] & H^{i-1}(M_{k-1}) \arrow[l] \arrow[d, "\cong"] & \cdots \arrow[l] \arrow[d, "\cong"] \\ \cdots & H^i(R) \arrow[l] & H^i(R,L_{k-1}) \arrow[l] & H^{i-1}(L_{k-1}) \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

5)

$$\begin{tikzcd} H^*(W_k,M_{k-1}) \arrow[r, "G^*", "\cong"'] \arrow[d, "\cap\mu_{W_k}", "\cong"'] & H^*(R,L_{k-1}) \arrow[d, "\cap\mu_R", "\cong"'] \\ H_*(W_k,M_k) & H_*(R,L_k) \arrow[l, "G^*"] \end{tikzcd}$$

8)

$$\begin{tikzcd} \cdots \arrow[r] & H_i(L_k) \oplus H_i(L_{k-1}) \arrow[r] \arrow[d, "\cong"] & H_i(bR) \arrow[r] \arrow[d, "(G/bR)_*"] & H_{i-1}(bL_k) \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[d, "\cong"] \\ \cdots \arrow[r] & H_i(M_k) \oplus H_i(M_{k-1}) \arrow[r] & H_i(bW_k) \arrow[r] & H_{i-1}(bM_k) \arrow[r] & \cdots \end{tikzcd}$$

Now suppose that (R,G) is a construction for $(W_k,M_{k-1};\ g)$. We want to show that c_g is the zero homomorphism . Represent x in $\pi_{k-1}(M_{k-1})$ by an embedded S^{k-1} . Suppose that

G restricted to a collar neighborhood of bR is equal to $(G/bR) \times I$, where we assume that G/bR is t-reqular to S^{k-1}. Then the class determined by $g^{-1}(S^{k-1}) = (G/bR)^{-1}(S^{k-1})$ in P_{k-1} is the same as the class determined by

$$G^{-1}(S^{k-1} \times 1/2) .$$

This means that

$$\begin{tikzcd} \pi_{k-1}(M_{k-1}) \arrow[d, "i_*"] \arrow[dr, "c_g"] & \\ \pi_{k-1}(W_k) \arrow[r, "c_G"'] & P_{k-1} \end{tikzcd}$$

is commutative. But $\pi_{k-1}(W_k) = 0$.

Therefore,

$$c_k(Q_{k-1}, F_{k-1}) = c_g = 0 .$$

Recall that the filtering pairs $(W_1+\dots+W_k, M)$ $k = 1,\dots,r$ give us a chain complex whose homology is $H_*(W,M)$. The k^{th} chain group is $H_k(W_1+\dots+W_k, W_1+\dots+W_{k-1})$, and the boundary map $C_k \xrightarrow{\partial} C_{k-1}$ is the boundary map of the triple

$$(W_1+\dots+W_k, W_1+\dots+W_{k-1}, W_1+\dots+W_{k-2}) .$$

Using excision we identify $H_k(W_k, M_{k-1})$ and C_k. Thus $c_k(Q_{k-1}, F_{k-1})$ determines a cochain in the cochain complex for $H^*(W,M)$.

Theorem 1.8 i) $c_k(F_{k-1}, Q_{k-1})$ is a cocycle. Let (c_k) be the cohomology class of c_k in $H^k(W,M;\ P_{k-1})$.

ii) (c_k) determines the zero homomorphism in $\mathrm{Hom}(H_k(W,M), P_{k-1})$ iffi c_g vanishes on the image of the composition

$$0 \longrightarrow H_{k-1}(M) \xrightarrow{i_0} H_{k-1}(W_1+\dots+W_{k-1}) \underset{\cong}{\xleftarrow{i_{k-1}}} H_{k-1}(M_{k-1}) \underset{\cong}{\xleftarrow{h}} \pi_{k-1}(M_{k-1}) .$$

Proof: i) We want to show that c_k vanishes on the image of

$$\begin{array}{ccccc} \partial : & H_{k+1}(W_{k+1}, M_k) & \longrightarrow & H_k(W_k, M_{k-1}) \\ & \| & & \| \\ & C_{k+1}(W,M) & \xrightarrow{\partial} & C_k(W,M) . \end{array}$$

But this follows from commutative diagram i) since by Lemma 1.6 b) i_{k-1} is an isomorphism.

To prove ii) note that (c_k) determines the zero homomorphism in $\mathrm{Hom}(H_k(W,M);\ P_{k-1})$ iffi c_k vanishes on the kernel of

$$\begin{array}{cccc} \partial : & H_k(W_k,M_{k-1}) & \longrightarrow & H_{k-1}(W_{k-1},M_{k-2}) \\ & \| & & \| \\ & C_k(W,M) & \xrightarrow{\partial} & C_{k-1}(W,M) \end{array} .$$

Thus ii) follows from commutative diagram ii) since

1) image j = kernel k and

2) i_o is an isomorphism by Lemma 1.6 a) .

i)

$$\begin{tikzcd}
H_k(W_k,M_{k-1}) \arrow[r, "b"] \arrow[d, "\cong"', "exc"] \arrow[rrr, bend left, "c_k"] & H_{k-1}(M_{k-1}) \arrow[r, "h"] \arrow[dr, "i_{k-1}"', "\cong"] & \pi_{k-1}(M_{k-1}) \arrow[r, "c_g"] & P_{k-1} \\
H_k(W_1+\dots+W_k, W_1+\dots+W_{k-1}) \arrow[rr] & & H_{k-1}(W_1+\dots+W_{k-1}) & \\
H_{k+1}(W_1+\dots+W_{k+1}, W_1+\dots+W_k) \arrow[u, "b"] \arrow[urr, "0"'] & & & \\
H_{k+1}(W_{k+1},M_k) \arrow[u, "exc"] \arrow[uuu, bend left, "\partial"] & & &
\end{tikzcd}$$

ii)

$$\begin{tikzcd}
H_{k-1}(W_{k-1},M_{k-2}) \arrow[d, "\cong"', "exc"] & & & & \\
H_{k-1}(W_1+\dots+W_{k-1}, W_1+\dots+W_{k-2}) \arrow[d, leftarrow, "b"] & & & H_{k-1}(M) \arrow[dl] \arrow[d, "i_o", "\cong"'] & \\
H_k(W_1+\dots+W_k, W_1+\dots+W_{k-1}) \arrow[rr] & & H_{k-1}(W_1+\dots+W_{k-1}) \arrow[ull, "k"'] & H_{k-1}(W_1+\dots+W_{k-2}) \arrow[l, "j"] & \\
 & & & & 0 \arrow[ul] \\
H_k(W_k,M_{k-1}) \arrow[uu, "\cong", "exc"'] \arrow[r, "b"] \arrow[rrr, bend right, "C_k"'] \arrow[uuuu, bend left, "\partial"] & H_{k-1}(M_{k-1}) \arrow[uur, "\cong", "i_{k-1}"'] & \pi_{k-1}(M_{k-1}) \arrow[l, "\cong"] \arrow[dr, "c_g"] & & \\
 & & & P_{k-1} &
\end{tikzcd}$$

Now we want to show that the cohomology class determined by $c_k(Q_{k-1},F_{k-1})$ depends precisely on the construction over the (k-2) skeleton of (W,M).

We need two preliminary definitions and lemmas.

Definition 1.8 Write W_k as

$$M_{k-1} \times I \cup_A \left(\bigcup_{i=1}^{r} D_i^k \times D_i^{n+1-k} \right).$$

Then the $\bar{D}_i^k = D_i^k \cup (bD_i^k \times I)$ $i = 1,\ldots,r$ with orientations generate $H_k(W_k;M_{k-1})$. Let DW_k denote $W_k \cup_{M_{k-1}} W_k$. Define the doubling homomorphism

$$d:\ H_k(W_k,M_{k-1}) \longrightarrow H_k(DW_k)$$

on generators by

$$d(\bar{D}_i^k) = (i_1\bar{D}_i^k - i_2\bar{D}_i^k)$$

where (x) denotes the homology class of a chain x and i_1 and i_2 are the two inclusions of W_k in DW_k.

Lemma 1.9 The following diagram is commutative.

$$\begin{tikzcd} & H_k(W_k,M_{k-1}) \arrow[d, "d"] \\ H_k(W_k) \arrow[ur, "j"] \arrow[r, "i_1 - i_2"'] & H_k(DW_k) \end{tikzcd}$$

Proof: Let x be a cycle representing (x) in $H_k(W_k)$. We may choose x so that

$$x = a_1\bar{D}_1^k + \ldots + a_r\bar{D}_r^k + y$$

where y is a chain on M_{k-1}.

Then $dj(x) = d(a_1(\bar{D}_1^k) \ \ldots \ a_r(\bar{D}_r^k))$

and $(i_1 - i_2)(x) = a_1(i_1\bar{D}_1^k - i_2\bar{D}_1^k) \ \ldots \ a_r(i_1\bar{D}_r^k - i_2\bar{D}_r^k)$.

Therefore $dj(x) = (i_1 - i_2)(x)$. Q.E.D.

Let (Q_{k-1}, F_{k-1}) be a construction for $(W,M;\ f)$ over the $(k-1)$ skeleton of (W,M). Suppose that (R,G) and (R',G') are two constructions for $(W_k, M_{k-1};\ g)$ $(g = F_{k-1}/L_{k-1} = bQ_{k-1}\text{-int } L)$ defining extensions of (Q_{k-1}, F_{k-1}) over the k-skeleton of (W,M). Then by using Van Kampen's Theorem, the Mayer Vietoris sequence, and the Hurewicz Theorem we see that

$$R \cup_{L_{k-1}} R' \quad \text{and} \quad W_k \cup_{M_{k-1}} W_k = DW_k$$

are $(k-1)$-connected and that

$$G \cup_g G' : (R \cup_{L_{k-1}} R', b(R \cup_{L_{k-1}} R)) \longrightarrow (DW_k, bDW_k)$$

has degree one. Thus Lemma 1.5 applies to $G \cup_g G'$.
Let $c_{G \cup G'}$ denote the homomorphism given to us by Lemma 1.5.

$$c_{G \cup G'} : \pi_k(DW_k) \longrightarrow P_k \ .$$

<u>Definition 1.9</u> (The difference cochain).
If (R,G) and (R',G') define extensions of (Q_{k-1},F_{k-1}) over the k-skeleton of (W,M), let $d(G,G')$ denote the composition

$$H_k(W_k,M_{k-1}) \xrightarrow{d} H_k(DW_k) \xleftarrow{\cong} \pi_k(DW_k) \xrightarrow{c_{G\cup G'}} P_k .$$

$d(G,G')$ is called <u>the difference cochain associated to the two extensions (R,G) and (R',G')</u>.

<u>Lemma 1.10</u> If $(R,G),(R',G')$ and (R'',G'') define three extensions of (Q_{k-1},F_{k-1}) over the k-skeleton of (W,M), then

$$d(G,G'') = d(G,G') + d(G',G'') .$$

<u>Proof:</u> Let W_k and D_i^k be as in Definition 1.8. Let $g = F_{k-1}/L_{k-1}$. By Theorem 1.7, $c_g = 0$. Therefore we may change g by a homotopy so that it has the nice form of $\tilde{g}$ in the proof of Theorem 1.7. We may suppose by the homotopy extension theorem that G, G', and G'' extend this new g.

Now we may approximate G by a map which is t-regular to $(\bar{D}_i^k, b\bar{D}_i^k)$ in (W_k,M_{k-1}). Then $G^{-1}(\bar{D}_i^k)$ is a framed submanifold of R whose boundary is PL-homeomorphic to S^{k-1}, and which is contractible in R. Let $G(\bar{D}_i^k)$ denote the element in P_k determined by $G^{-1}(\bar{D}_i^k)$. Make similar definitions for G' and G''.

Now $(G \cup_g G')^{-1}(i_1 \bar{D}_i^k \cup i_2 \bar{D}_i^k)$ is equal to $i_1 G^{-1}(\bar{D}_i^k) \cup i_2 \; G'^{-1}(\bar{D}_i^k)$. Thus it is clear that

$$d\,(G,G')\,(\bar{D}_i^k) = G(\bar{D}_i^k) + G'(-\bar{D}_i^k)$$

$$= G(\bar{D}_i^k) - G'(\bar{D}_i^k) \; .$$

The lemma follows immediately .

<u>Theorem 1.11</u> Let (R,G) and (R',G') define extensions of (Q_{k-1},F_{k-1}) over the k-skeleton of (W,M) . Denote the extensions by (Q_k,F_k) and (Q_k',F_k') . Then

$$c_{k+1}(Q_k,F_k) - c_{k+1}(Q_k',F_k') = \delta\, d(G,G') \; .$$

Conversely, suppose that (R,G) defines an extension (Q_k,F_k) of (Q_{k-1},F_{k-1}) and that u in $\mathrm{Hom}(H_k(W_k,M_{k-1});\,P_k)$ is given . Then there exists (R',G') so that

$$u = d\,(G,G') \; .$$

Thus we may change (Q_k,F_k) on the k-skeleton of (W,M) so that $c_{k+1}(Q_k,F_k)$ varies by an arbitrary coboundary .

Proof: For the first part we consider the diagram

$$\begin{tikzcd}
H_{k+1}(W_{k+1},M_k) \arrow[r, "b"] \arrow[d, "\cong"', "\text{exc.}"] \arrow[ddd, bend right=60, "\partial"'] \arrow[rrrdd, bend left=30, "c_{k+1}(Q_k,F_k)-c_{k+1}(Q_k',F_k')"] & H_k(M_k) \arrow[d] & \pi_k(M_k) \arrow[l, "\cong"'] \arrow[d] \arrow[rdd, "c_g-c_{g'}"] & \\
H_{k+1}(W_1\cdot\cdot W_{k+1},W_1\cdot\cdot W_k) \arrow[r] \arrow[d, "b"] & H_k(W_1\cdot\cdot W_k) & \pi_k(W_1\cdot\cdot W_k) \arrow[l, "\cong"'] \arrow[r, phantom, "\mathrm{III}"] & \\
H_k(W_1\cdot\cdot W_k,W_1\cdot\cdot W_{k-1}) & H_k(W_k) \arrow[l] \arrow[u, "\cong", "i_k"'] \arrow[d, "i_1-i_2"] \arrow[dl, "j"'] \arrow[d, phantom, "\mathrm{I}"'] & \pi_k(W_k) \arrow[l, "\cong"'] \arrow[u, "\cong"'] \arrow[d, "i_1-i_2"] \arrow[r, "c_G-c_{G'}"] & P_k \\
H_k(W_k,M_{k-1}) \arrow[u, "\cong", "\text{exc.}"'] \arrow[r, "d"] \arrow[rrru, bend right=30, "d(G,G')"'] & H_k(DW_k) & \pi_k(DW_k) \arrow[l, "\cong"'] \arrow[ru, "c_{G\cup G'}"'] \arrow[r, phantom, "\mathrm{II}"] &
\end{tikzcd}$$

where $g = G/L_k = Q_k -\text{ int } L$ and

$g' = G'/L_k' = Q_k' -\text{ int } L$.

Commutativity around region I is just Lemma 1.9 . Commutativity around region II is easily verified on representatives . Commutativity around region III is just the naturality property of c_g demonstrated in the proof of Theorem 1.7 .

Therefore the diagram is commutative and this proves the first part of Theorem 1.11 .

For the second part suppose that (R,G) is given and that $g = G/L_{k-1}$ has the form of $\bar{g}$ in Theorem 1.7 . Let (R_o , G_o) denote the construction given in the proof of Theorem 1.7 with $H = g \times I$.

If $x=(x_1,\ldots,x_r)$ denotes a sequence of elements in P_k, let (G_x,R_x) denote the construction of Theorem 1.7 with $H= g \times I$ and i replaced by i_x where

$$i_x^{-1} (\bar{D}_i^k \times p_i) \text{ represents } x_i .$$

and i_x is a h.e. of pairs. (For a proof that i_x exists see Lemma F in the appendix .) Then by the formula derived in the proof of Lemma 1.10 .

$$\begin{aligned} d(G_x,G_o)(\bar{D}_i^k) &= G_x(\bar{D}_i^k) - G_o(\bar{D}_i^k) \\ &= x_i - 0 \\ &= x_i . \end{aligned}$$

If (R,G) and u in $\mathrm{Hom}(H_k(W_k,M_{k-1}), P_k)$ are given, let $x = (G(\bar{D}_1^k) - u(\bar{D}_1^k),\ldots, G(\bar{D}_r^k) - u(\bar{D}_r^k))$ and set $(R',G') = (R_x,G_x)$. Then

$$\begin{aligned} d(G,G')(\bar{D}_i^k) &= (d(G,G_o) + d(G_o,G_x))(\bar{D}_i^k) \\ &= G(\bar{D}_i^k) - (G(\bar{D}_i^k) - u(\bar{D}_i^k)) \\ &= u (\bar{D}_i^k) . \end{aligned}$$

So $\qquad d(G,G') \quad = u$. This completes the proof of Theorem 1.11 .

We summarize the above results in the following theorem.

Theorem 1.12 Let $(W;M,M')$ be a smooth admissable cobordism of (M,bM) with the skeletal decomposition

$$(W;\ M,M') = (W_1+\ldots+W_r;\ M,M')$$

where $2r+1 < n$. Let $f\colon (L,bL) \longrightarrow (M,bM)$ be a homotopy equivalence of PL-manifold pairs. Then there exists an obstruction theory for the problem of making a stepwise cobordism construction for $(W,M;\ f)$ over the skeletons of (W,M). The obstructions (c_{k+1}) lie in

$$H^{k+1}(W,M;\ P_k)$$

where

$$P_k = \begin{cases} 0 & k \text{ odd} \\ Z_2 & k \equiv 2 (\mathrm{mod} 4) \\ Z & k \equiv 0 (\mathrm{mod} 4) \end{cases} .$$

Corollary: Let $(W,M;\ f)$ be as above and suppose that

1) $H^{4k+1}(W,M;\ Z) = 0$
2) $H^{4k-1}(W,M;\ Z_2) = 0$.

Then there exists a cobordism construction for $(W,M;\ f)$.

From the statement of Theorem 1.12 it appears that the obstruction theory depends on the particular skeletal decomposition of (W,M). This is not the case, however. In fact one can use Smale's theory of handle decompositions to pass from one decomposition to another and thereby relate the corresponding obstruction theories.

We will not do this because there is another interpretation of the obstruction theory in terms of an extension problem in homotopy theory (see Part II). From this point of view the independence of the theory from particular skeletal decompositions follows from a lemma in homotopy theory.

One can also ask if the obstructions described in Theorem 1.12 are actually representable as obstructions to making particular cobordism constructions. There are many examples (See Chapter 6). For instance the following lemma is easy to prove.

<u>Lemma 1.13</u> Let p be an element of P_k. Then there exists a smooth admissable cobordism $(W; M,M')$, a h.e. $f: (L,bL) \longrightarrow (M,bM)$, a construction (Q_k,F_k) for $(W,M; f)$ over the k-skeleton of (W,M), and an element x of $H_{k+1}(W,M)$ so that

$$c_{k+1}(Q_k,F_k) \cdot x = p \ .$$

<u>Proof:</u> Let $i_p\colon D_1^k \times D_1^n \longrightarrow D^k \times D^n$ $n > k$ be a homotopy equivalence of PL-manifold pairs so that $i_p^{-1}(D^k \times *)$ represents p. Then there is a product neighborhood

$$j\colon\ S^{k-1} \times D^n \longrightarrow b(D_1^k \times D_1^n)$$

so that $i_p \circ j$ is just the natural inclusion of $S^{k-1} \times D^n$ in $b(D^k \times D^n)$ so that $S^{k-1} \to bD^k \times *$. Define $L = D_1^k \times D_1^n \cup_j D_1^k \times D_1^n$ by identifying $b(D_1^k \times *) \times D^n$ with $j(S^{k-1} \times D^n)$. Let

$$M = D^k \times D^n \cup_{id} D^k \times D^n = S^k \times D^n,$$

and $W = D^{k+1} \times D^n$. Define $f\colon L \to M$ by $f = i_p \cup_j i$. Then (W,M,f) has a construction over the k-skeleton, (namely $(L \times I, f \times I)$), $H_{k+1}(W,M) = Z$, and $c_{k+1} \cdot (\text{gen}) = p$. Q.E.D.

Now we relate the obstructions in $H^{4r+1}(W,M;\ P_{4r})$ to (L,bL).

Let $p_i(N,Q)$ denote the i^{th} rational Pontryagin class of the PL-manifold N. Let $E\colon H^*(X;\ A) \to \operatorname{Hom}(H_*(X),A)$ be the evaluation map.

<u>Theorem 1.14</u> Let (Q_k, F_k) be a construction for $(W,M;\ f)$ over the k-skeleton of (W,M). Let $c_{k+1} = c_{k+1}(Q_k, F_k)$.

Then

a) if $k > 4r$, then $p_r(L,Q) = 0$

b) if $k = 4r$, then $p_r(L,Q) = 0$

iffi

$$E \cdot (c_{k+1}) = 0 .$$

<u>Proof:</u> Let (Q_{i-1}, F_{i-1}) be a construction for $(W,M;\ f)$ over the $(i-1)$-skeleton of (W,M), where $i = 4r$. Let

$$L_{i-1} = bQ_{i-1} - \operatorname{int} L$$

$$g = F_{i-1}/L_{i-1} .$$

Consider the diagram

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
H_i(M,Q) \arrow[r, "i_o"] \arrow[rrr, bend left, "e"] & H_i(W_1 \cup \dots \cup W_{i-1}, Q) & H_i(M_{i-1}, Q) \arrow[l, "\cong"'] & \pi_i(M_{i-1}) \otimes Q \arrow[l, "\cong"'] \arrow[r, "c_g"] \arrow[d, phantom, "\mathrm{II}"] & P_i \otimes Q \arrow[d, "\text{index}"] \\
H_i(L,Q) \arrow[u, "f", "\cong"'] \arrow[r, "i_o"] \arrow[rrr, bend right, "L_r(L)"'] & H_i(Q_{i-1}, Q) \arrow[u, "F_{i-1}", "\cong"'] \arrow[r, phantom, "\mathrm{I}"'] & H_i(L_{i-1}, Q) \arrow[l, "i", "\cong"'] \arrow[u, "g"', "\cong"] \arrow[r, "L_r(L_{i-1})"] & Q & Z \otimes Q \arrow[l]
\end{tikzcd}$$

where $L_r(X)$ denotes the r^{th} Hirzebruch polynomial of any PL-manifold X.

We show first that the commutativity of this diagram implies Theorem 1.14 .

a) Suppose that $4r < k$ and that $p_1(L,Q) = \dots = p_{r-1}(L,Q) = 0$. Let $i = 4r$ and choose (Q_{i-1}, F_{i-1}) so that $c_i(Q_{i-1}, F_{i-1}) = 0$. Then $c_g = 0$, and the commutativity of the diagram inplies $L_r(L) = 0$. Thus $p_r(L,Q) = 0$.

b) By Theorem 1.8 b), $E(c_{k+1}) = 0$ iffi $c_g e = 0$. By the diagram $c_g e = 0$ iffi $L_r(L) = 0$. By a) $p_i(L,Q) = 0$ for $4i < 4r = k$. Thus $p_r(L,Q) = 0$ iffi $L_r(L) = 0$.

Now the commutativity around region II is verified immediately on representatives. The commutativity around I is established as follows :

For each $k:(L,bL) \longrightarrow (S^{n-i},p)$ there exists K and k' so that

$$\begin{tikzcd}
(L,bL) \arrow[drr, "k"] & & \\
(Q_{i-1},bL) \arrow[u, hook] \arrow[rr, "K"] & & (S^{n-i},p) \\
(L_{i-1},bL_{i-1}) \arrow[u, hook] \arrow[urr, "k'"'] & &
\end{tikzcd}$$

is commutative. $(H^j(Q_{i-1},L;\pi_{j-1}(S^{n-i})) = 0$ for $j > n-i$.)
Suppose that k, k', and K are t-regular to y in $S^{n-i}-p$. Then $K^{-1}(y)$ is a cobordism between $k^{-1}(y)$ and $k'^{-1}(y)$. Thus if u generates $H^{n-i}(S^{n-i},p;\ Q)$

and LDx denotes the Lefschetz Dual of x, then

$$\begin{aligned} L_i(L)(LDk^*u) &= \text{index } k^{-1}(y) \\ &= \text{index } k'^{-1}(y) \\ &= L_i(L_{i-1})(LDk'^*u) \\ &= L_i(L_{i-1})(i^{-1}i_o\ LDk^*u) . \end{aligned}$$

Therefore I is commutative for elements in $H_i(L,Q)$ which are of the form LDk^*u where $k:(L,bL) \longrightarrow (S^{n-i},p)$.

It follows from a theorem of Serre that these generate $H_i(L,Q)$.

This completes the proof of Theorem 1.14 .

Appendix to Section 1

In this appendix we clarify a few points raised in Section 1.

1) Proof of Lemma's 1,2,and 3 for k 3,4.

Proof: $k=3: P_3 = 0$ by (27). The proof of Lemma 3 for $k = 3$ is the same as that for $k = 4$. Lemma 2 is clear.

$k = 4$: There is a homomorphism $P_4 \xrightarrow{\text{Index}} Z$. By (14) this is onto $16(Z)$.

Suppose Index $(M^4) = 0$. Since $bM^4 = S^3$, $M_1^4 =$ cone $bM^4 \cup M^4$ is a PL-manifold which is smoothable. By (28) M_1^4 is cobordant to S^4 by W_1 where $\pi_1(W_1) = 0$ and $\widetilde{H}_i(W_1) = 0$ for $i \neq 2$. Let $W = W_1$-(tubular neighborhood of arc connecting S^4 and M_1^4) and extend the embedding of M_4 to an embedding of $(W; M_4, bM_4, D^4, bD^4, bD^4 xI)$ in $(D^{4+k} xI; (D^{4+k}, bD^{4+k})x0, (D^{4+k}, bD^{4+k})x1, bD^{4+k} xI)$. Since $H^i(W, M_4; \pi_{i-1}(SO)) = 0$, the framing of M_4 extends over W. Thus

$$P_4 \xrightarrow{\text{Index}} Z$$

is a monomorphism.

Using a nice Morse function on W whose critical points are on the interior of W we can write $W = W_o \cup D^4 xI \bigcup_{i=1}^{n} D_i^5$, where W_o is the trace of framed surgeries beginning on M^4 whose spheres have dimension ≤ 3. Now $bW_o - M^4 = L = D^4 \cup_i S_i^4$.

So we can make L connected by framed surgery to get a four disk. Thus Lemma 3 is verified for $k = 4$. The proof for $k = 3$ is identical.

2) <u>Proof of Lemma 1.5</u>

<u>Proof:</u> Let x be in $\pi_k(M')$. Choose an $S^k \subset \text{int } M'$ to represent x. S^k has a normal microbundle by (6). Make f t-regular to S^k, assume $f^{-1}(S^k)$ is connected and let $T = f^{-1}(S^k) - \text{int } D^k$. ($f^{-1}(S^k)$ is non-void because f has degree one.) T is a contractible subcomplex of M, so by general position we may assume that T is contained in the interior of a closed n-cell D^n of M. Choose an embedding of $S^{k-1} \times I$ in $D^n - T$ so that $S^{k-1} \times 0$ goes into bT and $S^{k-1} \times 1$ lies in bD^n. Let $T^* = T \cup \text{image } (S^{k-1} \times I)$.

Now f induces a null-homotopic map $T \to S^k \subset M'$. We can change f by a homotopy f_t so f_t is t-regular to S^k, $f_t^{-1}(S^k) = f^{-1}(S^k)$, and $f_1(T) = p$ in S^k. By the t-regularity of f_1 there is an embedding $F: T \times D^{n-k} \to D^n \subset M$ extending the inclusion $T \subset M$ so that $f_1 F = ip_2$ where $i: D^k \to M'$ is a normal disk at p. Using (8) we can shange F (and f_1) by an ambient isotopy of D^n(fixed on bD^n) and put a smooth structure on T so that F becomes a smooth embedding.

Thus we may assume that T^* is a smooth f.a.c. submanifold of D^n. Define

$$c_k(x) = (\text{framed cobordism class of } T^*) \text{ in } P_k.$$

Now we show that $c_k(x)$ is well-defined. Suppose $f_o: M \to M'$ is t-regular to $S^k \subset M'$ and $f_o^{-1}(S^k)$ is connected. Suppose f_t is a homotopy between f_o and f_1. Then we can change f_t for t in $(o,1)$ so that $f_t xI: MxI \to M'$ is t-regular to S^k and so that $(f_t xI)^{-1}(S^k)$ is connected. (The connected part follows from the techniques described below.) Then if we have made constructions to produce T_1^* and T_o^* from $f_1^{-1}(S^k)$ and $f_o^{-1}(S^k)$ as above these can be extended to $(f_t xI)^{-1}S^k)$ to produce a framed cobordism between T_1^* and T_o^* in $D^n xI$.

Thus $c_k(x)$ is unchanged if we alter f_1 by a homotopy. Now any two choices of $S_k \subset M'$ to represent x, the t-regular approximation to f, the homotopy making $f^{-1}(S^k)$ connected, $D^k \subset f^{-1}(S^k)$, the isotopy of D^n, etc. are related by homotopies of f. Thus $c_k(x)$ is well-defined for each x in $\pi_k(M')$.

That $c_k(x)$ is a homomorphism is verified by an easy geometric argument with represèntives.

Now we show that $c_k(x) = 0$ iffi we can change f by a homotopy f_t so that f_1 is t-regular to S^k; and $f_1^{-1}(S^k) = S^k$.

The "only if" follows from property 1) of Lemma 1.5 and the construction of c_k.

Now suppose that $c_k(x) = 0$. Then by Lemma 1.2 there is a framed cobordism (W', G) of (T^*, F) in $D^n xI$ so that

i) $W' \cap D^n x1$ is a k-disk

ii) W' is trace of surgeries on r-spheres, $r \leq k-1$, and

iii) W' is the product cobordism on a neighborhood of bT^*

We may suppose that the neighborhood in iii) contains T^*-T; so $(f^{-1}(S^k)-T)xI \cup W'$ defines a cobordism of $f^{-1}(S^k)$, W in $M \times I$, and $(W \cap M \times 1)$ is a PL-homeomorphic to S^k. Define f_t in $(M\text{-int } D^n) \times I$ by $f \times t$, on W by $f(W) = p$, and on $\bar{N}$, a product neighborhood of W by ip_2G^{-1}.

Now we want to extend f_t over $D^n xI-N$ so that $f_1^{-1}(S^k) = W \cap Mx1$. Using a nice Morse function on W we can write W as the composition of traces of single surgeries and reduce the extension problem to this case. We may have to change D^n(and thus the embedding of W in MxI) to make the extension.

So let W be the trace of a framed surgery on some $S^r \subset \text{int } T$, $r \leq k-1$.

1) The embedding of $S^r \subset T$ extends to an embedding of D^{r+1} in Mxo so that

i) D^{r+1} intersects $f^{-1}(S^k)$ transversely along S^r

ii) f may be changed by a homotopy which is constant on a neighborhood of $f^{-1}(S^k)$ so that a neighborhood of D^{r+1} is mapped to the normal disk at p.

<u>Proof:</u> i) follows from general position since M is $k-1$ connected. There is no obstruction to deforming f as required by ii if $r+1 \leq k-1$ since M' is $(k-1)$-connected.

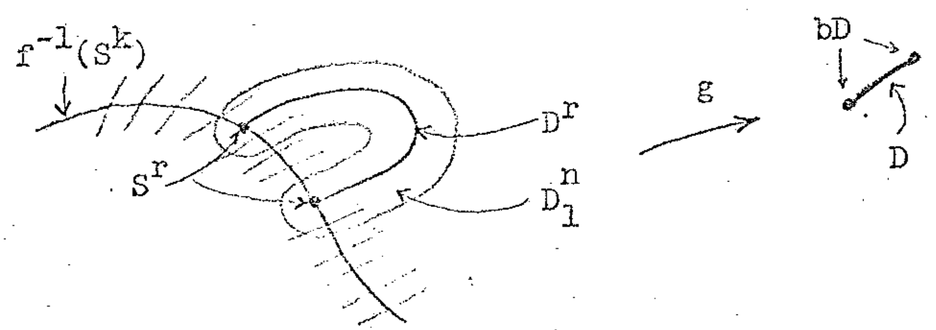
If $r = k-1$, the obstruction to doing this may be identified with an element of $\pi_k(M'-S^k)$ which is isomorphic to $\pi_k(M')$ under the inclusion $M'-S^k \subset M'$ (k is small wrt n). Now $f_*: \pi_k(M) \to \pi_k(M')$ is onto since f has degree one. Thus we may alter $D^k = D^{r+1}$ by adding some S^k to the interior of D^k (connected sum) to make the obstruction zero.

Now let D_1^n be an n-cell which intersects $f^{-1}(S^k)$ in a product neighborhood of S^r and which contains D^{r+1} in its interior. Then we may choose our embedding of W in $M \times I$ so that W is the product cobordism with the product framing outside $D_1^n \times I$.

Let $N = L \times D^{n-k}$ be a product neighborhood of $L = W \cap (D_1^n \times I)$ and define a partial $f_t = g$ as above on the complement of $(D_1^n \times I)-N$ where D_1^n plays the role of D^n. Let

$$X = (D_1^n \times I)-(L \times \operatorname{int} D^{n-k}), A = bX-\operatorname{int}(X \cap M \times 1),$$

and $D = D^{n-k}$ be the normal disk at p. We may assume that g maps A to bD.



The obstructions to extending

$$A \xrightarrow{g} bD$$

over X lie in

$$H^{i+1}(X,A; \pi_i(bD)) .$$

Now L is just a k+1 disk so using excision, the exact sequence, and Mayer Vietoris we see that

$$\begin{aligned} H^{i+1}(X,A) &= H^{i+1}(D_1^n xI, N \cup bD_1^n xI \cup D_1^n x0) \\ &= H^i(N \cup (bD_1^n xI \cup D_1^n x0)) \\ &= H^i(L \cup_{S^r} (n+1)\text{-disk}) \\ &= H^{i-1}(S^r) \end{aligned}$$

which is zero if $i \neq 1, r+1$. $\pi_i(bD) = 0$ if $i < n-k-1$. But $r+1 \leq k < n-k-1$. So the desired extension can be made.

We proceed in this way to construct the homotopy required by Lemma 1.5.

We can carry out these steps independently for a number of S_i^k 's so this completes the proof.

3) Theorem 1.7 for $k = 1,2$.

Proof: There is an easy direct proof that G is a homotopy equivalence. Or one can appeal to Lemma H of Part II, Chapter 2.

4) Lemma (Homotopy theory) Let

$$f: X \longrightarrow Y$$

be a homotopy equivalence of CW complexes. Let $a: S^{n-1} \to X$. Then if $X_1 = X \cup_a D^n$, $Y_1 = Y \cup_{f \cdot a} D^n$, and $f_1 = f \cup_a \text{id.}$, then

$$f_1: X_1 \longrightarrow Y_1$$

is a homotopy equivalence.

Proof: This also follows from Lemma H Chapter 2, Part II.

5) Smale Theory: No Smale Theory other than Lemma 1.6 was needed to develope the obstruction theory. However if one uses explicit minimal handlebody decompositions for (W,M) then attractive geometric proofs of the theorems can be given.

6) Lemma F: Let (P, bP) in (D, bD) $(D = D^{n+k})$ represent an element of P_n, $k+2 > n$. Then there exists a homotopy

equivalence

$$(D, bD) \xrightarrow{t_p} (D^n x D^k,\ b(D^n x D^k))$$

which is t-regular to $D^n x0$ and such that

$$t_p^{-1}(D^n x0) = P \subset D \quad .$$

<u>Proof:</u> Assume there is a small disk $D_1^{n+k} \subset D$ so that $P \cap D_1^{n+k} = (D^n x0) \cap D_1^{n+k} =$ n-disk. Define t_p to be the identity on D_1^{n+k}. Let $S_1 = bD_1^{n+k}$, $f_1 = t_p/S_1$. Then $D - D_1^{n+k} = S_1 xI$ and $P \cap (S_1 xI)$ defines a framed cobordism of $f_1^{-1}(b(\text{n-disk}))$. Now we proceed as in Lemma 1.5 to construct a homotopy $H: S_1 xI \longrightarrow S_1 xI$ so that H is t-regular to $(D^n x0) \cap S_1 xI$ and $H^{-1}(D^n x0) \cap S_1 xI = P \cap S_1 xI$. Then $H \cup t_p$ is the desired map.

The condition on k is needed to make the obstruction groups vanish. We can actually construct t_p for highly connected generators of P_n when $k \sim \frac{n}{2}$, by the same argument. By a different (and much more complicated) argument we can construct t_p for these generators when $k \geqslant 3$.

Application to the Study of Normal Invariants

Let M be a closed connected PL n-manifold with PL normal k-disk bundle $v = v(M)$, $k \gg n$. Let $T(v) = v \cup (\text{cone on } bv)$ be the Thom complex of v. Then an important invariant of M is any homotopy element c_M in $\pi_{n+k}(T(v))$ which can be obtained by collapsing S^{n+k} onto $T(v)$ where $v \subset S^{n+k}$. c_M is a very strong invariant of M. In fact, if M is simply connected and $n \geqslant 6$, then c_M determines the PL-homeomorphism type of M.

We make this last statement precise in Theorem 1.15 .

<u>Definition 1.10</u> A homotopy equivalence $g: L \longrightarrow M$ <u>preserves normal invariants</u> if g is covered by a bundle map

$$v(L) \xrightarrow{b(g)} v(M)$$

and there exist normal invariants c_L and c_M so that

$$\begin{tikzcd} & T(v(L)) \arrow[d, "T(b(g))"] \\ S^{n+k} \arrow[ur, "c_L"] \arrow[r, "c_M"'] & T(v(M)) \end{tikzcd}$$

is homotopy commutative, where

$$T(b(g) = b(g) \cup (\text{cone on } b(g)/bv(L)) .$$

<u>Theorem 1.15</u> If M is simply connected and $n \geqslant 6$, then $g: L \to M$ preserves normal invariants iffi g is homotopic to a PL-homeomorphism.

<u>Proof:</u> This is essentially the uniqueness theorem in the PL Browder-Novikov theory. We indicate the proof below.

In this section we will try to construct PL-homeomorphisms from homotopy equivalences, $g: L \longrightarrow M$. Thus by Theorem 1.15 this problem is equivalent (when M is simply connected and $n \geqslant 6$) to the problem of showing that g preserves normal invariants.

We first show how the latter problem is related to a certain cobordism construction. Since Theorem 1.15 can be stated for manifolds with boundary we consider that case.

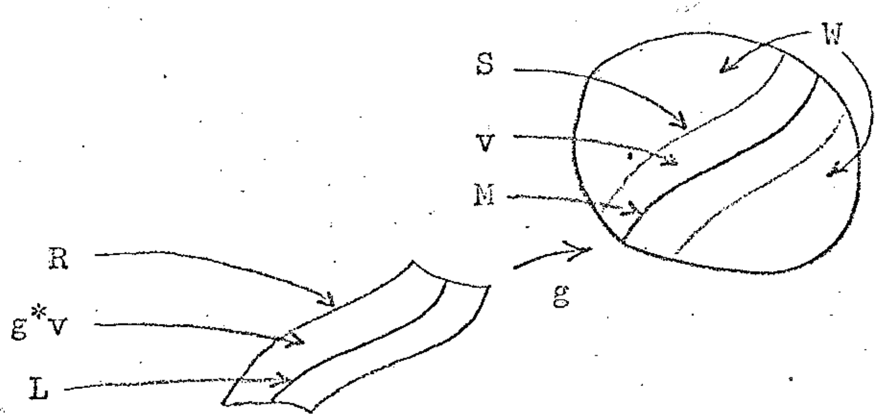
So let $g:(L, bL) \longrightarrow (M, bM)$ be a homotopy equivalence and assume that $(v(M), v(bM))$ is embedded in (D^{n+k}, bD^{n+k}). Then <u>the collapsing map</u>

$$(D^{n+k}, bD^{n+k}) \xrightarrow{c_M} (Tv(M), Tv(bM))$$

<u>defines a normal invariant for (M, bM)</u>. Let $W = D^{n+k}-$ (open disk bundle of $v(M)$) and $S =$ sphere bundle of $v(M)$. If $R = g^*S =$ sphere bundle of $g^*(v(M))$, then we have a homotopy equivalence

$$f:(R, bR) \longrightarrow (S, bS),$$

given by the bundle map $b(g)$ covering g.



Thus we can consider the problem of making a cobordism construction for $(W,S;f)$.

<u>Theorem 1.16</u> Let $g:(L^n,bL) \longrightarrow (M^n,bM)$ be a homotopy equivalence and let $v(M)$ and $b(g)$ be as above. Consider the propositions

i) g is homotopic to a PL-homeomorphism

ii) there is a cobordism construction for $(W,S;f) =$

$(D^{n+k}$-interior $v(M)$, sphere bundle $v(M)$; $b(g)$/sphere bundle $g^*v(M))$

iii) g preserves normal invariants.

Then i) $\Longrightarrow$ ii) $\Longrightarrow$ iii), and iii) $\Rightarrow$ i) if $n \geq 6$ and $\pi_1(M) = \pi_1(bM) = 0$.

<u>Proof:</u> i) $\Longrightarrow$ ii): Suppose g is homotopic to a PL-homeomorphism. Then by the covering homotopy theorem so is $b(g)$. Let

$$H: (R,bR) \times I \longrightarrow (S,bS)$$

be a homotopy between $b(g)$ and a PL-homeomorphism c. Then

$$(Q,F) = (W \cup_c R\times I,\ \mathrm{id} \cup_c H)$$

<u>is a cobordism construction for</u> $(W,S;f)$.

ii) $\Longrightarrow$ iii): If (Q,F) is a construction for $(W,S;F)$ consider

$$F \cup b(g):\ Q \cup b^*v(M) \longrightarrow W \cup v(M).$$

It follows from Lemma H of Chapter 2 Part II that $F \cup b(g)$ is a homotopy equivalence of manifolds with boundary. Now there are natural collapsing maps so that

$$\begin{tikzcd}
(W \cup v(M),\ b(W \cup v(M))) \arrow[r, "c_M"] & (Tv(M),\ Tv(bM)) \\
(Q \cup g^*v(M),\ b(Q \cup g^*v(M))) \arrow[u, "F \cup b(g)"] \arrow[r, "c_L^!"] & (Tg^*v(M),\ Tg^*v(bM)) \arrow[u, "T(b(g))"']
\end{tikzcd}$$

is commutative. $W \cup v(M)$ is just D^{n+k} so we may choose a homotopy inverse c for $F \cup b(g)$ which is a PL-homeomorphism. Then $(g^*v(M)) = v(L)$ is a normal bundle for L, $c_L = c_L^! c$ is a normal invariant for L, and

$$\begin{tikzcd}
(D^{n+k},\ bD^{n+k}) \arrow[r, "c_M"] \arrow[dr, "c_L"'] & (Tv(M),\ Tv(bM)) \\
& (Tv(L),\ Tv(bL)) \arrow[u, "T(b(g))"']
\end{tikzcd}$$

is homotopy commutative. Thus <u>g preserves normal invariants</u> .

iii) $\Rightarrow$ i) if $n \geq 6$ and $\pi_1(M) = \pi_1(bM) = 0$:

We first consider the case $bM \neq 0$. Suppose there is a homotopy

$$H: (D, bD)\times I \longrightarrow T(v(M), v(bM)) \quad , D = D^{n+k}$$

between c_M and $T(b(g)) \cdot c_L$.

Note that H is t-regular to

$$(M, bM) \xrightarrow{\text{O-section}} (Tv(M), Tv(bM))$$

on $D^{n+k}\times 0$ and $D^{n+k}\times 1$. Thus we can change H slightly in $D^{n+k}\times(0,1)$ so that H is t-regular to (M, bM) on all of $D^{n+k}\times I$.

Now we apply the technique described by Browder and Hirsch in (1) to change H in $D^{n+k}\times(0,1)$ so that $H^{-1}(M)$ defines an h-cobordism between $M' = H^{-1}(M) \cap D^{n+k}\times 0$ and $L' = H^{-1}(M) \cap D^{n+k}\times 1$. Here we use $bM \neq 0$ when n is odd.

Let $J: L\times I \rightarrow H^{-1}(M)$ be a PL-homeomorphism which carries $L\times i$ onto $H^{-1}(M) \cap D^{n+k}\times i$, $i = 0,1$. Let $c_i = J/L\times i$, $i = 0,1$. Then $H \cdot J$ defines a homotopy between

$$(c_M/M)c_0 \quad \text{and} \quad (T(b(g))/\text{O-section})(c_L/L)c_1$$

or

$$(c_M/M)c_0 \quad \text{and} \quad g(c_L/L)c_1 \quad .$$

So g is homotopic to a PL-homeomorphism.

If M is closed let $M_o = M\text{-int } D^n$ and $L_o = L\text{-int } D^n$. Change g by a homotopy so that $g_o = g/L_o$ induces a homotopy equivalence

$$g_o : (L_o, bL_o) \to (M_o, bM_o)$$

and g/D^n is a PL-homeomorphism. (see Lemma 21)

Then if g preserves normal invariants we get a homotopy

$$H: S^{n+k} \times I \longrightarrow T(v(M))$$

between c_M and $T(b(g))c_L$ which is t-regular to $M \xrightarrow{\text{0-section}} T(v(M))$. Then we can remove a tubular neighorhood in $S^{n+k} \times I$ of an arc in $H^{-1}(M)$ connecting L and M in $bH^{-1}(M)$ to show that g_o preserves normal invariants. Then g_o is homotopic to a PL-homeomorphism and thus g is. (Lemma 21).

Now the obstruction theory of Section I may be applied to the construction problem for $(W,S;f)$. We need to satisfy the conditions of Definition 1.4.

1) First write W as the composition of admissable cobordisms $W_1 \cup W'$ where $W_1 = S \times I \cup$ (2-handles) and $S' = bW_1\text{-int } S$ and W' are each simply connected. We can do this using standard surgery techniques in D^{n+k} since $\pi_1(W) = \pi_1(D^{n+k}\text{-}n \text{ complex}) = 0$ and $\pi_1(S)$ is finitely generated. Since $P_1 = 0$ we can make a construction (Q_1, F_1) for $(W_1, S; f)$. Then if $f' = F_1/S'$, we have reduced the problem to making a construction for $(W', M'; f')$ where $\pi_1(W') = \pi_1(S') = 0$.

2) (W', S') is a smoothable admissable cobordism.

a) W' is an $n+k$-PL submanifold of D^{n+k}. Thus W' can be smoothed by (8).

b) Using excision and the exact sequence of the pair (D^{n+k}, M^*) where $M^* = v(M) \cup W_1 = v(M) \cup$ (2 handles) we see that

$$\begin{aligned} H^{i+1}(W', S') &= H^{i+1}(D^{n+k}, M^*) \\ &= H^i(M^*) \\ &= H^i(M \cup 2 \text{ handles}) \\ &= 0 \quad \text{for } i > n. \end{aligned}$$

Furthermore since $W' = D^{n+k}$-(n complex), W' is $(k-1)$-connected.

Thus if we take k large the conditions of Definition 1.4 are satisfied. This proves

<u>Theorem 1.17</u> Let $g:(L, bL) \to (M, bM)$ be a homotopy equivalence of connected PL n-manifolds, $n \geqslant 1$. Then the cobordism obstruction theory may be applied to the problem of showing that g preserves normal invariants. The obstructions may be considered to lie in

$$H^i(M^*, P_i), \quad 0 < i < n$$

where M^* is any simply connected finite complex obtained from M by attaching 2-disks.

We state Theorem 1.17 for non-simply connected manifolds because there is some hope of proving an analog of iii) $\Longrightarrow$ i) of Theorem 1.16 for certain non-simply connected manifolds (see 30).

We will apply Theorem 1.17 to construct PL-homeomorphisms from homotopy equivalences. Other applications may be made to "construct normal invariants" needed for certain embedding and isotopy theorems (see 2,3).

<u>Corollary 1:</u> Let $g:(L,bL) \longrightarrow (M,bM)$ be a homotopy equivalence of PL n-manifolds and suppose that $n \geq 6$, $\pi_1(M) = 0$, and $\pi_1(bM) = 0$. Then there exists an obstruction theory for the problem of deforming g into a PL-homeomorphism. The obstructions lie in

$$H^i(M,P_i), \quad 0 < i < n \ .$$

<u>Corollary 2:</u> Let $g:(L,bL) \longrightarrow (M,bM)$ be a homotopy equivalence, M as above, and suppose that

$$H^{4i+2}(M,Z_2) = 0 \qquad 4i+2 < n$$

$$H^{4i}(M,Z) = 0 \qquad 0 < 4i < n \ .$$

Then g is homotopic to a PL-homeomorphism.

Corollary 2 is almost best possible if one only demands that g be a homotopy equivalence. See Examples, Chapter 6 Part II. There are counterexamples to Corollary 2 for most values of k and d when M has the homotopy type of

$$S^{2k} \quad \text{or} \quad S^{2k-1} \cup_{\deg.d} e^{2k} \ .$$

<u>Corollary 3:</u> Let $g:(L,bL) \longrightarrow (M,bM)$ be as above and let $p_i(X)$ denote the $i^{\underline{th}}$ rational Pontryagin class of the PL-manifold X. Then if

1) $H^{4i+2}(M,Z_2) = 0 \quad . \quad 4i+2 < n$
2) $H^{4i}(M,Z)$ is free
3) $g^*(p_i(M)) = p_i(L)$

then g is homotopic to a PL-homeomorphism.

<u>Proof:</u> By Corollary 1 the only possible non-zero obstruction to deforming g into a PL-homeomorphism lie in

$$H^{4i}(M,Z) \quad .$$

These are related by the isomorphisms in b) above to those in making a cobordism construction for $(W,S;f)$. Condition 3) implies that the rational Pontryagin classes of the tangent bundle of $g^*v(M)$ are zero. Thus Theorem 1.14 applies to show that the above obstructions are zero.

<u>Corollary 4:</u> If $h:L \longrightarrow M$ is a homeomorphism, $H^{4i+2}(M,Z_2) = 0$; $4i+2 < n$, and $H^{4i}(M,Z)$ is free, then h is homotopic to a PL-homeomorphism. (dim $M \geqslant 6$, $\pi_1(M)$ $\pi_1(bM) = 0$).

<u>Proof:</u> According to Novikov (20) Corollary 2 applies to h.

The condition $H^{4i}(M)$ is free in Corollary 4 may be weakened somewhat. See Corollary 6 to Theorem 26 in Part II. It may be weakened even more if g is known to be a PL-tangential equivalence. This is still not enough, however, to conclude even that M and L are PL-homeomorphic. For example when M is complex projective 4-space this is not true. The crucial point here is the fact that

$$H^6(CP^4, Z_2) \neq 0.$$

In general the obstructions in $H^{4i+2}(M, Z_2)$ are hard to handle, and little is known.

Triviality theorems for these obstructions (in dimensions such that $\Theta_{4i+1}(\partial\pi) = Z_2$) can be related by geometric arguments to hoped for properties of the Hurewicz homomorphism for PL/O or B_{PL}, but this is not too helpful because these problems are quite hard.

Also there is a Bockstein operation relating the obstructions in the above theory to those in an analogous smooth obstruction theory. This relationship would be useful when $\Theta_{4i+1}(\partial\pi)$ is a non-trivial direct summand of Θ_{4i+1}.

Some of these relationships can be readily seen from the viewpoint of Part II.

Now we give a brief skeletal description of the obstruction theory. Suppose that <u>g is homotopic to a PL-homeomorphism over the k-skeleton of M</u> in the sense of

Definition 22. Let M_i denote a thickened i-skeleton of M, with $M_i \subset \operatorname{int} M_{i+1}$.

Now using the fact that g/M_k is a PL-homeomorphism we can make a partial cobordism construction for $(W,S;f)$ over the k-skeleton of (W,S).

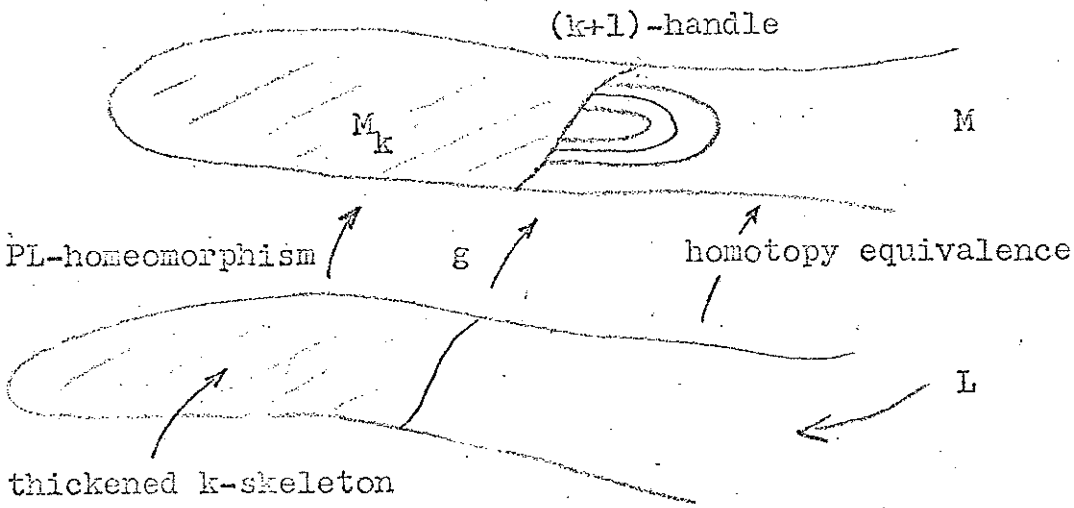
By making the argument in Theorem 1.16 relative we can prove that this construction can be extended over the $(k+1)$ skeleton of (W,S) iffi g can be changed by a homotopy in the complement of M_k so that it becomes a PL-homeomorphism over the $(k+1)$-skeleton of M.

In this way we show that the obstruction theory of Section I corresponds precisely to an obstruction theory for the problem of deforming g to a PL-homeomorphism "over the skeletons of M." (Such an obstruction theory is derived in a different manner in Part II.)

One can see the geometric significance of the obstructions by viewing the cobordism construction for $(W,S;f)$ as a process for building a disk, D^{n+k} around $g^*(v(M))$ and extending the bundle map covering g to a homotopy equivalence of (D^{n+k}, bD^{n+k}) which carries the complement of $g^*(v(M))$ to the complement of $v(M)$.

This point of view provides the geometric motivation for proving lemmas about the obstructions and provides a good model for working with the obstruction theory.

Another geometric point of view may be obtained by working directly in the manifold. Recall that g is a PL-homeomorphism over the k-skeleton of M, and M_i denotes a thickened i-skeleton of M.



We can think of forming M_{k+1} from M_k by attaching (k+1)-handles to bM_k. Now change g in the complement of M_k so that it is t-regular to the framed core disks of the handles and then look at g^{-1} of these. We get a sequence of framed submanifolds of L, which intersect bM_k in a sequence of k-spheres.

These framed submanifolds may be constructed so that they are contractible in L and thus determine elements in P_{k+1} (by using $L\times D^r$ if necessary).

A careful look at the construction for (W,S;f) arising from g/M_k shows that these submanifolds correspond in P_{k+1} to the values of the corbordism obstruction cochain on a basis for $H_{k+1}(M_{k+1},M_k)$.

This computation of the obstruction is useful when g is known to be a tangential equivalence. In fact, the essence of Corollary 7 Theorem 26 can be seen this way.

PART II

Piecewise Linear Structures on Poincaré Spaces

A connected CW pair (X,A) is called a relative Poincaré space of dimension $(n+1)$, if $H_{n+1}(X,A) = Z$, $\partial : H_{n+1}(X,A) \to H_n(A)$ is an isomorphism and the vertical maps in the diagram

$$\begin{array}{ccccccccc}
\cdots \to & H^k(X,A) & \to & H^k(X) & \to & H^k(A) & \to & \cdots \\
& \downarrow{\scriptstyle \cap g} & & \downarrow{\scriptstyle \cap g} & & \downarrow{\scriptstyle \cap \partial g} & & \\
\cdots \to & H_{n+1-k}(X) & \to & H_{n+1-k}(X,A) & \to & H_{n-k}(A) & \to & \cdots
\end{array}$$

are isomorphisms.

If we suppose that A and X are simply connected, then Spivak (33) has shown that (X,A) has a unique stable normal bundle in the category of homotopy objects. That is, there is a unique (up to fibre homotopy equivalence) spherical fibre space over (X,A) whose (relative) Thom space has spherical fundamental homology class.

Let $f: X \to B_F$ be the classifying map for this spherical fibre space.

$$\begin{array}{ccc}
 & & B_{PL} \\
 & \overset{\bar{f}}{\nearrow} & \downarrow \\
X & \xrightarrow{f} & B_F
\end{array}$$

The problem of factoring f through the classifying space B_{PL} is essentially the problem of finding a

PL-manifold with boundary (M, bM) which is homotopically equivalent to (X,A). In fact, if $n \geq 5$ and (X,A) is simply connected $A \neq 0$, Browder and Hirsch (Compare (26) and (1)) have shown that if $\bar{f}$ is a lifting of f, then there exists a PL-manifold pair (M, bM) and a map $g: (M, bM) \to (X, A)$ so that g is a homotopy equivalence and $\bar{f}g$ classifies the stable PL normal bundle of M.

In the next three sections we will classify the pairs (M, g) by the different liftings $\bar{f}$.

We assume then that at least one lifting exists. Thus by the theorem of Browder and Hirsch we may replace (X,A) by a PL manifold with boundary (M, bM).

We will work in the category C whose objects are pairs (K,L) consisting of a locally finite simplicial complexes K and a subcomplex L and whose morphisms are PL-maps

$$f: (K,L) \longrightarrow (K',L').$$

($f: K \to K'$ is PL if there exists a rectilinear subdivision of $\bar{K}$, $\bar{K} \to K$ so that $\bar{K} \to K \xrightarrow{f} K'$ maps each simplex of K linearly into a simplex of L.)

By a homotopy of f we mean a homotopy of f in the category, that is the subspace is mapped to the subspace during the homotopy. Thus

$$f: (K,L) \longrightarrow (K',L')$$

is a homotopy equivalence if there exists

$$g: \quad (K',L') \longrightarrow (K,L)$$

so that fg is homotopic to the identity of (K',L') (keeping L' in L') and gf is homotopic to the identity of (K,L) (keeping L in L).

Remark Let $f: (K,L) \longrightarrow (K',L')$ be such that $f: K \longrightarrow K'$ and $f/L: L \longrightarrow L'$ are homotopy equivalences. Then f is a homotopy equivalence in C.

Proof: We prove this in the category of CW-complexes. The result then follows by simplicial approximation.

By taking the mapping cylinder of f we may suppose that f is an inclusion so that $L' \cap K = L$.

Step 1 K is a strong deformation retract of $L' \cup K$. This follows from the fact that $L \subset L'$ is a homotopy equivalence and $L' \cap K = L$.

Step 2 $\pi_i(K', L' \cup K) = 0$ for all i. This follows from the exact sequence for the triple $K \subset L' \cup K \subset K'$. Thus $L' \cup K$ is a strong deformation retract of K'.

Step 3 Compose the deformation of step 2 with that of step 1 to get a strong deformation retraction of (K',L') onto (K,L) (preserving subspaces).

Let $\mathcal{M} \subset C$ be the subcategory of compact PL-manifold pairs (M,bM) with or without boundary and PL-maps. We will define what we mean by a PL-structure on (M,bM) and then compute these on certain subcategorys $\bar{\mathcal{M}}_n$, $n \geq 5$.

<u>Definition 1</u> A <u>piecewise linear homotopy structure on (M, bM)</u> is a PL-manifold pair (L, bL) and a homotopy equivalence $g\colon (L, bL) \to (M, bM)$. To be brief we call the pair (L, g) a PL-structure on M.

<u>Definition 2</u> We say that the PL-manifold M^n is <u>properly</u> embedded in the PL-manifold W^{n+k}, $k = 0, 1$ if

1) for $k = 0$, M is PL-embedded in the interior of W or M is PL embedded in W so that $bM \cap bW$ is an $(n-1)$ dimensional submanifold of bM.

2) for $k = 1$, M is PL-embedded in bW.

Note that if M is properly embedded in W, $W\text{-int } M$ is a PL-manifold with boundary in case 1) and $bW\text{-int } M$ is a PL-manifold with boundary in case 2).

<u>Definition 3</u> Let $a = (Q, g)$ be a PL-structure on W and suppose that M is properly embedded in W. We say that <u>a induces a PL-structure on M</u> if there is a PL-manifold L properly embedded in Q so that

$$\begin{tikzcd}
 & L \arrow[r, "g/L"] \arrow[dl, hook] & M \arrow[dr, hook] & \\
Q \arrow[rrr, "g"] & & & W \\
 & Q\text{-int } L \arrow[r, "g/Q\text{-int } L"] \arrow[ul, hook] & W\text{-int } M \arrow[ur, hook] &
\end{tikzcd}$$

is commutative and

$(L, g/L)$ is a PL- structure on M

$(Q\text{-int } L,\ g/Q\text{-int } L)$ is a PL-structure on W-int M.
in case 1).

$(bQ\text{-int } L,\ g/bQ\text{-int } L)$ is a PL-structure on bW-int M
in case 2).

We write a/M for the induced PL-structure on M.

We will show later that in most cases any structure on W may be changed slightly so that it induces a PL-structure on a properly embedded submanifold M.

<u>Definition 4</u> Let a be a PL-structure on M which is properly embedded in W. <u>A PL structure b on W is said to extend a</u> if b induces a PL-structure on M and $b/M = a$.

<u>Remark</u> We note here that if M^n is properly embedded in W^{n+1} and $a = (L, g)$ is a PL-structure on M, then the problem of extending a to a PL-structure on all of W is precisely the cobordism problem discussed in section 1 for admissable pairs (W, M).

<u>Definition 5</u> (Concordance) Let $a_i = (M_i, f_i)$ $i = 0,1$ be two PL-structures on (M, bM). Then a_o and a_1 induce a PL-structure on $(M \times 0) \cup (M \times 1)$ which is properly embedded in $M \times I$. We say that <u>a_0 is concordant to a_1</u> if this PL-structrue extends to a PL-structure on all of $M \times I$.

Concordance defines an equivalence relation* on the set of PL- structures on M. Let $PL(M)$ denote the set of equivalence classes. Let 0 in $PL(M)$ denote the concordance class of (M, id). (*) See Lemma H in Chapter 2.

<u>Definition 6</u> a)(The category $\mathcal{M}_n$) For $n \geq 6$ let $\mathcal{M}_n$ denote the category

1) whose objects are PL n-manifolds (M, bM) such that $bM \neq 0$, $\pi_1(M) = 0$ and $\pi_1(bM) = 0$.

2) whose morphisms are <u>embeddings</u> $i: M_1 \longrightarrow \text{int}\, M_2$ so that $\pi_1(M_2 - M_1) = 0$.

For $n < 6$ let $\mathcal{M}_n$ be the void category.

b)(The category $\overline{\mathcal{M}}_n$) For $n \geq 6$ let $\overline{\mathcal{M}}_n$ be the enlarged category defined by conditions 1) and 2) without the assumption that $bM \neq 0$. Let $\overline{\mathcal{M}}_5$ be the category of closed simply connected PL 5-manifolds and PL- homeomorphisms. Let $\overline{\mathcal{M}}_n$ be the void category for $n < 5$.

An application of Van Kampen's shows that $\mathcal{M}_n$ and $\overline{\mathcal{M}}_n$ are categories.

We will compute $PL(M)$ for M in $\mathcal{M}_n$ (and then $\overline{\mathcal{M}}_n$). Our applications to the problem of deforming a homotopy equivalence into a PL-homeomorphism and the study of pseudo-isotopy of PL- homeomorphisms will make use of the PL- h cobordism theorem for $\overline{\mathcal{M}}_n$.

<u>Theorem</u>. (h-cobordism) If M is in $\mathcal{M}_n$, then the relative h-cobordism theorem is true for (M, bM).

(The relative h-cobordism theorem asserts that if (W, W') is an h-cobordism of (M, bM) then W' is equivalent to the product cobordism $bM \times I$ and any such equivalence extends to an equivalence of W with the product cobordism $M \times I$.)

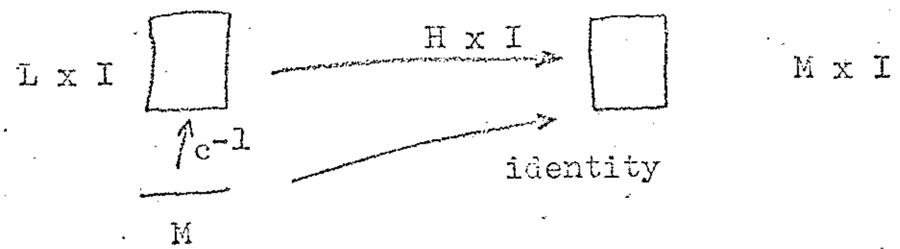
If M is smoothable this theorem follows from Smale (21). In the general PL-case it is indicated in work of Mazur (13) and Zeeman (32).

We will use this theorem in the applications to straighten out concordances.

<u>Lemma 1</u> Let $g : (L, bL) \to (M, bM)$ be a homotopy equivalence. Then if g is homotopic to a PL-homeomorphism, (L, g) is concordant to (M, id).

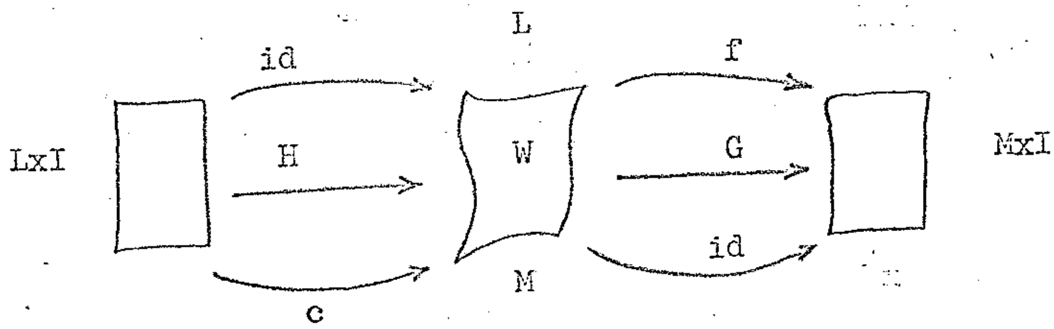
Conversely, if (L, g) is concordant to (M, id) and M belongs to $\mathcal{M}_n$ then f is homotopic to a PL-homeomorphism.

<u>Proof:</u> Suppose there exists $H : (L, bL) \times I \to (M, bM)$ so that $H(x, 0) = f(x)$ and $H(x, 1)$ is a PL-homeomorphism c.



Then $(W,G) = (L \times I \cup_c M,\ H \times I \cup c^{-1}\text{ identity})$ gives a concordance between (L,f) and (M,id). Thus if f is homotopic to a PL-homeomorphism (L,f) is concordant to zero.

Now suppose (W,G) is a concordance between (L,f) and (M,id).



Now W defines an h-cobordism between (L,bL) and (M,bM). By the h-cobordism theorem for M we may choose a PL-homeomorphism

$$H\colon\ L \times I \longrightarrow W$$

so that $H/L \times 1$ is the identity map of L and $c = H/L \times 0$ is onto M. Then $G \circ H$ is a homotopy between f and the PL-homeomorphism c. Q.E.D.

Let PL be the function which assigns to each M in $\mathfrak{M}_n$ the set $PL(M)$. We will now show how a morphism of $\mathfrak{M}_n$

$$i\colon\quad M_1 \rightarrow M_2$$

induces a map

$$i^*\colon\quad PL(M_2) \rightarrow PL(M_1)\ .$$

<u>Theorem</u> (Browder) Let $i: M_1 \to M_2$ be a morphism of $\bar{\mathcal{M}}_n$ and suppose $g: (L_2, bL_2) \to (M_2, bM_2)$ defines a PL- structure on M_2. Then we can change g by a homotopy so that g induces a PL-structure on $M_1 \subset M_2$. The induced PL-structure is unique up to a concordance which is embedded in $L_2 \times I$.

<u>Proof:</u> This is a restatement in our terminology of Browder's relative codimension one embedding theorem (2). He has a further hypothesis on H_2 which has since been removed by J. B. Wagoner (25).

Therefore given $i: M_1 \to M_2$ in $\bar{\mathcal{M}}_n$ we may define

$$i^*: PL(M_2) \to PL(M_1)$$

using the Browder Codimension One Theorem. This is well-defined on concordance classes by the uniqueness part of the Browder Theorem. (The construction of the concordance only uses the fact that $L_2 \times I$ is an h-cobordism.)

<u>Corollary 1:</u> If we let S be the category of pointed sets and base point preserving functions then the assignment

$$M \to PL(M)$$
$$i \to i^*$$

makes

$$\bar{\mathcal{M}}_n \xrightarrow{PL} S$$

into a contravariant functor.

Proof: Let 0 be the preferred element of $PL(M)$. On representatives we verify that

$$i^*(0) = 0.$$

To show naturality let $M_0 \longrightarrow M_1$ and $M_1 \longrightarrow M_2$ be morphisms of $\mathcal{M}_n$ and let (L_2, g) define a PL-structure on M_2. Change g to g' by a homotopy so that $g'/L_1 = g'^{-1}(M_1)$ induces a PL-structure on M_1. Change $f = g'/L_1$ by a homotopy so that $f/L_0 = f^{-1}(M_0)$ induces a PL-structure on M_0. Extend the homotopy of f to a homotopy of g', then g' induces a PL-structure on M_0. Then the naturality is verified on representatives.

We remark that the Browder theorem actually implies that

$$i^*: \ PL(M_2) \longrightarrow PL(M_1)$$

may be defined for proper embeddings

$$i : M_1^n \longrightarrow M_2^n$$

such that $L = bM_2 \cap bM_1 \neq 0$ if the further conditions

$$\pi_1(bM_2 - L) = \pi_1(bM_1 - L) = 0$$

and L is in $\mathcal{M}_{n-1}$ are satisfied.

If $i_1 : M_0 \rightarrow M_1$ and $i_2 : M_1 \rightarrow M_2$ are two such embeddings, then one can show just as in the corollary that

$$(i_2 i_1)^* = i_1^* \, i_2^*$$

in case $bM_0 \cap bM_1 \cap bM_2 = 0$.

In the next few sections we will show that $PL / \mathcal{M}_n$ is naturally equivalent to a functor which is defined on a category of topological spaces and continuous maps (containing $\mathcal{M}_n$). We then apply Brown's theory (4) to represent PL as

$$[\ ,\ Y]$$

for some space Y. With this approach we can get a fairly good hold on homotopy properties of Y .

Another approach might be the following. Define $\mathcal{M}_n \to \mathcal{M}_{n+1}$ by $M \to M \times I \subset M \times 3I$, $i \to i \times I$. Let $\mathcal{M} =$ dir lim $\mathcal{M}_n$, and replace embeddings by isotopy classes of embeddings. Let $\mathcal{C}$ be the category of finite simply connected complexes and continuous maps and define a covariant functor

$$\mathcal{C} \xrightarrow{RN} \mathcal{M}$$

by mapping objects to regular neighborhoods in the interior of $D^n \subset D^n \times I \subset \ldots$ and maps to isotopy classes of embeddings which are covered by ambient isotopies of $D^n \subset D^n \times I \subset \ldots$.

One can show that PL is well-defined on $\mathcal{M}$ (see Theorem 23). Therefore we have a contravariant functor H

$$\begin{tikzcd} \mathcal{C} \arrow[r, "RN"] \arrow[rr, bend right, "H"'] & \mathcal{M} \arrow[r, "PL"] & S \end{tikzcd}$$

from $\mathcal{C}$ to S which we can try to represent as $[\ \ ,Y]$ using Brown's theory.

The homotopy groups of Y can be computed directly using surgery, Theorem 23 implies that PL is determined by its behaviour on image (RN), thus we have obtained quite a bit of information about PL.

One difficulty is the fact that $\mathcal{C}$ contains only simply connected finite complexes. So there may be a problem applying Brown's theory.

This can probably be overcome. A more serious drawback is the fact that this definition of Y is somewhat remote, and a closer study of its homotopy properties and its relation to B_{PL} seems harder.

This approach can be used to study the concordance classes of smoothings of a PL- manifold.

The Classifying Bundle of a PL-Structure

Now we consider a bundle theory which will be used to study the functor

$$\mathcal{M}_n \xrightarrow{PL} S .$$

Definition 7 Let X be a locally finite simplicial complex. An $F/PL)_n$-bundle over X is a pair (E,t) consisting of a PL n-disk bundle $E \to X$ and an F-trivialization t. That is, we have

$$\begin{tikzcd} E \arrow[r, "t"] \arrow[d, "D^n"] & D^n \\ X & \end{tikzcd}$$

where t induces a homotopy equivalence of each fibre pair with (D^n, bD^n).

Definition 8 Two $F/PL)_n$-bundles (E_i, t_i) $i = 0,1$ are said to be equivalent if there is an $F/PL)_n$-bundle (E,T) over $X \times I$ so that

$$\begin{tikzcd} E_i \arrow[r, "b_i"] \arrow[d] \arrow[rr, bend left, "t_i"] & E \arrow[r, "T"] \arrow[d] & D^n \\ X \arrow[r, "\times i"] & X \times I & \end{tikzcd} \qquad i = 0,1$$

is commutative where b_0 and b_1 are PL bundle maps.
Let $B_n(X)$ denote the set of equivalence classes of $F/PL)_n$-bundles over X.

<u>Definition 9</u> (The Induced Bundle)

If $f: X \to Y$ is a piecewise linear map, define $B_n(f): B_n(Y) \to B_n(X)$ on representatives by

$$\begin{tikzcd} f^*E \arrow[r, "\bar{f}"] \arrow[d] & E \arrow[r, "t"] \arrow[d] & D^n \\ X \arrow[r, "f"] & Y & \end{tikzcd} ,$$

that is,
$$(E,t) \longrightarrow (f^*E,\ t\,\bar{f}) .$$

Its easy to show that $B_n(f)$ is well defined on equivalence classes.

<u>Definition 10</u> (<u>The Whitney Sum</u>) If x_1 in $B_n(X)$ is represented by (E_1,t_1) and x_2 in $B_m(X)$ is represented by (E_2,t_2) define the Whitney sum $x_1 \oplus x_2$ in $B_{n+m}(X)$ on representatives by

$$\begin{tikzcd} E_1 \oplus E_2 \arrow[r, "\bar{d}"] \arrow[d] \arrow[rr, bend left, "t_1 \oplus t_2"] & E_1 \times E_2 \arrow[r, "t_1 \times t_2"] \arrow[d] & D^n \times D^m \\ X \arrow[r, "diag."] & X \times X & \end{tikzcd} ,$$

where $E_1 \oplus E_2 = (\text{diag})^* E_1 \times E_2$ and $t_1 \oplus t_2$ is the indicated composition.

Let 0 in $B_n(X)$ be represented by $(X \times D^n, p_2)$. Let $\mathcal{C}$ be the category of countable connected locally finite simplicial complexes and PL-maps and S the category of pointed sets and base point preserving functions.

<u>Lemma 2</u> The association

$$X \longrightarrow B_n(X)$$

$$(X \xrightarrow{f} Y) \longrightarrow (B_n(Y) \xrightarrow{B_n(f)} B_n(X))$$

defines a contravariant functor from $\mathcal{C}$ to S.

$$\mathcal{C} \xrightarrow{c} S$$

B_n satisfies the following properties:

1) $B_n(pt) = \{0\}$.

2) $B_n(S^n)$ is countable. $B_n(S^1) = \{0\}$, $n > 2$.

3) If $X = \bigcup X_i$, X_i increasing subcomplexes of X, and if there exists x_i in $B_n(X_i)$ so that $x_i/X_{i-1} = x_{i-1}$, then there exists x in $B_n(X)$ so that $x/X_i = x_i$.

4) If $X = X_1 \cup X_2$ with $X_1 \cap X_2 = A$, and x_1 in $B_n(X_1)$, x_2 in $B_n(X_2)$ satisfy

$$x_1/A = x_2/A$$

then there exists x in $B_n(X)$ so that $x/X_i = x_i$, $i = 1,2$.

5) $B_n(f)$ only depends on the homotopy class of f.

<u>Proof:</u> The first statement is clear. Now for the properties.

1). $B_n(pt) = \{0\}$ because any homotopy equivalence of (D^n, bD^n) is homotopic to a PL-homeomorphism.

2) The statement about $B_n(S^1)$ follows from Lemma 1.5 and Lemma 19.

$B_n(S^k)$ is countable because

a) $B = \pi_{k-1}$ (PL automorphisms of D^n) is countable, (D^n is a finite simplicial complex).

b) T = the set of homotopy classes of F-trivializations of the trivial n-disk bundle over S^k is countable.

c) A subset of $B \times T$ maps onto $B_n(S^k)$.

3) Consider $X' = (X_1 \times I) \cup X_2 \times I \cup \ldots$ where $X_i \times 1$ is attached to $X_i \subset X_{i+1} \times 0$. If we choose representatives v_i for the x_i and apply the definition of equivalence we see that there is an $F/PL)_n$-bundle v over X' so that $v/X_i \times 0 = v_i$. Now there is a homotopy equivalence $X' \xrightarrow{p} X$ so that

$$\begin{tikzcd} X' \arrow[rr, "p"] & & X \\ & X_i \arrow[ul, "\times 0"] \arrow[ur, hook] & \end{tikzcd}$$

is commutative. So 3) follows from 5). (Let x in $B_n(X)$ be such that $p^*x = (v)$.)

4) Consider $X' = X_1 \cup A \times I \cup X_2$ where $A \times i$ is identified to A in X_{i+1}. Now proceed as in 3).

5) If f is homotopic to g by H then $H^*(v)$ gives an equivalence between f^*v and g^*v.

Let a belong to $PL(M)$. We want to "classify" a by an element in $B_k(M)$, k large.

<u>Definition 11</u> Let $f: (M,bM) \to (L,bL)$ a map of PL n-manifold pairs. A <u>k-tubular neighborhood of f</u> is a pair (E,i) consisting of a PL k-disk bundle over M ($E \xrightarrow{p} M$) and a PL embedding of $(E,E/bM)$ in $(L,bL) \times \operatorname{int} D^k$ so that

$$\begin{array}{ccc} (E,E/bM) & \xrightarrow{i} & (L,bL)\times \operatorname{int} D^k \\ \downarrow p & & \downarrow p_1 \\ (M,bM) & \xrightarrow[f]{} & (L,bL) \end{array}$$

is homotopy commutative.

In what follows $\bar{f}$ denotes a bundle map covering f, and f' denotes a homotopy inverse for f if f happens to be a homotopy equivalence

<u>Lemma 3</u> Let $g:(L,bL) \to (M,bM)$ be a homotopy equivalence. Then g has a k-tubular neighborhood (E,i), if $k \geq n+3$. Furthermore, i can be chosen so that

$$(M,bM) \times 0 \subset i(E',E'/bM)$$

where E' denotes the associated open disk bundle.

<u>Proof:</u> Approximate g by an embedding $\tilde{i}$: $(L,bL) \to (M,bM) \times \operatorname{int} D^k$. By the stable PL-tubular neighborhood theorem (6) we can find a bundle $E \xrightarrow{p} L$

and an embedding i extending $\tilde{\imath}$ (p/0-section). Then (E,i) is a tubular neighborhood of g.

Let $j:(M,bM) \to (E',E'/bM)$ be an embedding approximating $\tilde{\imath}g'$. Then j and the inclusion

$$(M,bM) \xrightarrow{x0} (M,bM) \times \operatorname{int} D^k$$

are homotopic and therefore isotopic since $k \geq n+3$. By (31) we may choose an ambient isotopy I of the identity of $(M,bM) \times D^k$ relating $x\,0$ and j. Then $(E, I(x,1)^{-1}\, i)$ has the desired property.

<u>Lemma 4</u> Let $g:(L^n,bL) \to (M^n,bM)$ be a homotopy equivalence. Let (E,i) be a k-tubular neighborhood of g' which has the additional property of Lemma 3. Then if L is compact, there is a k-disk $0 \subset D_1^k \subset \operatorname{int} D^k$ so that

$$\text{fibre} \subset E \xrightarrow{i} M \times D^k \to D^k$$

induces a homotopy equivalence t_x

$$(p^{-1}(x),\ b(p^{-1}(x)) \xrightarrow{t_x} (D^k, D^k - D_1^k)$$

for each x in L.

<u>Proof:</u> Using the compactness of L it is easy to see that D_1^k may be chosen (independent of x) so that t_x is well-defined for each x in M.

Now each fibre disk represents the Lefshetz Dual of a generator of $H^n(E,E/bM) = H^n(L,bL)$. Now $i(L,bL)$ is homologous to $(M,bM) \times 0$ so it is clear that the composition

$$\text{fibre} \xrightarrow{i} M \times D^k \xrightarrow{p_2} D^k$$

induces the desired relative homology isomorphism in dimension k. The lemma follows.

From Lemma 4 we see that a "good" k-tubular neighborhood of $g:(L,bL) \longrightarrow (M,bM)$ (with the additional property of Lemma 3) yields an $F/PL)_k$ bundle, namely

$$(E, rp_2i)$$

where

r is a h. e. $(D^k, D^k - D^k_1) \longrightarrow (D^k, bD^k)$.

<u>Definition 12</u> Let (L,g) be a PL- structure on M and g' a homotopy inverse for g. A <u>classifying bundle</u> for (L,g) is an $F/PL)_k$-bundle (E,t) where

i) t is given by the composition

$$E \xrightarrow{j} L \times D^k \xrightarrow{r} D^k$$

where (E,j) is a "good" tubular nghd of $M \xrightarrow{g'} L$ and,

ii) r is an F-trivialization of $L \times D^k \xrightarrow{p_1} L$ such that $r^{-1}(0) = L \times 0$ and r agrees with $L \times D^k \xrightarrow{p_2} D^k$ on a neighborhood of $L \times 0$.

<u>Theorem 6:</u> Suppose $k > n + 3$ and (L,g) is a PL-structure on M^n. Then the equivalence class in $B_k(M)$ of any k-dimensional classifying bundle for (L,g) depends only on the concordance class of (L,g).

The correspondence

$$PL(\text{structure}) \longrightarrow (\text{classifying bundle})$$

defines a natural transformation of contravariant functors an M_n.

<u>Proof:</u> The first statement follows immediately from Lemma 4' below applied to $(M \times I,\ M \times 0 \cup M \times 1)$.

If $M_1 \xrightarrow{i} M_2$ is a morphism of $\mathcal{M}_n$, and (L_2, g_2) is a PL-structure on M_2 which restricts to a PL-structure (L_1, g_1) on M_1, then any classifying bundle for (L_1, g_1) restricts to a classifying bundle for $(bL_1,\ g_1/bL_1)$. Using Lemma 4' we can extend the latter classifying bundle over M_2-int M_1 so that it is classifying for

$$(L_2\text{-int } L_1,\ \ g_2/\ L_2\text{-int } L_1).$$

The conditions of Lemma 4' imply that these fit together to give a classifying bundle for (L_2, g_2). (See first part of proof of Theorem 9 for more remarks on this.) The naturality is then verified for these representatives.

<u>Lemma 3'</u> Let M^n be embedded in bW^{n+1}. Suppose (L,g) defines a PL-structure on M which extends to a PL-structure (Q,G) on W. Let (E_1,i_1) be a "good" k-tubular neighborhood of g, $k > n+3$. Then there exists a "good" k-tubular neighborhood of $G,(E,j)$, and a bundle map $E_1 \xrightarrow{b} E$ covering $L \subset Q$ so that

$$\begin{tikzcd} E/(Q;L,bL,L',bL') \arrow[r, "j"] \arrow[d, "p"] & (W;M,bM,M';bM') \arrow[d, "p_1"] \\ (Q;L,bL,L',bL') \arrow[r, "G"] & (W;M,bM,M',bM') \end{tikzcd}$$

is homotopy commutative and

$$j/(E/L) \circ b = i_1 .$$

$$(L' = bQ\text{-int } L, \quad M' = bW\text{-int } M)$$

Furthermore, we may assume that there is a homotopy between $p_1 j$ and Gp which extends (using b) a given homotopy between $p_1 i_1$ and gp.

<u>Proof:</u> Let h be a given homotopy between $p_1 i_1$ and gp. Recall that G defines a map of 5-tuples,

$$(Q;L,bL,L',bL') \xrightarrow{G} (W;M,bM,M',bM')$$

which is a homotopy equivalence on each factor.

1) There exists (E, j_1), a tubular neighborhood of G, so that

i)

$$\begin{array}{ccc} E/(Q;L,bL,L',bL') & \xrightarrow{j_1} & (W;M,bM,M',bM') \times \operatorname{int} D^k \\ \downarrow p & & \downarrow p_1 \\ (Q;L,bL,L',bL') & \xrightarrow{G} & (W;M,bM,M',bM') \end{array}$$

is homotopy commutative by a homotopy extending h, and

ii) $(E/L,\ j_1/(E/L))$ defines a "good" tubular neighborhood of $g = G/L$,

$$g: (L,bL) \longrightarrow (M,bM) .$$

Proof of 1):

i) Extend $h \times p_2 i_1/$0-section to a homotopy H'_t of $(\times 0)G$. Change H'_1 by a homotopy which is fixed on the 0-section of E_1 to an embedding. Let H_t be the sum of these two homotopies.

Apply the PL-tubular neighborhood theorem to find a PL-bundle E and an embedding

$$j_1: E/(Q;L,bL,L',bL') \longrightarrow (W;M,bM,M',bM') \times \operatorname{int} D^k$$

so that j_1 / 0-section is the embedding

$$H_1/(\text{0-section}): (Q;L,bL,L',bL') \longrightarrow (W;M,bM,M',bM') \times \operatorname{int} D^k .$$

There is no obstruction to finding a homotopy between $p_1 j_1$ and Gp which extends h.

Now we deform j_1 so that ii) is satisfied.

Now i_1 and $j_1/E/(L,bL)$ define two tubular neighborhoods of i_1(O-section) $= j_1$(O-section) .

So by the PL-tubular neighborhood theorem there is an isotopy I_t of the identity of $(M,bM)\times \text{int } D^k$ which is constant at infinity and so that I_1 carries $j_1(E/(L,bL))$ onto $i_1(E_1/(L,bL))$. Thus

$$(E/L, \quad I_1 j_1)$$

defines a "good" tubular neighborhood of g.

Now we apply the IEP (Isotopy Extension Theorem) successively to extend I_t to an isotopy K_t of the identity of $(W;M,bM,M',bM')\times \text{int } D^k$. Then $(E,K_1 j_1)$ is the desired tubular neighborhood.

2) a) There is an isotopy K_t' of

$$(M',bM') \xrightarrow{\times 0} (M',bM')\times \text{int } D^k$$

so that

i) K_t'/bM' constant isotopy

ii) $K_1'((M',bM')\times 0) \subset j_1(E'/(L',bL'))$

(where E' is the associated open disk bundle of E).

b) K'_t defines an isotopy of

$$bW \xrightarrow{x0} bW \times \text{int } D^k$$

($bW = M \cup M'$) which is the constant isotopy on M. There exists an extension of K'_t to K_t, an isotopy of

$$(W, bW) \xrightarrow{x0} (W, bW) \times \text{int } D^k .$$

c) There exists an isotopy I_t of

$$(W, bW) \xrightarrow{K_1} (W, bW) \times \text{int } D^k$$

so that

i) $I_t / bW =$ constant isotopy

ii) $I_1 (W, bW) \subset j_1(E', E'/bQ)$.

Proof of 2): a) First we produce a homotopy H'_t of

$$(M', bM') \xrightarrow{x0} (M, bM) \times \text{int } D^k$$

with the desired properties.

Now $bM' \times 0 = bM \times 0$ which is contained in $j_1(E'/bL) \subset j_1(E'/bL')$, by 1b). Therefore we may consider the problem of deforming

$$M' \times 0 \text{ into } j_1(E'/L).$$

while keeping $bM' \times 0$ fixed. This problem gives rise to a sequence of obstructions in

$$H^i((M',bM') \times 0;\ \pi_i(M' \times \operatorname{int} D^k,\ j_1(E'/L)))\ .$$

However the coefficient groups all vanish by 1a). Therefore H_t' exists.

To get K_t' we "approximate" H_t' using general position.

Proof of b): Apply the IEP.

Proof of c): $K_1(bW) \subset j_1(E'/bQ)$ by construction so we may proceed just as in the proof of 2a).

3) There is an isotopy J_t of the identity of $(W;M,bM,M',bM') \times D^k$ so that

a) $J_t/(M,bM) \times D^k =$ constant isotopy

b) $J_1((W;M,bM,M',bM')\times 0) \subset j_1(E'/(Q;L,bL,L',bL'))$,

Proof of 3):

Step 1) Apply the IEP to get an isotopy M_t' of the identity of $M' \times D^k$ which is fixed on the boundary and covers $\bar{K}_t'$.

Step 2) M_t' extends to an isotopy bW_t' of the identity of $b(W \times D^k)$-int $(M' \times D^k)$. bW_t' may be extended to an isotopy W_t' of the identity of $W \times D^k$ using the IEP.

Step 3) Apply 2c) to $K_1 = W_1'/(W\times 0)$. We get an isotopy of $W_1'/\ W\times 0$ which carries $W\times 0$ into $j_1(E')$. We apply the IEP to extend this to an isotopy W_t of

$W \times D^k$ which is fixed on the boundary.

Then

$$J_t = \begin{cases} W'_{2t} & 0 \leq t \leq \frac{1}{2} \\ W_{2t-1} \cdot W'_1 & \frac{1}{2} \leq t \end{cases}$$

is the desired isotopy.

Now 3) means that we can take a tubular neighborhood (E, j_1) of

$$(Q; L, bL, L', bL') \xrightarrow{G} (W; M, bM, M', bM')$$

which restricts to a "good" tubular neighborhood of

$$(L, bL) \xrightarrow{g} (M, bM)$$

and change it by an isotopy (of 5-tuples) which is fixed on L so that it becomes "good" everywhere (Isotop j_1 by $J_t^{-1} \cdot j_1$.)

We apply the PL-stable tubular neighborhood theorem to produce a bundle map $E_1 \xrightarrow{b} E/L$ covering the identity and an isotopy L_t between

$$(E_1, E_1/bL) \xrightarrow{i_1} (M, bM) \times \text{int } D^k \quad \text{and}$$

$$\begin{tikzcd}(E_1, E_1/bL) \arrow[r, "b"] \arrow[rr, bend right, "i_2"'] & (E/L, E/bL) \arrow[r, "j_1"] & (M, bM) \times \text{int } D^k.\end{tikzcd}$$

Now

$$L_t \times I\colon E_1 \times I \longrightarrow (M, bM) \times I \times \text{int } D^k$$

defines a tubular neighborhood of

$$(LxI;\ Lx0,\ Lx1,\ bLx(0,1)) \xrightarrow{f} (MxI;\ Mx0,\ Mx1,\ bMx(0,1))$$

which is "good" when restricted to $Lx0 \cup Lx1$, $(f = p_1(L_t xI))$. We apply 3) to this situation to isotop $L_t xI$ to a PL- embedding

$$H:\ E_1 xI \rightarrow (M, bM)\ x\ I\ x\ \text{int}\ D^k$$

so that

$$H/E_1 xs = i_{s+1} \qquad s = 0,1$$

and

$$H(E_1'/(LxI;\ Lx0,\ Lx1,\ bLxI)) \subset (MxI;\ Mx0,\ Mx1,\ bMxI)\ x\ 0.$$

Now let us suppose that over some product neighborhood N of (L,bL) in (Q,bQ) E is identified to the product bundle $(E/(L,bL))xI$ so that j_1 on this subset is just $(j_1/(E/(L,bL)))xI$: $E/N \longrightarrow N'x\ \text{int}\ D^k$: $N' = (M,bM)\ x\ I$ is a collar neighborhood of (M,bM) in (W,bW). Suppose also that I_t is fixed on $N'x\ \text{int}\ D^k$.

So define $E \xrightarrow{j} Wx\ \text{int}\ D^k$ by

$$j(x) = \begin{cases} I_1^{-1} \cdot j_1(x) & x \text{ in } E/Q-N \\ H\ (b^{-1}x\ I) & x \text{ in } E/N \end{cases}$$

Then

a) $j(E'/(Q;L,bL,L',bL')) \supset (W;M,bM,M',bM')x\ 0$

b) $j/(E/L) \cdot b = H(x,0)(b^{-1}x0)\ b = i_1$

c) jp_1 and Gp are homotopic (as maps of 5-tuples) by a homotopy extending $bh_t b^{-1}$. These are clear from the construction . This completes the proof of Lemma 3'.

<u>Lemma 4'</u> Let M be embedded in bW^{n+1}. Let (L,g) be a PL-structure on M which extends to a PL-structure (Q,G) on W. Let (E_1,r_1i_1) be a k-dimensional classifying bundle for (L,g). $k>n+3$. Then (Q,G) has a classifying bundle (E,rj) so that

1) (E,rj) restricts to a classifying bundle for $(L',G/L')$

2) $j/(E/M)\cdot b = i_1$, $r/L\times D^k = r_1$, where $b: E_1 \to E$ is a bundle map covering $M \subset W$.

3) There is a homotopy making

$$\begin{array}{ccc} E/(W;M,bM,M',bM') & \xrightarrow{j} & (Q;L,bL,L',bL')\times \text{int } D^k \\ \downarrow p & & \downarrow p_1 \\ (W;M,bM,M',bM') & \xrightarrow{G'} & (Q;L,bL,L',bL') \end{array}$$

commutative which extends (using b) a given homotopy making

$$\begin{array}{ccc} E_1/(M,bM) & \xrightarrow{i_1} & (L,bL)\times \text{int } D^k \\ \downarrow q & & \downarrow p_1 \\ (M,bM) & \xrightarrow{g'} & (L,bL) \end{array}$$

commutative.

Proof: By Lemma H below there is homotopy inverse for G,

$$G' : (W; M, bM, M', bM') \longrightarrow (Q; L, bL, L', bL')$$

which we may assume restricts to a given inverse g' for $g = G/L$.

Now (E_1, i_1) is a "good" tubular neighborhood of g'. So by Lemma 3' there is a "good" tubular neighborhood (E, j) of G' and a bundle map $E_1 \to E$ so that 3) and the first part of two are satisfied. Also (E, j) restricts to a "good" tubular neighborhood of G'/M'. Thus if we can find a F-trivialization

$$r: \ Q \times D^k \to D^k$$

such that $r^{-1}(0) = Q \times 0$, r agrees with p_2 on a neighborhood of $Q \times 0$, $r/L \times D^k = r_1$, and so that rj defines an F-trivialization of E, then the proof of Lemma 4' will be completed.

It follows from Definition 12 that there is a small concentric disk D_1^k about the origin in D^k so that $r_1/L \times D_1^k = p_2$, and $r_1^{-1}(D_1^k) = L \times D_1^k$. Suppose the radius of D_1^k is 2a. Let

$$r_1' : D^k \longrightarrow D^k$$

be defined by $d \longrightarrow a(|d|)d$, where

$$a(|d|) = \begin{cases} 1 & |d| \leq a \\ \dfrac{1-2a}{2a^2}\, d + \dfrac{4a-1}{2a} & a \leq |d| \leq 2a \\ \dfrac{1}{|d|} & |d| \geq 2a \end{cases}$$

Then $r_a : L \times D^k \longrightarrow D^k$ defined by $r_a(x,d) = r_1^!(d)$ is an F-trivialization of $L \times D^k \rightarrow L$ which satisfies all the appropriate properties and is easily extendable to

$$Q \times D^k \longrightarrow D^k.$$

Define $H: L \times D^k \times I \longrightarrow D^k$ by

$$H(x,d,t) = ((1-t) + t\, a(|d|))d \quad ; \quad |d| \leq 2a$$

$$= \left| r_1(x,d) \right| \left(1-t+\frac{t}{|r_1(x,d)|}\right) \frac{r_1(x,(1-t+\frac{2at}{d})\, d)}{\left|r_1(x,(1-t+\frac{2at}{d})\, d)\right|} \, ; \; |d| \geq 2a$$

Then if $|d| = 2a$,

$$H(x,d,t) = \left(1-t \;\; \frac{t}{|r_1(x,d)|}\right) \;\; r_1(x,d)$$

$$1-t+\frac{td}{|d|} \quad . \qquad \text{So}$$

$H(x,d,t)$ defines a homotopy(thru F-trivializations) between r_a and r_1. Furthermore,

$$\left|H(x,d,t)\right| \geq \left|H(x,d,0)\right| = \left|r_1(x,d)\right| .$$

Now choose a so small that $D_1^k \subset (D^k - r_1 i_1(\text{sphere bundle of } E_1))$. Then $H_t \cdot i_1$ defines a homotopy between $r_1\, i_1$ and $r_a\, i_1$. Let r_a also denote

$$Q \times D^k \xrightarrow{p_2} D^k \xrightarrow{r_1'} D^k \quad .$$

Then define

$$r\colon Q \times D^k \longrightarrow D^k$$

by

$$r = \begin{cases} H_t & \text{on collar neighborhood of } (L, bL) \text{ in } (Q, bQ) \\ r_a & \text{complement of collar neighborhood} \end{cases} .$$

Then its easy to check that r has the desired properties.

<u>Lemma H:</u> Let

$$(X; A_1, \ldots, A_n) \xrightarrow{\quad f \quad} (Y, B_1, \ldots, B_n)$$

be a mapping of $(n+1)$-tuples. Suppose that X and Y are complexes, A_i and B_i are subcomplexes, and $X = \cup A_i$, $Y = \cup B_i$. If S is a non-void subset of $\{1, \ldots, n\}$ let $A_S = \bigcap_{i \text{ in } S} A_i$, $B_S = \bigcap_{i \text{ in } S} B_i$. Suppose f induces a homotopy equivalence

$$f_S\colon A_S \longrightarrow B_S \qquad \text{for each non-void } S.$$

Then f is a homotopy equivalence of $(n+1)$-tuples .

Proof: Suppose first that $n = 2$. We want to show that

$$f: X \longrightarrow Y$$

is a homotopy equivalence. By forming the mapping cylinder of f we can suppose that f is an inclusion so that

$$B_S \cap X = A_S \quad .$$

Using the exact sequence of an appropriate triple (see Remark chapter 1) we can show that

$$\pi_*(B_i, A_i \cup B_1 \cap B_2) = 0 \qquad i = 1,2 \quad .$$

Thus $X \cup B_1 \cap B_2$ is a strong deformation retract of Y, and thus so is X.

By induction we obtain that $f: X \longrightarrow Y$ is a homotopy equivalence for any n.

Now we can finish the proof by a similar argument with the mapping cylinder. (See 17)

C Is a Natural Equivalence

Let M^n be properly embedded in W^{n+k}, $k=0,1$. Let (L,g) be a PL-structure on M^n. We want to consider the problem of extending this PL-structure over W.

Suppose $k=0$ and M^n is embedded in the interior of W. Then bM^n is properly embedded in $b(W-\text{int } M)$ and $(bL, g/bL)$ defines a PL-structure on bM^n. The problem of extending (L,g) over W is equivalent to the problem of extending $(bL, g/bL)$ over $W-\text{int} M$.

Suppose $k=0$ and $bM^n \cap bW = N \neq 0$. If N and $bM-\text{int } N$ belong to $\mathcal{M}_{n-1}$, we can apply the Browder Theorem to change g by homotopy so that (L,g) induces a PL-structure say (L',g') on $bM-\text{int } L$. The problem of extending (L',g') over $W-\text{int } M$ is equivalent to the problem of extending the concordance class of (L,g) over W.

Thus it suffices in most instances to consider the case where M is embedded in bW.

<u>Definition 13</u> (An admissable pair) Let M be void or let M be a PL n-manifold which is embedded in bW^{n+1}. Then (W,M) is called an <u>admissable pair</u> if $n \geqslant 5$, $bW - M \neq 0$ if n is odd, and $\pi_1(W) = \pi_1(bW-M) = 0$.

Now we give an equivalent formulation of the extension problem for admissable pairs.

<u>Theorem 7</u> Let M^n be embedded in bW^{n+1}. Let $a=(L,g)$ be a PL-structure on M^n with k-dimensional classifying bundle $v=(E_1,t_1)$. Then if a extends to a PL-structure on W, $v\oplus O^r$ extends to a $F/PL)_{k+r}$-bundle over W for some r. Conversely, if $v\oplus O^r$ extends and (W,M) is admissable, then $a=(L,g)$ extends to a PL-structure on W.

<u>Proof:</u> Suppose that (L,g) extends to a PL-structure (Q,G) on W. Then $v\oplus O^r$ is a classifying bundle for (L,g) and if r is large Lemma 4' applies to give the desired extension.

Now the converse. Suppose $(E_1,t_1)\oplus O^r$ extends over W. Then we claim that there is an $F/PL)_s$-bundle (E,T) over W so that

1) $t=T/(E/M)$ is t-regular to O in D^s

2) $t^{-1}(0)\xrightarrow{\text{incl}}E/M\xrightarrow{p}M$ defines a PL-structure on M

3) there is a PL-homeomorphism $c:L\to t^{-1}(0)$ so that $p(\text{incl})c$ is homotopic to g, (p is the projection map of E).

Proof: Since (E_1, t_1) is a classifying bundle for (L,g), $(E_1,t_1) \cong (E_1, ri_1)$ where (E_1, i_1) defines a "good" tubular neighborhood of $g':(M,bM) \to (L,bL)$. If we change notation and replace E_1 by $E_1 \oplus$ (trivial D^r bundle), i_1 by $i_1 \times \text{id } D^r$, and r by $r \times \text{id } D^r$, then $(E_1,t_1) \oplus 0^r = (E_1, ri_1)$ where (E_1, i_1) defines a "good" tubular neighborhood of g' and $r = p_1$ in a neighborhood of $L \times 0$.

By hypothesis there is bundle (E,T) over W and a bundle map $(E_1, ri_1) \xrightarrow{b} (E,T)$ covering the inclusion $M \subset W$. (That is, $E_1 \xrightarrow{b} E/M$ is a bundle map covering the identity and $ri_1 = T/(E/M) \circ b$.)

Let $t = T/(E/M)$. Now b is a PL-homeomorphism, so $t = ri_1 b^{-1}$. Thus it is clear that t is t-regular to 0 in D^s. Since $i_1(E_1^{\cdot}) \supset Lx0 \subset Lx \text{ int } D^s$ we may define

$$c = bi_1^{-1}(x0) .$$

Then

$$p(\text{incl.}) \cdot c = p(\text{incl.}) bi_1^{-1}(x0)$$

$$= qi_1^{-1}(x0) \qquad \text{(where } q \text{ is the projection map of } E_1)$$

which is homotopic to g (as a map of pairs) since (E_1, i_1) is a tubular neighborhood of g'.

Thus it is clear that Theorem 7 follows from Lemma 8.

Lemma 8 Let (W^{n+1}, M^n) be an admissable pair and let (v,T) be an $F/PL)_k$ bundle over W with projection map p, $k > n+3$. If $M \neq 0$, suppose that $t = T/(v/M)$ is t-regular to 0 in D^k and

$$t^{-1}(0) \subset v/M \xrightarrow{p} M$$

defines a PL-structure (L,g) on M. Then there exists

$$H: \; v \times I \longrightarrow D^k$$

so that $H(x,0) = T(x)$; $H(x,t)$ is an F-trivialization of v; $h(m,t) = T(m)$ for m in v/M, t in I; H is t-regular to 0 in D^k; and if $H(x,1) = T_1 : v \to D^k$, then

$$T_1^{-1}(0) \subset E \xrightarrow{p} W$$

defines a PL-structure on W extending (L,g).

Proof: Suppose that T is t-regular to 0 in D^k on all of v. Then

$$T^{-1}(0) = (Q', bQ') \subset v/(W, bW) \xrightarrow{p} (W, bW)$$

is a first approximation to a PL-structure on W which extends the PL-structure on M defined by (L,g).

By this we mean that L is embedded in bQ' and G' induces g, g', bG' so that

$$\begin{tikzcd}
 & bQ'\text{-int } L \arrow[r, "g'"] & bW\text{-int } M \arrow[dr, phantom, sloped, "\subset"] & \\
bQ' \arrow[ur, phantom, sloped, "\supset"] \arrow[rrr, "bG'"] & & & bW \\
 & L \arrow[ul, phantom, sloped, "\subset"] \arrow[r, "g"] & M \arrow[ur, phantom, sloped, "\subset"] &
\end{tikzcd}$$

is commutative. However we do not know that g', bG', and G' are homotopy equivalences.

First we examine the algebra of the situation. Let $L' = bQ'\text{-int } L$ and $M' = bW\text{-int } M$. Then we have

$$\begin{tikzcd}
L' \arrow[r, "g'"] & M' \\
bQ' \arrow[u, phantom, "\cap"] \arrow[r, "bG'"] & bW \arrow[u, phantom, "\cap"] \\
Q' \arrow[u, phantom, "\cup"] \arrow[r, "G'"] & W \arrow[u, phantom, "\cup"]
\end{tikzcd} .$$

It follows from Lefschetz Duality (LD) in $v/(W,bW)$ that there exists $\mu_{Q'}$ in $H_{n+1}(Q',bQ')$ so that $G'_{!*}\mu_{Q'} = \mu_W$, a generator of $H_{n+1}(W,bW)$. Thus we have the diagram

$$(D)\qquad \begin{tikzcd}
 & 0 \arrow[d] & & 0 & & 0 & \\
\arrow[r] & A_* \arrow[rr, "k"] \arrow[d, "i"] & & B_* \arrow[rr, "j"] \arrow[d, "i"] & & C_* \arrow[r, "b"] \arrow[d, "i"] & {} \\
\arrow[r] & H_*(L') \arrow[rr] \arrow[d, "g_{!*}"] & & H_*(Q') \arrow[rr] \arrow[d, "G'_{!*}"] & & H_*(Q',L') \arrow[r] \arrow[d, "G'_{0!*}"] & {} \\
\arrow[r] & H_*(M') \arrow[rr] \arrow[d] \arrow[u, dashed, bend left, "\alpha"] & & H_*(W) \arrow[rr] \arrow[d] \arrow[u, dashed, bend left, "\beta"] & & H_*(W,M') \arrow[r] \arrow[d] \arrow[u, dashed, bend left, "\gamma"] & {} \\
 & 0 & & 0 & & 0 &
\end{tikzcd}$$

where $A_*=(\ker g')_*$, $B_*=(\ker G')_*$, $C_*=(\ker G'_o)_*$ and i denotes the various inclusions. α, β, and γ are defined by

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
\alpha: & H_*(M') \arrow[d] & H^*(M'bM') \arrow[l, "\cap\mu_{M'}"', "\cong"] \arrow[r, "g'^*"] \arrow[d] & H^*(L',bL') \arrow[r, "\cap\mu_{L'}"] \arrow[d] & H_*(L') \arrow[d] \\
(\bar{D})\ \beta: & H_*(W) \arrow[d] & H^*(W,bW) \arrow[l, "\cap\mu_W"', "\cong"] \arrow[r, "G'^*"] \arrow[d] & H^*(Q',bQ') \arrow[r, "\cap\mu_{Q'}"] \arrow[d] & H_*(Q') \arrow[d] \\
\gamma: & H_*(W,M') \arrow[d] & H^*(W,M) \arrow[l, "\cap\mu_W"', "\cong"] \arrow[r, "G'^*_o"] \arrow[d] & H^*(Q',L) \arrow[r, "\cap\mu_{Q'}"] \arrow[d] & H_*(Q',L') \arrow[d] \\
 & {} & {} & {} & {}
\end{tikzcd}$$

where the vertical maps come from the exact sequence for a pair or a triple. (e.g. $H^*(M',bM') \underset{\text{exc}}{\cong} H^*(bW,M)$.)

From all this and the fact that

$$g:\ (L,bL) \longrightarrow (M,bM)$$

is a homotopy equivalence we derive the following

<u>Lemma A</u>

1)

$$\cdots \longrightarrow A_k \xrightarrow{k} B_k \xrightarrow{j} C_k \xrightarrow{b} A_{k-1} \longrightarrow \cdots$$

is exact.

2)
$$\begin{aligned}
H_k(L') &= A_k \oplus H_k(M'), & g'_*\alpha &= 1 \\
H_k(Q') &= B_k \oplus H_k(W), & G'_*\beta &= 1 \\
H_k(Q';L') &= C_k \oplus H_k(W,M'), & G'_{o*}\gamma &= 1
\end{aligned}$$

3) A_* satisfies Poincare' Duality wrt dimension n (i.e. $A_k = A^{n-k} \cong \mathrm{Hom}(A_{n-k}, Z) \oplus \mathrm{Ext}(A_{n-k-1}, Z)$)

B_* and C_* satisfy Lefschetz Duality wrt dimension $n+1$ (i.e. $B_{k+1} = C^{n-k}, B^{k+1} = C_{n-k}$)

<u>Proof of Lemma A:</u> First we show that

$$g^!_*\alpha = 1, \quad G^!_*\beta = 1, \text{ and } \quad G^!_{o*}\gamma = 1 .$$

Let x be in $H_k(M')$, then

$$x = \mu_{M'} \cap u \quad \text{for some } u \text{ in } H^{n-k}(M', bM').$$

But

$$\begin{aligned} g^!_*(\mu_{L'} \cap (g'^* u)) &= g^!_* \mu_{L'} \cap u \\ &= \mu_{M'} \cap u \\ &= x . \end{aligned}$$

Therefore $g^!_*\alpha = 1$. Similarly $G^!_*\beta = 1$ and $G^!_{o*}\gamma = 1$. This proves 2) and shows that $g^!_*$, $G^!_*$, and $G^!_{o*}$ are onto as indicated in (D).

Now 1) follows from 2) and the commulativity of (D) and ($\bar{D}$) by a diagram chase.

To prove 3) we have that

$$H_i(L') = A_i \oplus H_i(M').$$

We also have

$$\begin{tikzcd}
\arrow[r] & H_i(bL') \arrow[r] \arrow[d, "\cong"] & H_i(L') \arrow[r, "j"] \arrow[d, "g'_*"] & H_i(L', bL') \arrow[r] \arrow[d, "g'_{0*}"] & {} \\
\arrow[r] & H_i(bM') \arrow[r] & H_i(M') \arrow[r] \arrow[d] & H_i(M', bM') \arrow[r] \arrow[u, bend left, dashed, "\alpha_0"] \arrow[d] & {} \\
& & 0 & 0 &
\end{tikzcd}$$

The vertical isomorphisms come from the fact that

$$g/bL: \quad bL \longrightarrow bM \qquad \text{is}$$

a homotopy equivalence, $bM = bM'$ and $bL = bL'$. α_0 is defined (like α was) so that $g'_0\alpha_0 = 1$. Thus $\ker g'_*)_i = \ker g'_0)_*$, and $\ker g'_0)_*$ is a direct summand of $H_i(L', bL')$. Therefore $H_i(L', bL') = A_i \oplus H_i(M', bM')$.

We use this to show that

$$\begin{aligned}
A_i \oplus H_i(M') &= H_i(L') \\
&= H^{n-i}(L', bL') \\
&= \operatorname{Hom}(H_{n-i}(L', bL'), Z) \oplus \operatorname{Ext}(H_{n-i-1}(L', bL'), Z) \\
&= \operatorname{Hom}(A_{n-i} \oplus H_{n-i}(M', bM') \\
&\qquad \oplus \operatorname{Ext}(A_{n-i-1} \oplus H_{n-i-1}(M', bM'), Z) \\
&= \operatorname{Hom}(A_{n-i}, Z) \oplus \operatorname{Ext}(A_{n-i-1}, Z) \oplus H^{n-i}(M', bM') \\
&= A^{n-i} \oplus H_i(M')
\end{aligned}$$

Since everything is finitely generated this means that $A_i = A^{n-i}$.

To prove the second statement of 3) we use

$$\begin{tikzcd}
\to H_k(bQ',L') \arrow[r] \arrow[d, "\cong"', "\bar{g}_*"] & H_k(Q',L') \arrow[r] \arrow[d, "G'_{o*}"] & H_k(Q',bQ') \to \arrow[d, shift left, "G'_*"] \\
\to H_k(bW,M') \arrow[r] & H_k(W,M') \arrow[r] & H_k(W,bW) \to \arrow[u, dashed, shift left, "\beta_o"]
\end{tikzcd},$$

define β_o so that $G'_*\beta_o = 1$ and conclude that $H_k(Q',bQ') = H_k(W,bW) \oplus C_k$. (By excision $\bar{g}_*$ is an isomorphism since g_* is.)

Therefore

$$\begin{aligned}
B_k \oplus H_k(W) &= H_k(Q') \\
&= H^{n+1-k}(Q',bQ') \\
&= \mathrm{Hom}(H_{n+1-k}(Q',bQ');Z) \oplus \mathrm{Ext}(H_{n-k}(Q',bQ'),Z) \\
&= \mathrm{Hom}(C_{n-k+1} \oplus H^{n+1-k}(W,bW),Z) \\
&\qquad \mathrm{Ext}(C_{n-k} \oplus H_{n-k}(W,bW),Z) \\
&= C^{n-k+1} \oplus H^{n+1-k}(W,bW) \\
&= C^{n-k+1} \oplus H_k(W) .
\end{aligned}$$

This completes the proof of Lemma A.

<u>Lemma S:</u> Suppose that $S^r \overset{i}{\subset} \mathrm{int}\ L'$ ($S^r \overset{i}{\subset} \mathrm{int}\ Q'$) represents an element of A_r (B_r). Then a neighborhood of $S^r \subset \mathrm{int}\ L'$ ($S^r \subset \mathrm{int}\ Q'$) is smoothable.

Proof: Consider the case $S^r \overset{i}{\subset} \text{int } L'$. $S^r \overset{i}{\subset} v/\text{int } M'$ is contractible. So by general position we may suppose that a neighborhood of $S^r \overset{i}{\subset} v/\text{int } M'$ is contained in a top dimensional cell of $v/\text{int } M'$. Then

$$t_{L'}/S^r \oplus v_{L'}/S^r = \begin{array}{l}\text{normal microbundle} \\ \text{of } S^r \subset \text{cell}\end{array} \oplus t_{S^r}$$

where $t_{L'} =$ tangent microbundle of L'

$v_{L'} =$ normal microbundle of L' in $v/\text{int } M'$.

By the transverse regularity of T, $v_{L'}$ is trivial. Thus $t_{L'}/S^r$ is stably trivial. Similarly $t_{Q'}/S^r$ is stably trivial.

Therefore we may smooth a neighborhood of $S^r \subset \text{int } L'$ or $S^r \subset \text{int } Q'$. Q.E.D.

Suppose (Q,bQ) is a PL-framed submanifold of $v/(W,bW)$ which contains L in its boundary. We say that $R \subset v \times I$ is an elementary framed cobordism on the interior of (Q,L'), where $L' = bQ - \text{int } L$, if

1) there is a closed $(n+k)$-cell U in the interior of v so that

a) $(U,bU) \cap Q = (S^r x D^s, b(S^r x D^s))$

where $r+s=n+1$

b) $(U' x I) \cap R = (U' x I) \cap Q x I$

where $U' = v - \text{int } U$

c) $(U x I, b(U x I)) \cap R = (D^{n+2}, bD^{n+2})$,

where D^{n+2} is the $(n+2)$-disk.

2) there is an extension of ((framing of $(Q \cap U')) \times I) \cup$ (framing of $Q \times 0$) to a framing of (R, bR) in $v/(W, bW)$.

We say that $R \subset v \times I$ is an elementary framed cobordism on the boundary of (Q', L') if there is a closed $n+k$-cell U in v so that

i) $U \cap v/bQ = U \cap v/\mathrm{int}\, M' = U_1$, an $(n+k-1)$ cell in L'.

ii) a), b), c) and 2) hold for U.

iii) $(U_1, bU_1) \cap bQ = (S^r \times D^{s-1}, b(S^r \times D^{s-1}))$ and $(U_1 \times I, b(U_1 \times I)) \cap bR = (D^{n+1}, bD^{n+1})$.

Let $Q_1 = R \cap v \times 1$. Then it is clear that

$$(Q_1, bQ_1) \subset v/(W, bW) \xrightarrow{p} (W, bW)$$

has degree one, and

$$(L_1', bL_1') \subset v/(M', bM') \longrightarrow (M', bM')$$

has degree one. Thus Lemma A applies to $Q_1 \subset v \times 1$.

An e.f. (elementary framed) cobordism on the interior or the boundary of (Q, L') can arise by doing framed surgery on the interior of Q or by doing framed surgery on bQ and then thickening.

<u>Lemma T:</u> Let $T: v \to D^k$ be an F-trivialization of v such that T is t-regular to 0 in D^k and $T^{-1}(0) = Q$. Suppose (Q, bQ) is framed in $v/(W, bW)$ by T. Let $R \subset v \times I$ be an elementary framed cobordism on the interior or on the boundary of (Q, L'). Then there is an F-trivialization of $v \times I$

$$H: v \times I \longrightarrow D^k$$

so that H is t-regular to 0 in D^k, $H^{-1}(0) = R$, and $H = T \times I$ on $L \times I$.

<u>Proof:</u> Consider the case when R is an e.f. cobordism on the interior of (Q, L'). We define

$$H: \quad v \times I \longrightarrow D^k$$

as follows

1) define H on $U' \times I$ by $T \times I$

2) map $D^{n+2} = R \cap (\bar{U} \times I)$ to 0 and map a product neighborhood N of D^{n+2} using the framing of R.

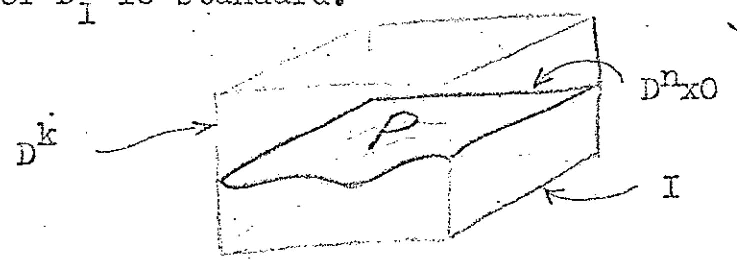
3) if $N = D^{n+2} \times D^k$, we may assume that $(\bar{U} \times 0 \cup b\bar{U} \times I) - (D^{n+2} \times \text{int } D^k)$ is mapped to bD^k. Extend H over $U \times I$ using obstruction theory to map $U \times I - (D^{n+2} - \text{int } D^k)$ to bD^k. (See Lemma 1.5 in the appendix to section I for details.)

Thus H can be constructed to satisfy the geometric conditions. That H is an F-trivialization follows from these conditions and the nature of R using Lefshetz Duality as in Lemma 4.

To prove Lemma T when R is an e.f. cobordism on the boundary of (Q,L') we apply the above process to define H on v/bW and then we are essentially reduced to the interior case.

Let (P,bP) in (D^{n+1+k},bD^{n+1+k}) represent an element of P_{n+1}, n odd. Let $Q\#_b P$ denote the oriented connected sum of Q and P along there boundaries.

1) Assume that D^n_1 in $bP = b(D^{n+1}xD^k)$ coincides with $D^n_1 = D^n x0 \cup bD^n xI$ where $D^{n+1} = D^n xI$, and that the framing of D^n_1 is standard.



2) Assume that W and Q coincide on some collar neighborhood $D^n xI$ of $D^n x1$ in int L', that v/D^{n+1} is identified with $D^{n+1}xD^k = D^{n+1+k}$, and that the framing of $Q \cap D^{n+1+k}$ is the standard framing of D^{n+1} in D^{n+1+k}.

If we let $Q' = (Q - \text{int } D^{n+1}) \cup P$, then $Q' = Q\#_b P$ and (Q',bQ') is naturally framed in $v/(W,bW)$.

Lemma P: Let (Q,F) be a framed submanifold $v/(W,bW)$ and (P,G) a framed submanifold of (D^{n+1+k}, bD^{n+1+k}) representing an element of P_{n+1}, n odd. Let $F\#_b G$ denote the natural framing of $Q\#_b P$ in v, where Q and P are connected in the interior of L'. Then (Q,F) and $(Q\#_b P, F\#_b G)$ are related by a sequence of e.f. cobordisms in v x I.

Proof: Assume Q has the form of 2) above. Define a cobordism R between Q and $Q\#_b P$ in v x I by

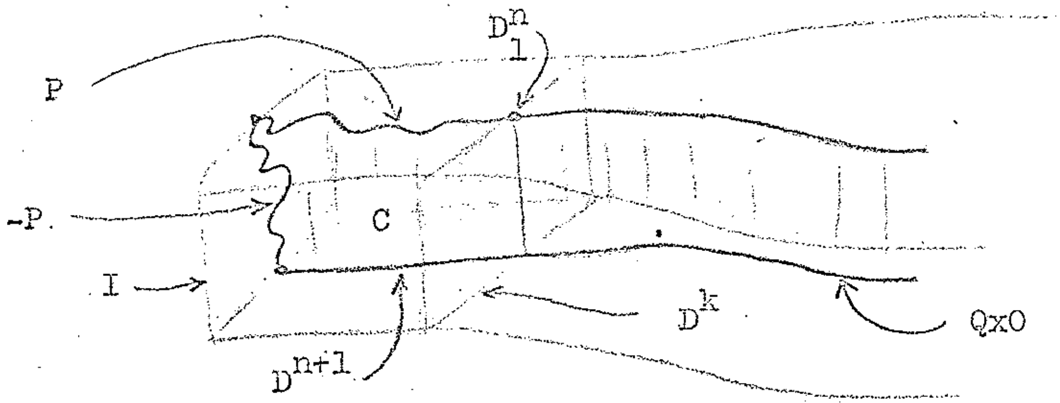
$$R = \begin{cases} Q \times I & \text{in } (v - \mathrm{int}D^{n+1+k}) \times I \\ C & \text{in } D^{n+1+k} \times I \end{cases}$$

where C is a submanifold of $D^{n+1+k} \times I$ which is constructed as follows:

a) Write $D^{n+1+k} = D^n \times I \times D^k$ and assume P is embedded in $D^{n+1+k} \times 1$ as above.

b) Then $-P$ is naturally embedded (by interchanging coordinates) in $D^k \times (D^n \times 1) \times I$ so that $P \cap (-P) = bP \cap b(-P) = bP - \mathrm{int}\, D_1^n$.

c) $P \cup (-P) \subset (D^n \times I \times D^k) \times 1 \cup D^k \times (D^n \times 1) \times I \subset$ $b(D^{n\ 1\ k} \times I)$ is just $P\#_b(-P)$ which is framed cobordant to the n+1-disk in $D^{n+k+1} \times I$. Let C be such a framed cobordism so that $bC = C \cap b(D^{n+k+1} \times I) = (P\#_b(-P)) \cup ((D_1^n \times 0) \times I) \cup (D^{n+1})$.



Assume that the framing of C is standard on $D^{n+1} \cup D^n_1 x0$.

d) Then the product framing of $QxI \cap (v\text{-}intD^{n+1+k})xI$ extends using the framing of C to a framing H of R in vxI. Furthermore $(R,H) \cap (vx0) = (Q,F)$ and $(R,H) \cap (vx1) = (Q\#_b P, F\#_b G)$.

e) We factor R into e.f. cobordism on the boundary and interior using general position and a Morse function f on C such that

$$f(x) = \begin{cases} 0 & x \text{ in } D^{n+1} \\ y & x = (d,y) \text{ in } D^n_1 xI \\ 1 & x \text{ in } P \end{cases}$$

and f has distinct critical values. The critical points of f on the interior of C yield interior e.f. cobordisms, and those in (-P) yield e.f. cobordisms on the boundary.

Now we use Lemmas A,S, and P to construct a sequence (R_i,F_i), $i=0,\ldots,r$, of e.f. cobordisms in $v.x\,[i,i+1]$ so that $(R_i,F_i) \cap v\ x\ (i+1) = (R_{i+1},F_{i+1}) \cap v\ x\ (i+1)$, $(R_o,F_o) = (Q',F')xI$ and if $Q_r = R_r \cap v\ x\ (r+1)$ and $L'_r = bQ_r-(intL)\ x\ (r+1)$,

$$(Q_r, L_r') \subset v/(W, M') \longrightarrow (W, M')$$

is a homotopy equivalence. This construction (with trivial modifications) is done by Browder and Hirsch (1) in their proof of Wall's Theorem (26) in the PL-case. Lemmas A, S, and P provide the main ingredients of the construction.

Then we use Lemma T r-times to build a homotopy of T with the required properties. If T_r denotes the end of the homotopy then

$$(Q_r;\ L_r', bL_r', L\times(r+1)) \subset v/(W;\ M', bM', M) \xrightarrow{p} (W;\ M', bM', M)$$

induces a homotopy equivalence on each factor. It follows from Lemma H of Chapter 2 that

$$bQ_r \subset v/bW \xrightarrow{p} bW$$

is a homotopy equivalence. Thus

$$Q_r = T_r^{-1}(0) \subset v \times (r+1) \xrightarrow{p} W$$

defines a PL-structure on W extending (L,g). This completes the proof of Lemma 8.

Let M be properly embedded in W. We want to consider the different possible extensions of a given PL-structure on M to PL-structures on W. We will classify the extensions for admissable pairs such that $bW-M \neq 0$.

Definition 14. Let M be properly embedded in W and let $a=(L,g)$ be a PL-structure on M. Let $b_i=(Q_i,G_i)$ define extensions of a to PL-structures on W, $i=0,1$. (So there are embeddings $e_i: L\subset Q_i$ so that $G_i e_i = g$.) Let $b_0 \cup axI \cup b_1$ denote the PL-structure[1] on $Wx0 \cup MxI \cup WxI \subset WxI$ defined by

$$(Q_0 \cup_{e_0} LxI \cup_{e_1} Q_1,\ G_0 \cup_{e_0} gxI \cup_{e_1} G_1) \quad .$$

We say that b_0 and b_1 are concordant relative to M if $b_0 \cup axI \cup b_1$ extends to a PL-structure on all of WxI.

This defines an equivalence relation[1] on the set of extensions of a. Let $PL(W,M;a)$ denote the set of equivalence classes.

Definition 15. Let (X,A) belong to C, and let $v=(E,t)$ be an $F/PL)_k$ bundle over A. Let $v_i=(E_i,T_i)$ define extensions of (E,t) to $F/PL)_k$ bundles over X. (i.e. there are PL-bundle maps $b_i:E\to E_i$ covering the

---

[1] See Lemma H of previous chapter.

inclusion $A \subset X$ so that $t = T_i b_i$.) $i = 0,1$. Then

$$v_o \cup v \times I \cup v_1 = (E_o \cup_{b_o} E{\times}I \cup_{b_1} E_1,\ T_o \cup_{b_o} t{\times}I \cup_{b_1} T_1)$$

defines on $F/PL)_k$ bundle over $X{\times}0 \cup A{\times}I \cup X{\times}1 \subset X{\times}I$. We say that (E_o, T_o) and (E_1, T_1) are <u>equivalent relative to M</u> if $v_o \cup v{\times}I \cup v_1$ extends to an $F/PL)_k$-bundle over $X{\times}I$.

This defines an equivalence relation on the set of extensions of v over X. Let $B(X;A,v)$ denote the set of equivalence classes.

We remark that there are maps

$$B(X;A,v) \xrightarrow{p} B(X)$$

$$PL(W;M,a) \xrightarrow{p} PL(W)$$

defined on representatives by considering an extension of v(or a) as a bundle over X(or a PL-structure on W).

If $F: (X,A) \to (Y,B)$ is a PL map, then the induced bundle operation induces $F^*: B(Y,B;v) \to B(X,A;f^*v)$ where $f = F/A$, and

$$\begin{tikzcd} B(Y,B;v) \arrow[r, "F^*"] \arrow[d, "p"] & B(X,A;f^*v) \arrow[d, "p"] \\ B(Y) \arrow[r, "F^*"] & B(X) \end{tikzcd}$$

is commutative.

If $I:(Q,L) \subset (W,M)$ is an embedding so that the Browder theorem applies to $L \subset M$ and $Q \subset W$ in a relative manner then $I^*: PL(W,M;a) \longrightarrow PL(Q,L;i^{\#}a)$ may be defined, where $i = I/L$ and $i^{\#}a$ is a specific restricted PL-structure on M. An analogous diagram results.

<u>Theorem 9</u>. Let (W^{n+1}, M^n) be a PL-manifold pair. Let $a = (L,g)$ be a PL-structure on M with k-dimensional classifying bundle $v = (E,t)$, $k > n+3$. Then there exists a correspondence

$$PL(W,M;a) \xrightarrow{C_a} B_k(W,M;v)$$

so that

$$\begin{array}{ccc} PL(M') & \xrightarrow{C} & B_k(M') \\ \uparrow p & & \uparrow p \\ PL(W,M;a) & \xrightarrow{C_a} & B(W,M;v) \\ \downarrow p & & \downarrow p \\ PL(W) & \xrightarrow{C} & B(W) \end{array}$$

is commutative, and

C_a is injective if (W^*, M^*) is admissable

C_a is onto if (W,M) is admissable .

$((W^*, M^*) = (W \times I,\ W \times 0 \cup M \times I \cup W \times 1).)$

<u>Proof:</u> Let $b = (Q,G)$ define an extension of $a = (L,g)$ over W. Let (E, ri) be the classifying bundle for (L,g), g' a homotopy inverse for g, and H a homotopy between $g'p$ and $p_1 i$.

Then there exists a classifying bundle v_b for (Q,G) which extends (E, ri) in the sense of 2) and 3) of Lemma 4 and which has an associated homotopy which extends h as in 3) of Lemma 4'. Define $C_a = C_a(L,g,E,r,i,g',h)$ on representative by

$$b \longrightarrow v_b \quad .$$

Note then that the desired commutativity holds if C_a is well-defined.

<u>Claim:</u> C_a is well-defined and injective.

<u>Proof:</u> If b_o and b_1 denote two extensions of a over W with classifying bundles v_o and v_1 extending v, then $v_o \cup v \times I \cup v_1$ is a classifying bundle for $b_o \cup a \times I \cup b_1$.

To see this, write $b_s = (Q_s, G_s)$, $v_s = (E_s, r_s i_s)$, $v = (E, ri)$, $s = 0,1$. Let j_s denote the embedding of L in Q_s and b_s the embedding (as a subbundle) of v in v_s, $s = 0,1$. Let g' and G_s' denote homotopy inverses for g and G_s, $s = 0,1$. By Lemma H we may choose G_s' so that $(G_s'/j_s L) \circ j_s = g'$.

Then $i^* = i_o \cup_{b_o} i \times I \cup_{b_1} i_1$ defines an embedding of $E^* = E_o \cup_{b_o} E \times I \cup_{b_1} E_1$ in $Q^* \times \operatorname{int} D^k =$

$$(Q_o \cup_{j_o} L \times I \cup_{j_1} Q_1) \times \operatorname{int} D^k \quad .$$

E^* is a bundle over M^* with projection $p^* = p_o \cup p \times I \cup p_1$.

If we let $G'^* = G_o' \cup g' \times I \cup G_1'$, and apply Lemma 4' 3)

$$\begin{tikzcd}
(E^*, E^*/bM^*) \arrow[r, "i_*"] \arrow[d, "p^*"] & (Q^*, bQ^*) \times \mathrm{int}\, D^k \arrow[d, "p_1"] \\
(M^*, bM^*) \arrow[r, "G^{*\prime}"] & (Q^*, bQ^*)
\end{tikzcd}$$

is homotopy commutative. (Let h be a fixed homotopy between $p_1 i$ and $g'p$, H_s the extended homotopies between $p_1 j_s$ and $G'p_s$. Then $H^* = H_0 \cup h \times I \cup H_1$ is a homotopy between $p_1 i^*$ and $G^{*\prime}p^*$).

Also $i^*(E^{*\prime}, E^{*\prime}/bM^*) \supset (Q^*, bQ^*) \times 0$, so (E^*, i^*) defines a "good" tubular neighborhood of G^*.

Thus if $r^* = r_0 \cup r \times I \cup r_1$, then $v_0 \cup v \times I \cup v_1$ $(E^*, r^* i^*)$ a classifying bundle for $b_0 \cup a \times I \cup b_1$.

Thus by Theorem 7 C_a is well-defined for all pairs (W,M), and it is injective when (W^*, M^*) is admissable.

<u>Claim:</u> C_a is onto.

<u>Proof:</u> Let $v = (E_0, T)$ represent x in $B(W,M;v)$. We will use Lemma 8 to construct a PL-structure $b = (Q,G)$ on W extending $a = (L,g)$ where Q is embedded in E_0. Then we carefully deform E_0 over itself until it has the form of some v_b. The process is complicated by the subspace M.

Let $b\colon E \to E_0$ be a bundle map covering the inclusion $M \subset W$ so that

$$*(T/(E/M))\, b = t\,.$$

If $t = ri$ we make the following assumptions about $(L,g;E,r,i;g',h)$

A) $g\colon L \to M$ is given by the composition

$$L \xrightarrow{\times 0} L \times \operatorname{int} D^k \xleftarrow{\;i\;} E \xrightarrow{\;p\;} M\,.$$

B) there is an embedding

$$L \times D^k \xrightarrow{\;e\;} E$$

so that i) ei is the identity on a neighborhood of the 0-section of E

ii) $rie = p_2$ on some neighborhood of $L \times 0$ in $L \times D^k$.

C) $g'\colon M \to L$ is given by the composition

$$M \xrightarrow{\text{0-section}} E \xrightarrow{\;i\;} L \times \operatorname{int} D^k \xrightarrow{\;p_1\;} L$$

D) h is given by the composition

$$E \times I \xrightarrow{\;H\;} E \xrightarrow{\;i\;} L \times \operatorname{int} D^k \xrightarrow{\;p_1\;} L,$$

where H is a deformation retraction of E onto its zero section.

The proof that C_a is onto proceeds in steps.

1) There is a PL $(n+1)$-manifold Q, an F-trivialization $T_o: E_o \to D^k$ of E_o, an embedding

$$L \xrightarrow{k} bQ,$$

and an embedding

$$Q \times D^k \xrightarrow{e_o} E_o$$

so that

i) $e_o(Q \times 0) = T_o^{-1}(0)$

ii) the composition

$$\begin{tikzcd} Q \arrow[r, "\times 0"] \arrow[rrr, bend right, "G"'] & Q \times D^k \arrow[r, "e_o"] & E_o \arrow[r, "p_o"] & W \end{tikzcd}$$

defines a PL-structure on W extending

$$(kL, gk^{-1}).$$

iii) $T_o e_o = p_2$ and $be = e_o(k \times id.)$ (1)

iv) (E_o, T_o) is equivalent in $B(W,M;v)$ to (E_o, T).

Proof: We apply Lemma 8 and change T to T_o by a homotopy which is fixed on E_o/M so that T_o is t-regular to 0 in D^k and

$$\begin{tikzcd} Q = T_o^{-1}(0) \subset E_o \arrow[r, "p_o"] & W \end{tikzcd} \qquad (Q \xrightarrow{G} W)$$

defines a PL-structure on W which extends the PL-structure defined by

---

(1) on a neighborhood of $Q \times 0$.

$$(tb^{-1})^{-1}(0) \subset E_o/M \longrightarrow M .$$

Let $k\colon L \subset bQ$ be defined by the composition

$$\begin{tikzcd} L \arrow[r, "x0"] \arrow[rrr, bend right, "k"'] & L \times \operatorname{int} D^k \arrow[r, "i"] & E \arrow[r, "b^{-1}"] & (E_o/M \cap bQ) \end{tikzcd}$$

Then from A) we get that

$$G/kL = gk^{-1} .$$

Now iv) is clear from our construction, so only iii) is left to prove.

Let e_o be defined on $kL \times D^k$ by $be(k^{-1} \times id)$. Then on $kL \times D^k$ we have that

$$\begin{aligned} T_o e_o &= T_o be(k^{-1} \times id) \\ &= rie(k^{-1} \times id) \\ &= p_2(k^{-1} \times id), \text{ by B)} \\ &= p_2, \text{ on some neighborhood of} \\ &\qquad L \times 0 \text{ in } L \times D^k. \end{aligned}$$

Because of the t-regularity of T_o we can extend the embedding of $Q \subset E_o$ to an embedding

$$Q \times D^k \xrightarrow{e_o'} E_o$$

so that $T_o e_o' = p_2$. Thus our partial e_o and $e_o'/kL \times D^k$ represents $kL \times D^k$ as tubular neighborhoods of $e_o'\, k\, L$.

The PL-tubular neighborhood theorem and the equation

$$T_o(\text{partial } e_o) = T_o(e_o^!/kL \times D^k) = p_2$$

imply they are related by an ambient isotopy of the identity of $E_o/(M,bM)$ which is fixed on $e_o^!kL$.

We can use this isotopy to alter T_o and $e_o^!$ on a collar neighborhood of $E_o/(M,bM)$ in $E_o/(W,bW)$ and $(kL,bkL) \times D^k$ in $(Q,bQ) \times D^k$ so that iii) is satisfied.

2) Let E_o^e denote the "enlarged" bundle whose space is $E_o \cup$ (sphere bundle) $\times I$ and whose maps (the projection and the F-trivialization) are induced by

$$E_o^e \xrightarrow{\text{id} \cup p_1 = r} E_o \quad .$$

Then there exists an isotopy M_t of the identity of $E_o^e/(M,bM)$ so that

i) $M_1/(E_o/(M,bM))$ is the composition s,

$$E_o/(M,bM) \xrightarrow{b^{-1}} E/(M,bM) \xrightarrow{i} (L,bL) \times D^k \xrightarrow{e_o(k \times id)} E_o/(M,bM) \subset E_o^e/(M,bM)$$

ii) $M_t(E_o^!/(M,bM)) \supset e_o((L,bL) \times 0)$, t in I.

<u>Proof:</u> s and the inclusion $E_o/M \subset E_o^e/M$ each represent $E_o/(M,bM)$ as a tubular neighborhood of the zero section of $E_o^e/(M,bM)$. Since s may be written as

$$(\text{inclusion})\ b\ e\ i\ b^{-1},$$

these embeddings agree on a neighborhood of the zero-section of $E_o/(M,bM)$. Thus it follows from the PL-tubular neighborhood theorem that there is an ambient isotopy M_t' of the identity of $E_o^e/(M,bM)$ relating them. We may alter M_t' so that ii) also holds using the technique of 3) in the proof of Lemma 3'.

3) There is an isotopy W_t of the identity of $E_o^e/(W;M,bM,M',bM')$ so that

i) $W_t/(E_o^e/(M,bM)) = M_t$

ii) $W_1(E_o/(W;M,bM,M',bM')) \subset e_o((Q;L,bL,L',bL')\mathrm{x}\,\mathrm{int}\, D^k)$

iii) $e_o((Q;L,bL,L',bL')\mathrm{x}\ 0) \subset W_t(E_o'/(W;M,bM,M',bM'))$

for each t in I

Proof: First we construct W_t so that i) and ii) are satisfied. This construction proceeds in steps —

a) Extend M_t to an isotopy W_t' of the identity of $E_o^e/(W;M,bM,M',bM')$.

b) Deform (first by a homotopy, then by an isotopy) $W_1'(W;M,bM,M',bM')$ over $E_o^e/(W;M,bM,M',bM')$ so that it is contained in $e_o((Q;L,bL,L',bL')\mathrm{x}\ \mathrm{int}\ D^k)$. Keep (M,bM) fixed.

c) Cover the isotopy by an ambient isotopy to get W_t.

We get W_t to satisfy iii) by doing a),b), and c) again in $E_o \times I$. See proof of Lemma 3', 2) and 3), for details.

4) Let $E_1 = W_1(E_o)$. Let $q_1 : E_1 \to W$ and $j : E_1 \to Q \times D^k$ be defined by $p_o W_1^{-1}$ and $W_1(E_o) \subset E_o \xleftarrow{e_o} Q \times D^k$ respectively. Let $b_1 = M_1 \cdot b$. Then there exists

$$r_1 : \ Q \times D^k \longrightarrow D^k$$

so that

i) $r_1 = p_2$ on a neighborhood of $Q \times 0 \ \ r_1^{-1}(0)$,

ii) $E_1 \xrightarrow{q_1} W$ is a PL-bundle and $E_1 \xrightarrow{r_1 j} D^k$ is an F-trivialization ,

iii) b_1 is a bundle map covering k, in fact $r_1(k \times id) = r$ and $jb_1 = (k \times id)i$,

iv) $(E_1, r_1 j)$ represents $C_a(Q,G)$.

<u>Proof:</u> Let $0 \subset D_1^k \subset D^k$ be such that

$$2D_1^k \subset D^k - T_o(j_1(\text{sphere bundle of } E_1))$$

and $r/L \times D_1^k$ is the identity. Let $r_1' = r' p_2$ where $r' : (D^k, D^k - 2D_1^k) \longrightarrow (D^k, bD^k)$ be a homotopy equivalence which is the identity on D_1^k. Then $r_1' j$ is an F-trivialization of E_1. We can now alter r_1' on a collar neighborhood of $k(L, bL)$ in (Q, bQ) so that $r_1(k \times id) = r$. Thus i),ii),and iii) are satisfied by r_1.

It is clear that $E_1 \xrightarrow{q_1} W$ is a PL-k-disk bundle.

Now

$$jb_1 = e_o^{-1}(\text{inclusion})\ M_1\ b$$

$$= e_o^{-1}(\text{inclusion})\ e_o(k \times id)i\ , \text{ by 2)}$$

$$= (k \times id)i\ .$$

So only iv) needs to be proved.

Recall that $b_1Hb_1^{-1}$ is a deformation retraction of b_1E onto its zero section. The obstructions to extending $b_1Hb_1^{-1}$ to a deformation retraction of E_1 onto its zero section vanish since $\pi_*(E_1, bE \cup 0 \text{ section } E_1) = 0$. So let H_1 be a deformation retraction extending $b_1Hb_1^{-1}$.

Then if we let $G' = p_1j(0\text{-section})$, p_1jH_1 yields a homotopy between $G'q_1$ and p_1j. Thus (E_1, j) defines a "good" tubular neighborhood of

$$G' : (W; M, bM, M', bM') \to (Q; kL, kbL, L', bL')$$

which extends the "good" tubular neighborhood, $(b_1E, (k \times id)i\ b_1^{-1})$, of $kg' : (M, bM) \to (kL, kbL)$. in fact,

$$G'/(M, bM) = p_1j(0\text{-section})/(M, bM)$$

$$= p_1j\ b_1(0\text{-section})$$

$$= p_1(k \times id)i\ (0\text{-section})$$

$$= kp_1i\ (0\text{-section})$$

$$= kg'\ ,$$

$$j/b_1E = (k \times id)i\, b_1^{-1},$$

and

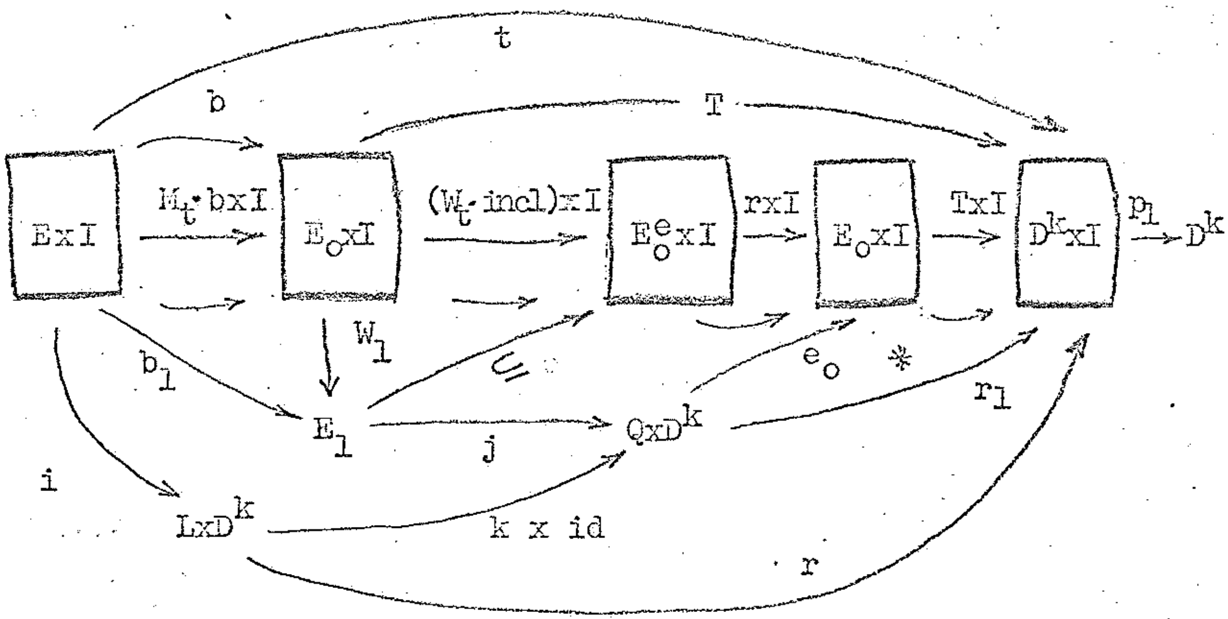
$$p_1jH_1/b_1E = p_1jb_1Hb_1^{-1} \quad .$$
$$= k(p_1 i\, H)b_1^{-1}$$

which is the prescribed homotopy between $k(p_1 i)b_1^{-1}$ and $k(g'p)b_1^{-1}$.

Thus $(E_1, r_1 j)$ satisfies all the properties of Lemma 4' and so represents $C_a(Q,G)$.

5) $(E_1, r_1 j)$ is equivalent in $B(W,M;v)$ to (E_o,T).

Proof: Consider the diagram



All regions are commutative except $*$ because of 2),3) and 4).

Let (B,T'_B) denote the pair

$$(E_o xI \cup_{W_1} E_1 \quad , \quad p_1(TxI)(rxI)((W_t \cdot \text{incl})xI))$$

Then B is a bundle over WxI which contains the bundle $E_o \cup_b ExI \cup_{b_1} E_1 = E^*$ so that T'_B restricted to E^* gives the F-trivialization $T \cup_b txI \cup_{b_1} T_1$. This follows from the diagram.

Now $*$ is commutative for a neighborhood of $Qx0$ in QxD^k. So 3 iii) implies T'_B induces a homotopy equivalence of each fibre pair with (D^k, D^k-0). Thus we can modify T'_B slightly on the complement of E^* so that it becomes an F-trivialization T_B. Then (B,T_B) provides the desired equivalence.

This proves that

$$C_a(L,\ g;\ E,\ r,\ i;\ g',\ h)$$

is onto. Now we show that this result is independent of our choices of (L,g) and (E,ri). This follows from

6) i) We can change a given (L,g) and (E,ri) by equivalences so that A and B are satisfied

ii) Suppose (L,g), (E,ri), g' and h satisfy A,B,C, and D. Suppose (L,g) and (E,ri) are respectively equivalent to (L_1,g_1) and (E_1,r_1i_1). Let C_a^* denote a map defined as above using Lemma 4' (for some choice of g_1' and h_1). Let C_a denote

$$C_a\ (L,\ g;\ E,\ r,\ i;\ g',\ h)\ .$$

Then there is a commutative diagram

$$\text{(D)}\qquad \begin{array}{ccc} PL(W,\ M;\ (L,g)) & \xrightarrow{C_a} & B_k(W,\ M;\ (E,ri)) \\ \Big\downarrow e & & \Big\downarrow e^* \\ PL(W,\ M;\ (L_1,g_1)) & \xrightarrow{C_a^*} & B_k(W,\ M;\ (E_1,r_1i_1)) \end{array}$$

where e and e^* are bijective maps.

<u>Proof:</u> i) Let $e\colon L x D^k \longrightarrow E$ be defined by the composition

$$L \times D^k \xrightarrow{id \times r} L \times \tfrac{1}{n}(D^k) \xrightarrow{i^{-1}} E\ .$$

where n is chosen so that i^{-1} is defined and $r\colon D^k \to \frac{1}{n}D^k$ is the identity on some neighborhood of 0. Then B) is satisfied.

We can change g by a homotopy so that A) is satisfied.

ii) Let (R,f) be a concordance between (L,g) and (L_1,g_1). Define

$$PL(W, M; (L,g)) \xrightarrow{e} PL(W, M; (L_1,g_1))$$

on representatives by

$$(Q,G) \longrightarrow (Q \cup_L R, c^{-1}(G \cup_g f)) ,$$

where

$$W \cup M\times I \xrightarrow{c^{-1}} W$$

is a PL-homeomorphism which restricts to $M\times I \xrightarrow{p_1} M$ on $M\times I$ and the identity on the complement of a collar neighborhood of M in W.

If (S,T) is an equivalence between (E,ri) and (E_1,r_1i_1) define

$$B_k(W,M;(E,ri)) \xrightarrow{e^*} B_k(W,M;(E_1,r_1i_1))$$

on representatives by

$$(E_o,T_o) \longrightarrow (c^*(E_o \cup_E S), \ \bar{c}(T_o \cup_t T))$$

where $\bar{c}$ is a bundle map covering c.

If we choose, $(S,T) = (S,r_2i_2)$ to be a classifying bundle for (R,f) which extends $(E,ri) \cup (E_1,r_1i_1)$ with respect to $g' \cup g_1'$ and $h \cup h_1$ using Lemma 4', then an easy but tedious argument shows that (D) is commutative.

This completes the proof of Theorem 9.

<u>Corollary:</u> C is a natural equivalence of contravariant functors on $\mathcal{M}_n$.

The Classifying Space F/PL

Theorem 9 implies that

$$PL \xrightarrow{C} B_k$$

is a natural equivalence of functors on $\mathcal{M}_n$, $k >> n$. We now study the functor B_k.

Definition 16 Let X be a space. Then $(\ ,X)$ denotes the contravariant functor on C which assigns to K in C the set of free homotopy classes of maps from K to X and to $f:K \to L$ the induced map $(L,X) \xrightarrow{f^*} (K,X)$ defined on representatives by $g \to gf$.

If X is in C and u is in $B_k(X)$, let u denote the natural transformation

$$(\ ,X) \xrightarrow{u} B_k$$

defined on representatives by $(K \to X) \longrightarrow (f^*v)$, where v is an $F/PL)_k$- bundle representing u.

Theorem 10 Let $\mathcal{C}$ be the category of countable connected locally finite simplicial complexes and PL-maps. Then if $k \geqslant 3$, there is a space $F/PL)_k$ in $\mathcal{C}$ and an element u_k in $B_k(F/PL)_k)$ so that

$$(\ ,F/PL)_k) \xrightarrow{u_k} B_k$$

is a natural equivalence of contravariant functors on $\mathcal{C}$.

Proof: One can carry through Brown's construction a la Milnor (19) to construct $F/PL)_k$ using the properties of B_k described in Lemma 2. We can forget base points because $B_k(S^1) = \{0\}$.

Lemma 11 Let v be an $F/PL)_k$-bundle over D^n and let v_k represent the universal bundle u_k. Suppose $f: bD^n \longrightarrow F/PL)_k$ is such that $f^* v_k$ is embedded (as a subbundle) in v. Then there exists an extension g of f over D^n so that the bundle

$$v \cup (f^* v_k \times I) \cup g^* v_k$$

over $D^n \cup bD^n \times I \cup D^n$ extends over $D^n \times I$.

Proof: Let A be a subcomplex of X. Let u be an $F/PL)_k$ bundle over A and $f: A \to X$ be such that $f^* v_k$ is equivalent in $B_n(A)$ to u. Then Theorem 10 implies that u extends over X iffi f extends over X.

In our case $f^* v_k$ extends over D^n so f has at least one extension g. Then $v_g = v \cup (f^* v_k \times I) \cup g^* v_k$ extends over $D^n \times I$ iffi a certain element C_g in $\pi_n(F/PL)_k)$ vanishes (C_g is the classifying map for v_g).

If $C_g \neq 0$, we merely alter g in the interior of D^n by adding $-C_g$. If g' is the new map, then $C_{g'} = 0$ and $v_{g'}$ extends over $D^n \times I$.

Theorem 12 Let X belong to $\mathscr{C}$ and let A be a subcomplex of X. Let v_k represent the universal bundle u_k over $F/PL)_k$, and let $f: A \to F/PL)_k$ be given. Let $(X,A;f)$ denote the homotopy classes (by homotopies fixed on A) of extensions of f over X. Then the induced bundle construction defines a one-to-one correspondence

$$(X,A;f) \xrightarrow{\;*\;} B_k(X,A;f^*v_k) .$$

Proof: Its easy to see that $*$ is well defined and injective.

Let v belong to $B_k(X,A;f^*v_k)$. Let $X_r = A \cup$ (r-skeleton of X). Suppose inductively that there is an extension, g_r, of f over X_r so that $g_r{}^*v_k$ is equivalent to v/X_r in $B_k(X_r,A;f^*v_k)$. This means that

$$v/X_r \cup f^*v_k \times I \cup g_r{}^*v_k$$

extends over $X_r \times I$. g_{r+1} extending g_r is constructed with this property by applying Lemma 11 to each r cell of $X_{r+1} - X_r$. If we let $g = \bigcup g_r$, then g extends f and $*(g) = v$. Thus $*$ is onto.

Now we relate $F/PL)_n$ to certain other universal spaces.

Definition 17 Let $F(n)$ denote the set of homotopy equivalences of S of degree $+1$ endowed with the compact open topology.

Let $B_{PL)_n}$ denote the universal space for PL n-disk bundles (19).

Let $B_{F(n)}$ denote the universal space for $(n-1)$-spherical fibre spaces (22).

Theorem 13 There exist maps $F(n) \xrightarrow{O_n} F/PL)_n$, $F/PL)_n \xrightarrow{b_n} B_{PL)_n}$, and $B_{PL)_n} \xrightarrow{J_n} B_{F(n)}$ so that

$$(\ \ , F(n)) \xrightarrow{O_n} (\ \ , F/PL)_n) \xrightarrow{b_n} (\ \ , B_{PL)_n}) \xrightarrow{J_n} (\ \ , B_{F(n)})$$

is an exact sequence of functors on $\mathcal{C}$.

Proof: Let $v_n = (E_n, t_n)$, p_n, and f_n denote universal bundles over $F/PL)_n$, $B_{PL)_n}$, and $B_{F(n)}$, respectively. Let b_n and J_n be defined by $J_n^* f_n$ is fibre homotopically equivalent to the sphere bundle of p_n, and $b_n^* p_n$ is PL-bundle equivalent to E_n.

Now it follows from (22) that $F(n)$ is homotopically equivalent to an element of $\mathcal{C}$. Define $F(n) \times D^n \longrightarrow D^n$ by $(f,d) \xrightarrow{t} (\text{cone on } f)(d)$. Then $(F(n) \times D^n, t)$ defines an $F/PL)_n$ - bundle over $F(n)$ which is classified by $O_n \colon F(n) \longrightarrow F/PL)_n$.

Exactness at $B_{PL})_n$: Let X be in $\mathcal{C}$ and f be in $(X, B_{PL})_n)$. Then $J_n(f^*p_n) = 0$ iffi f^*p_n admits an F-trivialization t. (f^*p_n, t) is classified by $g: X = F/PL)_n$ and $(b_n g)^* p_n = g^* b_n{}^* p_n = g^* E_n = f^* p_n$. So $b_n g \sim f$. Thus $J_n f = 0$ iffi $f \sim b_n g$.

Exactness at $F/PL)_n$: Let f belong to $(X, F/PL)_n)$. Then $b_n f = 0$ iffi $f^* E_n$ is the trivial bundle. Thus $b_n f = 0$ iffi $f^* v_n$ is equivalent to $(X \times D^n, t_o)$ for some trivialization t_o. Assume $t_o(x,d) = (\text{cone on } g(x)(d)$ where $g: X \to F(n)$. Then

$$
\begin{aligned}
(0_n g)^* (E_n, t_n) &= g^* (F(n) \times D^n, t) \\
&= (X \times D^n, t(g \times id)) \\
&= (X \times D^n, t_o)
\end{aligned}
$$

since $f(g(x),d) = (\text{cone on } g(x))(d)$. Therefore $0_n g$ is homotopic to f. The exactness follows from the

Lemma: Let $(X \times D^n, t)$ be an $F/PL)_n$-bundle. Let $g: X \to F(n)$ be defined by

$$(g(x))(y) = t(x,y) \quad , \quad y \text{ in } bD^n.$$

Then $(X \times D^n, t)$ is equivalent to $(X \times D^n, \text{cone on } g)$.

<u>Proof:</u> A homotopy between t and the cone on g is given by

$$H(x,d,s) = \begin{cases} |d| \; t(x, \frac{d}{|d|}) & \text{if } |d| \geq 1-s \text{ or } s = 1 \text{ and } |d| \neq 0 \\ (1-s)t(x, \frac{d}{1-s}) & \text{if } |d| < 1-s, s < 1 \\ 0 & \text{if } s = 1, \; d = 0. \end{cases}$$

We also have maps so that

$$\begin{tikzcd} F(n) \arrow[r, "O_n"] \arrow[d, "\text{susp.}"] & F/PL)_n \arrow[r, "b_n"] \arrow[d, "i_n"', "\text{W. sum}"] & B_{PL)_n} \arrow[r, "J_n"] \arrow[d, "\text{W. sum}"] & B_{F(n)} \arrow[d, "\text{W. join}"] \\ F(n+1) \arrow[r, "O_{n+1}"] & F/PL)_{n+1} \arrow[r, "b_{n+1}"] & B_{PL)_{n+1}} \arrow[r, "J_{n+1}"] & B_{F(n+1)} \end{tikzcd}$$

is homotopy commutative. These are defined respectively by suspension, Whitney sum with the trivial one-dimensional bundle, and Whitney join with the trivial 0-dimensional bundle.

<u>Definition 18</u> By forming mapping cylinders we can suppose that the vertical maps are inclusions. Let

$F = \bigcup F(n)$, $F/PL = \bigcup F/PL)_n$, $B_{PL} = \bigcup B_{PL)_n}$, and

$B_F = \bigcup B_{F(n)}$.

Then by successively applying the homotopy extension theorem we can make the diagram actually commutative. Let $O = \cup O_n$, $b = \cup b_n$, and $J = \cup J_n$.

We get a sequence

$$F \xrightarrow{O} F/PL \xrightarrow{b} B_{PL} \xrightarrow{J} B_F .$$

<u>Theorem 14</u> Let X be a finite complex. Then

a) F/PL is the classifying space for stable equivalences classes of $F/PL)_k$-bundles over X.

b) If $f : X \longrightarrow F/PL$ is the classifying map for the class of (E,t), then bf is the classifying map for the stable equivalence class of E.

c) The sequence

$$(X,F) \xrightarrow{O_*} (X,F/PL) \xrightarrow{b_*} (X, B_{PL}) \xrightarrow{J_*} (X,B_F)$$

is exact.

d) The Whitney sum operation induces a H-space structure on F/PL.

e) This H-space structure on F/PL makes $(X,F/PL)$ into an Abelian group.

<u>Proof:</u> If $A = \cup A_n$, then any map of a finite complex into A is homotopic to a map into A_n for some n.

Thus a) follows from Theorem 10, b) follows from the definition of b, and c) follows from Theorem 13.

i) b and J are homomorphisms (where the operations in (X, B_{PL}) and (X, B_F) are induced by Whitney sum)

ii) (X, B_{PL}) is an Abelian group, (see (15) and (12))

iii) (X, F) is finite. (The homotopy groups of F are finite (22).)

Now inverses are easily constructed using c). This proves e).

The Homotopy Properties of F/PL

<u>Lemma 15</u> (Stability) $\pi_i(F/PL)_n \xrightarrow{i_n} \pi_i(F/PL)_{n+1}$ is an isomophism if $i < n - 6$.

<u>Proof:</u> Its easy to check that

$$\begin{tikzcd} & B_n(S^i \times D^3) \arrow[d, "\oplus 0_1"] & \pi_i(F/PL)_n \arrow[l, "u_n"'] \arrow[d, "i_n"] \\ PL(S^i \times D^3) \arrow[ur, "C"] \arrow[dr, "C"'] & B_{n+1}(S^i \times D^3) & \pi_i(F/PL)_{n+1} \arrow[l, "u_{n+2}"'] \end{tikzcd}$$

is commutative, (here we identify $(S^i \times D^3, X)$ with $\pi_i(X)$ by $p_1 : S^i \times D^3 \to S^i$). Now u_n and u_{n+1} are isomorphisms and C is an isomorphism if $n - 3 > i + 3$ by Theorem 9.

<u>Definition 19</u> Let M_1 and M_2 be two <u>oriented</u> closed connected PL n-manifolds. Let $f_i : M_i \to F/PL$ and $D_i^n \subset M_i^n$ be such that $f_i(D_i^n) = p$ in F/PL, $i=1,2$. Let $(M_1, f_1) \# (M_2, f_2)$ denote the pair $(M_1 \# M_2, f_1 \# f_2)$, where $M_1 \# M_2$ denotes the <u>oriented</u> connected sum of M_1 and M_2 using D_1^n and D_2^n and $f_1 \# f_2 : M_1 \# M_2 \to F/PL$ is defined by $f_1 \cup f_2$.

Now F/PL is simply connected so we actually get a well-defined map

$$(M_1, F/PL) \times (M_2, F/PL) \xrightarrow{\#} (M_1 \# M_2, F/PL)$$

by choosing representatives which satisfy $f_i(D_i^n) = p$.

If $M_1 = M_2 = S^i$, then $\#$ is just the group operation in $\pi_i(F/PL)$.

<u>Theorem 16</u> Let M_i^n be simply connected closed <u>oriented</u> PL n-manifolds $i = 0,1$. Let f_i in $(M_i, F/PL)$ classify the $F/PL)_k$-bundles (E_i, t_i), $i = 0,1$; $k >> n$. Then there exists an element

$$n(M_i, f_i) \text{ in } \begin{cases} 0 & \text{if } n \text{ is odd} \\ Z_2 & \text{if } n \equiv 2 \pmod 4 \\ Z & \text{if } n \equiv 0 \pmod 4 \end{cases}$$

so that

i) if $n \geq 4$, $n(M_o, f_o) + n(M_1, f_1) = n((M_o, f_o) \# (M_1, f_1))$

ii) if $n \geq 5$, $n(M_i, f_i) = 0$ iffi t_i "splits", i.e. t_i is homotopic (through F-trivializations) to t_i' where t_i' is t-regular to 0 in D^k and the composition

$$t_i'^{-1}(0) \subset E_i \longrightarrow M_i$$

is a homotopy equivalence.

Proof: Let M_1 be an oriented n-manifold and suppose $M_1 \times D^k$ is embedded in E so that $(M_1 \times 0)$ is homologous to (M). Then for $n = 4i$, let

$$n(M,f) = \frac{1}{8}(\text{index } M_1' - \text{index } M).$$

It follows from (16) that $n(M,f)$ is an integer.

The additivity follows from an easy geometric argument using the additivity of the index.

Now we can apply the technique of Lemma 8 and change t_i by a homotopy through F-trivializations so that

$$f_i^{-1}(0) \subset E_i \xrightarrow{P_i} M_i$$

$$f_i$$

is $\left[\frac{n-1}{2}\right]$ -connected. If $n = 4i$, then $n(M_i, f_i)$ is the precise obstruction to making f_i $\frac{n}{2}$ -connected (and thus a homotopy equivalence) when $n \geq 5$, (see 1).

If n is odd and $n \geq 5$ there is no obstruction.

If $n = 4k + 2$, there is a Z_2 obstruction (the Kervaire invariant of the kernel of

$$H_{2k+1}(f_i^{-1}(0)) \longrightarrow H_{2k+1}(M).\ )$$

whose vanishing implies f_i can be made $\frac{n}{2}$-connected, $n \geq 5$. (see 1).

An easy application of the proof of Lemma 8 where (W,M) there corresponds to $(M \times I, M \times 0)$ here shows the converse. A similar application shows that the obstruction only depends on the equivalence class of (E,t).

Thus we can define $n(M,f)$ so that ii) holds using this obstruction. i) will also hold, in fact this is used in the proof of Lemma 8 (see 1).

<u>Theorem 17</u> Let W^{n+1} be a simply connected PL-manifold whose boundary components $M_1, M_2, \ldots, M_s$ are simply connected, $n \geq 4$. Let M_i be oriented by the boundary of an orientation of (W, bW). If $F: W^{n+1} \longrightarrow F/PL$, let $f_i = F/M_i$, $i = 1, \ldots, s$. Then

$$n(M_1,f_1) + n(M_2,f_2) + \ldots + n(M_s,f_s) = 0 .$$

<u>Proof:</u> A homology calculation takes care of the case $n = 4$ (indeed $n = 4i$). We proceed to the case $n > 4$.

It follows immediately from the definition of $n(M,f)$ that $n(-M,f) = -n(M,f)$, where $-M$ denotes M with the opposite orientation.

We use this to reduce Theorem 17 to the case when all except possibly one of the $n(M_i,f_i)$ are zero.

Choose an arc ℓ in int W connecting a point in M_1 to a point in M_2. If M is an oriented n-manifold let $(W;M_1,M_2) \# M$ denote the manifold obtained from W-(open tubular neighborhood of ℓ). and (M-open disk)x I by the obvious identification of their mutual $(S^{n-1} \times I)$'s. To be more precise do this in an orientation preserving manner so that the oriented components of $b((W;M_1,M_2) \# M)$ are $M_1 \# M$, $M_2 \# -M$, $M_3, \ldots, M_s$.

If $f: M \to F/PL$ is given, and $F: W \to F/PL$ and $fp_1: M \times I \to F/PL$ map the $(S^{n-1} \times I)$'s to p in F/PL, there is an obvious extension over $(W; M_1,M_2) \# M$, say $(F;f_1,f_2) \# f$. In general, we can change fp_1 and F so that this condition is satisfied and $(F;f_1,f_2) \# f$ is well defined up to homotopy (since F/PL is simply connected).

Now replace (W,F) by $(W',F') = ((W;M_1,M_2) \# (-M_1), (F;f_1,f_2) \# f_1)$. Then if bW' has components $M_1', M_2', M_3', \ldots, M_s'$ and $f_i' = F'/M_i'$, we have

$$n(M_1', f_1') = 0$$

$$n(M_2', f_2') = n(M_1, f_1) + n(M_2, f_2), \quad \text{and}$$

$$n(M_i', f_i') = n(M_i, f_i) \quad \text{for} \quad 2 < i \leq s .$$

We continue in this manner, and we get a pair (W^*,F^*) so that

i) W^* is simply connected

ii) bW^* consists of simply connected components
$$M_1^* , M_2^* , \ldots , M_s^*$$

iii) if $f_i^* = F^*/M_i^*$, then

$$n(M_i^* , f_i^*) = \begin{cases} 0 & i \neq s \\ n(M_1,f_1) + \ldots + n(M_s, f_s) & i = s \end{cases}$$

Let F^* determine the bundle (E^*, t^*). Let $(E,t) = (E^* , t^*) / bW^* - M_s$. Apply Theorem 16 to each component of $bW^* - M_s$ and "split" t. Use a collar neighborhood of $bW^* - M_s$ in W, to change t^* on the complement of E^*/ M_s so that t^*/E is "split". Now let $(W^*, M_s , bW^* - M_s, t^*)$ correspond to (W,M,M',T) of Lemma 8 to see that $t^* / (E^*/ M_s)$ can be "split."
Thus $n(M_s^*,f_s^*) = n(M_1,f_1) + \ldots + n(M_s,f_s) = 0$.

Let $\Omega_n^{PL}(F/PL)$ denote the n^{th} oriented PL-bordism group of F/PL . Since $\pi_1(F/PL) = 0$, each bordism class in $\Omega_n^{PL}(F/PL)$ contains a pair (M,f) where M is simply connected, $n \geq 4$. Furthermore any cobordism between two such simply connected pairs may be replaced by a simply connected cobordism. (Any oriented n-manifold is oriented cobordant (by one-dimensional surgery) to a simply connected manifold, $n \geq 4$.)
Thus Theorem 17 has the

<u>Corollary:</u> The correspondence $(M,f) \to n(M,f)$ defines a homomorphism

$$\Omega_n^{PL}(F/PL) \xrightarrow{K} P_n$$

for $n \geq 4$, where

$$P_n = \begin{cases} 0 & n \text{ odd} \\ Z_2 & n \equiv 2 \pmod 4 \\ Z & n \equiv 0 \pmod 4 \end{cases}$$

We remark that K has a natural definition for $n < 4$. This is unique for $n = 1,3$. For $n = 2$ define $K(M^2,f)$ to be 0 if t splits and 1 otherwise $(f \sim (E,t))$. The additivity of K for this case follows exactly as in the case $n = 6$ or $n = 14$. The fact that K is well-defined on cobordism classes follows from Lemma 19 and the fact that $\pi_2(F/PL) = \Omega_2^{PL}(F/PL) = Z_2$;

Then K has the property that if $n \neq 3,4$, $K(M^n,f) = 0$ iffi t "splits" where $f \sim (E,t)$.

We relate K and $\pi_*(F/PL)$ by

<u>Theorem 18</u> Let $\pi_n(F/PL) \xrightarrow{h'} \Omega_n^{PL}(F/PL)$ denote the "Ω^{PL}-Hurewicz Homomorphism." Then if $n \neq 4$, the composition

$$\pi_n(F/PL) \xrightarrow{h'} \Omega_n^{PL}(F/PL) \xrightarrow{K} P_n$$

is an isomorphism.

If $n=4$, Kh' is a monomorphism onto the even integers.

Proof: We make use of the following

Lemma 19: Let (E,t) denote an $F/PL)_k$-bundle over closed M^i, $k>>i$. Then t splits so that $t^{-1}(0)$ is PL-homeomorphic to M^i. iffi (E,t) is equivalent to the trivial $F/PL)_k$-bundle, $(M^i \times D^k, p_2)$.

the h.e. must be ~ to a PL-homeo

Proof: Extend the embedding of $M \times 0 = t^{-1}(0) \subset E \times 0$ to an embedding i of $(M \times I, M \times (0,1))$ in $(E' \times I, E' \times (0,1)$ so that $M \times 1$ goes onto the zero section of E. Now $M \times I$ has a trivial normal bundle in E' (because $M \times 0$ does) so by the PL tubular nghd theorem E is the trivial bundle, $M \times D^k \xrightarrow{p_1} M$. It follows from Lemma 3' , or by an easy direct argument, that there is an embedding e of $M \times I \times D^k$ in E' so that

1) $e/M \times I \times 0 = i$
2) $e/M \times 1 \times D^k =$ is a PL-homeomorphism C onto $E \times 1$
3) $t\,(e/(M \times 0 \times D^k)) = p_2$.

Then define $T: E \times I \to D^k$ on the image of e union $E \times 0$ using p_2 and t and on the complement of image of e union $E \times 0$, using obstruction theory so that T is an F-trivialization. Then $(E \times I \cup_C M \times D^k,\ T \cup_C p_2)$ gives an equivalence between (E,t) and $(M \times D^k, p_2)$. The converse follows from the PL covering htpy. theorem.

Now we prove Theorem 18.

Let (L_n, bL_n) be a framed, almost closed, almost parallelizable $\left[\frac{n-1}{2}\right]$ - connected n-manifold in (D^{n+k}, bD^{n+k}) which generates P_n (Definition 1.2). Let $\overline{L}_n$ denote $L_n \cup_{bL_n} (\text{cone})\, bL_n$. Let

$$h_n: (D^n, bD^n) \times D^k \longrightarrow (D^n, bD^n) \times D^k$$

be the map given by Lemma F so that h_n is t-regular to $D^n \times 0$ and $h_n^{-1}(D^n, bD^n) \times 0 = (L_n, bL_n) \subset (D^n, bD^n) \times D^k$.

We may assume that $h_n /$ (nghd of bL_n in $bD^n \times D^k$) is a PL-homeomorphism c which is the identity on $S^{n-1} \times 0 = bL_n$. Let

$$E = (D^n, bD^n) \times D^k \bigcup_{c(bL_n \times D^k)} (D^n, bD^n) \times D^k$$

and

$$t = \quad p_2\,(h_n \cup_c \text{id}).$$

An easy application of the PL-tubular neighborhood theorem shows E is the total space of a PL k-disk bundle over S^n. Then (E,t) is an $F/PL)_k$ bundle over S^n, t is t-regular to 0 in D^k, and $t^{-1}(0) = \tilde{L}_n$.

Suppose (E,t) is classified by $f\colon S^n \to F/PL$. Then if $n=4i$,

$$n(S^n,f) = \pm(\text{index } L_n)$$

$$= \begin{cases} \pm 1 & \text{if } i>1 \\ \pm 2 & \text{if } i=1. \end{cases}$$

If $n=4i+2$, $n \geqslant 6$, then

$$n(S^n,f) = 1, \quad \text{by } (1).$$

Thus Kh' is onto if $n \geqslant 5$ and onto $2(P_n)$ if $n=4$.

Suppose $f\colon S^n \to F/PL$ classifies (E,t) and $Kh'(S^n,f) = n(S^n,f) = 0$. If $n=4$, we may change[1] t by a homotopy so that $t^{-1}(0)$ is PL-homeomorphic to S^4. If $n \geqslant 5$, Theorem 16 and the Generalized Poincare Conjecture imply that t can be split so that $t^{-1}(0)$ is PL-homeomorphic to S^n. Then Lemma 19 implies f is homotopic to zero. Thus Kh' is a monomorphism, $n \geqslant 4$.

---

1 See Lemma 1.5

Lemma 1.5 and Lemma 19 imply $\pi_1(F/PL) = \pi_3(F/PL = 0$. π_2 (F/PL) can be computed in a number of ways. One way is to use the exact sequence of Theorem 14 to show that $\pi_2(F/PL)$ has 2 elements. Lemma 19 implies Kh' is an isomorphism. This completes the proof.

Corollary: $\pi_n(F/PL) = P_n$.

Now we apply Theorem 18 to study the k-invariants and Z_2-cohomology of F/PL.

Let X_i denote the space which is homotopically equivalent to F/PL over the i-skeleton and such that $\pi_k(X_i) = 0$ for $k > i$. Let k_{i+1} in $H^{i+2}\ (X_i, \pi_{i+1}(F/PL))$ be any k-invariant (or characteristic class) associated to the fibration

$$K(i+1, \pi_{i+1}(F/PL)) \rightarrow X_{i+1} \longrightarrow X_i$$

in a Postnikov tower for F/PL, (see 23).

The k-invariants are closely related to the Hurewicz homomorphism.

Theorem 20 Let h denote the Hurewicz homomorphism, $h: \pi_*(F/PL) \longrightarrow H_*(F/PL)$, and h_2 the (mod 2) Hurewicz homomorphism $h_2: \pi_*(F/PL) \longrightarrow H_*(F/PL; Z_2)$. Then h is a monomorphism, and in all positive even dimensions h_2 is non-trivial.

<u>Proof:</u> We first study the mod 2 Hurewicz homomorphism

$$\pi_n(F/PL) \xrightarrow{h_2} H_n(F/PL, Z_2).$$

Let $\Omega_n(F/PL)$ and $\mathcal{N}_n(F/PL)$ denote the oriented and unoriented (smooth) bordism groups of F/PL, respectively. Then h_2 may be factored as follows:

$$\begin{tikzcd}
 & & \Omega_n^{PL}(F/PL) \arrow[dr] & \\
\pi_n(F/PL) \arrow[r, "h''"] \arrow[urr, bend left, "h'"] \arrow[rrr, bend right=40, "h_2"'] & \Omega_n(F/PL) \arrow[ur] \arrow[rr, "c"] \arrow[dr, "r'"'] & & H_n(F/PL, Z_2) \\
 & & \mathcal{N}_n(F/PL) \arrow[ur] &
\end{tikzcd}$$

Now h' has a left inverse if $n \neq 4$ by Theorem 18. Therefore h'' has a left inverse, $n \neq 4$. The kernel of r' consists of elements divisible by 2, so this means that $r'h''$ is a monomorphism. If $f: S^n \to F/PL$, represents a generator of $\pi_n(F/PL)$, then we have shown that (S^n, f) is non-trivial in $\mathcal{N}_n(F/PL)$. Thus there exists a polynominal w in the Stiefel-Whitney classes of S^n and an element u in $H^*(F/PL, Z_2)$ so that $((f^*u) \cup w) \neq 0$ in $H^n(S^n, Z_2)$. This can only happen when w equals 1 and u is in $H^n(F/PL, Z_2)$. Thus h_2 is non-trivial. For $n = 4$ see Corollary 5.

This shows that h is a monomorphism (onto a direct summand) in dimensions $\neq 4i$. To see that h is a monomorphism in dimensions $4i$ we can look at the Pontryagin numbers of (S^{4i}, f).

For the facts about bordism theory used here see (5).

<u>Corollary 1</u> The n^{th} k-invariant of F/PL is zero if $n \neq 4i$.

<u>Proof:</u> We make use of Theorem 20.

There is an exact sequence

$$(S) \qquad H^i(X_i, \pi_i) \xrightarrow{i^*} H^i(K(i, \pi_i), \pi_i) \xrightarrow{t} H^{i+1}(X_{i-1}, \pi_i)$$

where i is the inclusion map, t is the transgression map, and $\pi_i = \pi_i(F/PL)$, (see 7). If u denotes the fundamental class of $K(i, \pi_i)$, then $u \xrightarrow{t}$ (k-invariant). Thus we need to look at image i^*. Consider

$$\begin{tikzcd} \pi_i(X_i) \arrow[d, shift right, "h"'] & \pi_i(K_i) \arrow[l, "i"'] \arrow[d, "h"] \\ H_i(X_i) \arrow[u, dashed, bend right, "r"'] & H_i(K_i) \arrow[l, "i^*"'] \end{tikzcd}$$

where r is defined using the homotopy equivalence with X over the i-skeleton and the result about h_2 in dimensions $\neq 4k$. If (r) in $H^i(X_i, \pi_i)$ determines the homomorphism $r\colon H_i(X_i) \longrightarrow \pi_i(X_i)$, then the homomorphism $H_i(K_i) \xrightarrow{r'} \pi_i(K_i)$ determined by $i^*(r)$ may be identified with $i^{-1} r\, i_*$. So $r'h$ is essentially rh. (See proof of Corollary 2 for details.) If $i \neq 4i$, hr is the identity so (r) goes to the fundamental class of $K(i, \pi_i)$ since its associated homomorphism is h^{-1}. The Corollary follows from the exactness of (S).

Corollary 2 The $4i^{\underline{th}}$ k-invariant of F/PL is a torsion element, and its reduction mod 2 is zero.

Proof: There is a class in $H^{4i}(X_{4i}, Z)$ which restricts to some non-zero multiple of the fundamental class of $K(4i, Z)$ where the latter space is the fibre of

$$X_{4i} \longrightarrow X_{4i-2} ,$$

in the Postnikov tower F/PL. This follows from the fact that

$$h: \pi_{4i}(F/PL) \longrightarrow H_{4i}(F/PL)$$

is a monomorphism. The first statement now follows from the exact sequence of Serre,

$$H^{4i}(X_{4i}, Z) \longrightarrow \underset{\substack{\cup\!\!\!\!\mid \\ i}}{H^{4i}(K(4i, Z), Z)} \xrightarrow{t} \underset{\substack{\cup\!\!\!\!\mid \\ k}}{H^{4i+1}(X_{4i-2}, Z)} .$$

For the second statement we proceed in the same manner using the mod 2 Serre sequence.

<u>Addendum:</u> Let

$$S_k = \frac{B_k\, j_k\, D_k}{\operatorname{ord}\,(\theta_{4k-1}(\partial\pi))}$$

where $B_k = (2k-1)!\,(k+1,2)$, the Bott index

$j_k =$ index $(\ker(\pi_{4k-1}(0) \xrightarrow{J} \pi_{4k-1}(F)))$

$D_k =$ index of the image of the composition

$$H^{4k}(B_{PL},Z) \xrightarrow{i} H^{4k}(B_o,Z) \xrightarrow{\text{coeff } P_k} Z\ .$$

Then the order of the $4k^{\underline{th}}$ k-invariant of F/PL divides S_k. $(S_1 = 48,\ S_2 = 360)$

<u>Proof:</u> One shows that there is a class u in $H^{4k}(B_{PL},Z)$ such that u pulls back to $S_k(\text{gen})$ under the composition

$$S^{4k} \xrightarrow{\text{gen. of } \pi_{4k}(F/PL)} F/PL \xrightarrow{b} B_{PL}\ .$$

In fact choose u so i^*u has coefficient D_k on P_k. Then the diagram

$$\begin{tikzcd}
H^{4k}(S^{4k}) & & H^{4k}(B_o) \arrow[ll, "(\text{gen ker}\,J)^*"'] \\
H^{4k}(S^{4k}) \arrow[u, "\text{ord}\,\theta_{4k-1}(\partial\pi)"'] & & H^{4k}(B_{PL}) \arrow[ll, "(b\circ\text{gen}\,\pi_{4k}(F/PL))^*"'] \arrow[u, "i^*"'] \\
S_k & & u \arrow[ll, dotted]
\end{tikzcd}$$

is commutative. So

$$(\text{ord}\ \theta_{4k-1}(\partial\pi)\ S_k = D_k B_k j_k.$$

Note that

$$S_k = \frac{B_k\ D_k\ \text{ord}\ \pi_{4k-1}(F)}{\text{ord}\ \pi_{4k-1}(PL/O)} .$$

<u>Corollary 3:</u> Let X be a finite complex, and $d = \left[\frac{\dim X}{4}\right]$. Let $P_* \subset (M, F/PL)$ be the subset consisting of maps $f: M \to F/PL$ such that $f^* \otimes Q = 0$ in all positive dimensions. Then P_* is finite. In fact,

$$\text{ord}\ P_* \leq \text{ord}\Big(\bigoplus_{i=0}^{d} \big(H^{4i+2}(X, Z_2) \oplus \text{Torsion}\ H^{4i}(X, Z)\big)\Big)$$

<u>Proof:</u> See proof of Corollary 4.

<u>Corollary 4:</u> Let X be a finite complex and let T_* denote the kernel of

$$(X, F/PL) \xrightarrow{b_*} (X, B_{PL}) .$$

Then $T_* \subset P_*$, and there exists a sequence of integers $S_1, S_2, \ldots$ such that

$$\operatorname{ord} T_* \leqslant \operatorname{ord}\Big(\bigoplus_{i=0}^{d} \big(H^{4i+2}(X,Z_2) \oplus (\text{Torsion } H^{4i}(X,Z) \otimes Z_{S_i})\big)\Big).$$

S_i is described in the Addendum to Corollary 2.

<u>Proof:</u> It follows from a spectral sequence argument that

$$H^*(B_{PL},Q) \xrightarrow{b_*} H^*(F/PL,Q)$$

is an isomorphism. Thus $T_* \subset P_*$. This also follows from below.

To get the estimate on the orders of T_* and P_* we look at the Postnikov tower of F/PL,

$$\begin{tikzcd}
 & F/PL \arrow[d] \\
 & \vdots \arrow[d] \\
 & X_{i+1} \arrow[d, "K(\pi_{i+1}(F/PL),\, i+1) = K_{i+1}"] \\
X \arrow[ur, "\hat{f}"] \arrow[r, "f"'] & X_i \arrow[d] \\
 & \vdots \arrow[d] \\
 & *
\end{tikzcd}$$

Now $X_{i+1} \xrightarrow{K_{i+1}} X_i$ is a principal fibration so the homotopy classes of liftings $\hat{f}$ of f are in one to one correspondence with (X, K_{i+1}). If $\hat{f}$ is one lifting then any other lifting is homotopic to the composition

$$X \xrightarrow{\hat{f} \times g} X_{i+1} \times K_{i+1} \xrightarrow{\mu} X_{i+1}$$

where $g: X \longrightarrow K_{i+1}$ and μ is the action.

If $\hat{f}^* u$ belongs to $H^{i+1}(X, Z)$, $\hat{f} \cdot g = \mu(\hat{f} \times g)$, and $i: K_{i+1} \longrightarrow X_{i+1}$ is the inclusion of a fibre, then it is easy to check that

$$\text{(e)} \qquad (\hat{f} \cdot g)^* u = (\hat{f}^* + (ig)^*)\, u$$

Now choose v_{4i} in $H^{4i}(B_{PL}, Z)$ so that

$$(b^* v_{4i}) \cap (h(\text{gen } \pi_{4i}(F/PL)) = S_i \quad .$$

Let u_{4i} denote the image of $(b^* v_{4i})$ in X_{4i}. Then $i^*(u_{4i}) = S_i$(fundamental class of K_{4i}).

Suppose by induction that the estimates of Corollary 3 and Corollary 4 are valid for maps of X into X_k where the sum is taken over cohomology dimensions $\leq k$.

To get an upper bound on the number of liftings of these to X_{k+1} when $k+1 = 4i+2$ we merely multiply these estimates by $\text{ord}(X, K_{4i+2}) = \text{ord } H^{4i+2}(X, Z_2)$.

If $k+1 = 4i$, $\hat{f}: X \to X_{k+1}$ is in P_* for X_{k+1} and covers $f: X \to X_k$ then $\hat{f}\cdot g$ covers f and is in P_* only if

$$(\hat{f}\cdot g)^*u = \hat{f}^*u + (ig)^*u$$

has finite order in $H^{4i}(X,Z)$ for all u in $H^{4i}(X_{i+1},Z)$. If we take $u = u_{4i}$ then we see that we need only multiply $\hat{f}$ by the subset of $(X,K(Z,4i))$ corresponding to some subset of $\mathrm{Torsion}(H^{4i}(X,Z))$ to construct all elements of P_* for X_{k+1} which project to f. This completes the inductive step for Corollary 3.

We prove Corollary 4 in the same way. The inductive step uses the fact that $\hat{f}$ is in T_* of X_{k+1} only if $\hat{f}^*u_{4i} = 0$. So if $(\hat{f}\cdot g)^*u_{4i} = 0$ then $g^*(i^*u_{4i}) = S_i(g^*i) = 0$. Thus we only need multiply $\hat{f}$ by certain maps $g: X \to K_{4i}$ (corresponding to elements in $H^{4i}(X,Z)$ of order S_i) to get the other liftings of f which are in T_* of X_{k+1}.

For k large,

$$P_*(X,X_k) \xrightarrow{p_k} P_*(X,F/PL)$$

is an isomorphism, where $p_k: F/PL \to X_k$ is the projection. Similarly for T_*. This completes the proof.

Corollary 5: Let Π denote the infinite product,
$K(Z_2,2) \times K(Z,4) \times \ldots \times K(Z_2,4i-2) \times K(Z,4i) \times \ldots$.
Then $H^*(F/PL;Z_2)$ and $H^*(\Pi;Z_2)$ are isomorphic as algebras.

Proof: We first fill the gap remaining in the proof of Theorem 20, namely that the reduction of k_4 to Z_2 coefficients is zero.

Let k denote the reduction of k_4 to Z_2 coefficients. Then k belongs to $H^5(K(Z_2,2);Z_2)$. If u denotes the fundamental class of $K(Z_2,2)$ then k must equal

$$a(u\, S_q^1\, u) + b(S_q^2\, S_q^1\, u)$$

where a and b are in Z_2. If m^* denotes a co-multiplication in $H^*(K(Z_2,2);Z_2)$ which is compatible with that coming from the H-space structure in F/PL then we must have

$$m^*(k) = k \otimes 1 + 1 \otimes k,$$

i.e. k must be primitive. Also we must have $S_q^1\,(k) = 0$ since k is the reduction (mod 2) of an integral class.

Now

$$m^*\,(u) = u \otimes 1 + 1 \otimes u$$

so

$$m^*\,(S_q^2\, S_q^1\, u) = S_q^2\, S_q^1\, u \otimes 1 + 1 \otimes S_q^2\, S_q^1\, u$$

(1) $$m^*\,(u\, S_q^1\, u) = (u \otimes 1 + 1 \otimes u)(S_q^1\, u \otimes 1 + 1 \otimes S_q^1\, u)$$

$$= (u\, S_q^1\, u \otimes 1 + 1 \otimes u\, S_q^1\, u) + (u \otimes S_q^1\, u + S_q^1\, u \otimes u).$$

Also

$$(2) \quad \begin{aligned} & S_q^1\,(u\,S_q^1\,u) = (S_q^1\,u)^2 \\ & S_q^1\,(S_q^2\,S_q^1\,u) = S_q^3\,(S_q^1\,u) = (S_q^1\,u)^2 \,. \end{aligned}$$

The first set of equations implies $(a,b) = (0,1)$ or $(a,b) = (0,0)$. The second set implies $(a,b) = (1,1)$ or $(a,b) = (0,0)$. Thus $(a,b) = (0,0)$ and $k = 0$. (I am indebted to J. Milnor for this argument.)

Now consider a fibration in the Postnikov tower for F/PL,

$$X_{i+1} \xrightarrow{K_{i+1}} X_i \quad .$$

Since there exists u in $H^{i+1}(X_{i+1},Z_2)$ which restricts to the fundamental class of K_{i+1} (reduced mod 2) the restriction map takes

$$H^*(X_{i+1},Z_2) \quad \text{onto} \quad H^*(K_{i+1},Z_2) \quad .$$

Thus the Serre spectral sequence (with Z_2 coefficients) collapses and $E_2 = E_\infty$.

Then it follows that $H^*(X_{i+1},Z_2)$ is isomorphic to

$$H^*(K_{i+1},Z_2) \otimes H^*(X_i,Z_2)$$

as an algebra if $H^*(X_i,Z_2)$ is a polynomial algebra (since $H^*(K_{i+1},Z_2)$ is a polynomial algebra.)

The Corollary follows by induction.

The PL-Structures on a Bundle over M

Let $E \xrightarrow{q} M$ be a PL k-disk bundle over the PL n-manifold M. Then E is a PL(n+k)-manifold and PL(E) is defined. We would like to consider the relationship between PL(E) and PL(M).

Definition 20 Define

$$PL(M) \xrightarrow{q^*} PL(E)$$

on representatives by

$$(L,g) \longrightarrow (g^*E, b(g)) ,$$

where b(g) is a bundle map covering g.

Definition 21 If M^n is closed, let M_o denote $M\text{-int } D^n$. If M^n is in $\overline{\mathcal{M}}_n$, let

$$PL(M) \xrightarrow{r} PL(M_o)$$

denote the map induced $M_o \subset M$. Let

$$PL(M_o) \xrightarrow{c} PL(M)$$

denote the map defined on representatives by

$$(L_o, g) \longrightarrow (L_o \cup (\text{cone on } bL_o), g \cup (\text{cone on } g/bL_o)).$$

<u>Lemma 21</u> Let M^n belong to $\mathcal{M}_n$ and let r and c be the maps defined in Definition 21. Then rc equals the identity of $PL(M_o)$ and cr equals the identity of $PL(M)$.

<u>Proof:</u> The first statement follows immediately on the level of representatives.

The second statement follows from the following:

Let $f: M \longrightarrow L$ be a homotopy equivalence of closed simply connected PL n-manifolds. Then f is homotopic to $f_1: M \to L$ where $f_1/D^n \subset M$ is a PL-homeomorphism and $f_1(M-D^n) \subset M-f_1(D^n)$.

This follows from an easy argument using t-regularity and framed cobordism of finite discrete sets.

<u>Lemma 22</u> Let

$$(M, F/PL) \xrightarrow{q^*} (E, F/PL)$$

be defined by $f \longrightarrow fq$. Then

$$\begin{tikzcd} PL(M) \arrow[r, "C"] \arrow[d, "q^*"] & (M, F/PL) \arrow[d, "q^*"] \\ PL(E) \arrow[r, "C"] & (E, F/PL) \end{tikzcd}$$

is commutative.

<u>Proof:</u> Let (L,g) be a PL-structure on M with classifying bundle (v,rj). (v,j) is a good tubular neighborhood of

$$g': M \longrightarrow L,$$

a homotopy inverse for g. Now in the diagram

$$\begin{tikzcd} v \arrow[r, "j"] \arrow[d, "p"] & L \times \operatorname{int} D^r \arrow[d, "p_1"] \\ M \arrow[r, "g'"] & L \end{tikzcd}$$

each map is a homotopy equivalence. Thus we may lift this diagram using $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$ to obtain

$$\begin{tikzcd} p^* E \arrow[r, "b(j)"] \arrow[d, "b(p)"'] & g^* E \times \operatorname{int} D^r \arrow[d, "b(p_1)"] \\ E \arrow[r, "b(g')"] & g^* E \end{tikzcd}$$

where $b(f)$ denotes a bundle map covering f. Now $b(p)$ is a bundle projection since p was, and $b(j)$ is an embedding (since j was). Now the open disk bundle $p^*E \longrightarrow E$ contains $g^*E \times 0$ since $j(E')$ contained $L \times 0$. Thus $(p^*E, b(j))$ is a good tubular neighborhood of $b(g')$.

For suitable $r_1 \colon D^k \longrightarrow D^k$ then $(p^*E,\ r_1p_2b(j))$ is a classifying bundle for $(g^*E,\ b(g))$; $C(q^*(L,g))$. So we want to show that it is equivalent to

$$(q^*V, rjb(g)) = q^*(C(L,g)).$$

We know that

$$\begin{tikzcd}
p^*(E) \arrow[r, "b(j)"] \arrow[d, "p^*(q)"] & g^*E \times D^k \arrow[r, "p_2"] \arrow[d, "g^*(q) \times \mathrm{id}."] & D^k \\
V \arrow[r, "j"] & L \times D^k \arrow[ur, "p_2"'] &
\end{tikzcd}$$

is commutative. Also, there is a bundle map I so that

$$\begin{tikzcd}
 & p^*E \arrow[dl, "p^*(q)"'] \arrow[r, "b(p)"] \arrow[d, "I"] & E \arrow[d, "\mathrm{id}"] \\
V & q^*V \arrow[l, "b(q)"] \arrow[r, "q^*(p)"'] & E
\end{tikzcd}$$

is commutative. (Let I be the identity.) Therefore

$$\begin{aligned}
C(q^*(L,g)) &= (p^*E,\ r_1p_2b(j)) \\
&= (p^*E,\ r_1p_2jp^*(q)) \\
&= (q^*V,\ r_1p_2jb(q)) \\
&= q^*\ (C(L,g))
\end{aligned}$$

since the choice of r doesn't affect the equivalence class.

<u>Theorem 23</u> Let M be in $\mathcal{M}_n$ and suppose $E \xrightarrow{q} M$ be a k-disk bundle over M. Define

$$PL(M) \xrightarrow{q^*} PL(E)$$

on representatives by

$$(g:L \longrightarrow M) \rightarrow (b(g):g^*E \longrightarrow E).$$

Then q^* is bijective.

<u>Proof:</u> We first show that (E, bE) is a simply-connected manifold pair. Now

$$bE = E/bM \cup_{E_s/bM} E_s$$

where E_s is the associated sphere bundle. From the homotopy sequence of the fibration we have

$$\pi_1(S^{k-1}) \longrightarrow \pi_1(E_s) \longrightarrow \pi_1(M) = 0$$

and

$$\pi_1(S^{k-1}) \longrightarrow \pi_1(E_s/bM) \rightarrow \pi_1(bM) = 0.$$

So for $k > 2$,

$$\pi_1(E_s) = \pi_1(E_s/bM) = 0.$$

For $k = 2$,

$\pi_1(E_s)$ and $\pi_1(E_s/bM)$ are generated by the fibre sphere which goes to zero in bE. For $k = 1$, $bE = b(M \times I) =$ double of M. So in all cases $\pi_1(bE) = 0$ by

Van Kampen's Theorem. ($\pi_1(E)$ and $\pi_1(E/bM)$ are clearly zero.)

By Theorem 9, $PL(E) \xrightarrow{c} (E, F/PL)$ and $PL(M) \xrightarrow{c} (M, F/PL)$ are bijective. $(M, F/PL) \xrightarrow{q^*} (E, F/PL)$ is bijective. So the theorem follows from Lemma 22.

Theorem 23 is germane to a current problem in differential and PL topology.

The Bundle Problem: Let M be a PL n-manifold, $E \xrightarrow{p} M$ a PL k-disk bundle over M, and

$$(W, bW) \xrightarrow{G} (E, bE)$$

a homotopy equivalence. The Bundle Problem for $(E, M; G)$ is the following: find a PL n-manifold L, a homotopy equivalence $g: (L, bL) \longrightarrow (M, bM)$, a bundle projection $W \xrightarrow{q} L$; and change G by a homotopy to b so that b is a bundle map covering g,

$$\begin{array}{ccc} W & \xrightarrow{b} & E \\ \downarrow q & & \downarrow p \\ L & \xrightarrow[g]{} & M \end{array} .$$

We note that if the h-cobordism theorem applies to (W, bW) then the Bundle Problem for $(E, M; G)$ is essentially the problem of changing G by a homotopy so that it is t-regular to the 0-section of E and

$$(G^{-1}(M),\ bG^{-1}(M)) \xrightarrow{G} E/(M, bM)$$

is a homotopy equivalence.

Corollary 1 Let M be in $\mathcal{M}_n$ and E be a k-disk bundle over M. Let

$$(W, bW) \xrightarrow{G} (E, bE)$$

be a homotopy equivalence. Then there is a solution to the Bundle Problem for $(W, M; G)$.

Proof: By Theorem 23 (W, G) is concordant to $(g^*E, b(g))$ for some $g: (L, bL) \longrightarrow (M, bM)$. We apply the h-cobordism theorem to straighten out this concordance, i.e. if (Q, J) is the concordance choose a PL-homeomorphism

$$H': W \times I \longrightarrow Q$$

so that

$$\begin{aligned} H'/W \times 0 \quad &= \quad \text{identity of } W \text{ and} \\ H'/W \times 1 \quad & \quad \text{is onto } g^*E. \end{aligned}$$

If we let

$$q = g^*(p) \cdot H'/W \times 1$$

$$b = b(g) \cdot H'/W \times 1$$

$$H = JH'$$

then q and b are as desired and H is a homotopy between G and b.

Corollary 1 is true in such generality because $bM \neq 0$. Now we consider the Bundle Problem for closed M.

Let $M_o = M\text{-int } D^n$ and $E_o = E/M_o$. We can apply the Browder Theorem to get a map

$$PL(E) \xrightarrow{b(r)} PL(E_o)$$

if $k \geqslant 3$, and M belongs to $\overline{\mathcal{M}}_n$. Its easy to check that

$$\begin{tikzcd} PL(E) \arrow[r, "b(r)"] \arrow[d, "q^*"] & PL(E_o) \arrow[d, "q_o^*"] \\ PL(M) \arrow[r, "r"] & PL(M_o) \end{tikzcd}$$

is commutative. Thus we obtain

<u>Corollary 2</u> If M belong to $\overline{\mathcal{M}}_n$, n odd and $k \geqslant 3$, then the Bundle Problem for $(M,E;G)$ has a solution.

Proof: Consider the commutative diagram

$$\begin{tikzcd} PL(E) \arrow[r, "C"] \arrow[d, "b(r)"] & (E,F/PL) \arrow[d, "r^*"] \\ PL(E_o) \arrow[r, "C"] & (E_o,F/PL) \end{tikzcd}$$

where r^* is induced by $E_o \subset E$.

Since the Hurewicz homomorphism is a monomorphism for F/PL, r^* is onto. Since $\pi_n(F/PL) = 0$ when n is odd r^* is injective. Thus $b(r)$ is bijective by Theorem 9.

r and q_o^* are bijective by Lemma 21 and Theorem 23. Thus q^* is bijective. The proof is now completed just as in Corollary 1.

The case when n is even is harder to deal with. We must have additionl hypothesis on G in order that the Bundle Problem for $(E,M;G)$ be solvable.

Let $f: M \longrightarrow F/PL$ and u belong to $H^i(F/PL,A)$ where $A = Q$ or Z_2. Let p denote a monomial in the rational Pontryagin classes of M and w a monomial in the Stiefel-Whitney classes of M. Then $(p \cup f^*u)(M)$ is called the Pontryagin number (u,p) of (M,f) (when $A = Q$) and $(w \cup f^*u)(M)$ is called the Stiefel-Whitney number (u,w) of (M,f)(when $A = Z_2$.)

Let $E \xrightarrow{p} M$ be a k-disk bundle over M in $\mathcal{M}_n$ and

$$(W, bW) \xrightarrow{G} (E, bE)$$

a homotopy equivalence, $k \geq 3$. Let $G' : (E, bE) \to (W, bW)$ be a homotopy inverse for G. Let $t(X)$ denote the tangent bundle of X, where X is a PL-manifold.

<u>Theorem 24</u> Let M be in $\mathcal{M}_n$ and $k \geq 3$. Then there exists

$$f: M \longrightarrow F/PL$$

so that

$$E \xrightarrow{p} M \xrightarrow{f} F/PL \xrightarrow{b} B/PL$$

classifies $(G')^*(t(W)) \oplus - t(E)$ and so that the Bundle Problem for $(E, M; G)$ is solvable when

i) $n = 4i$ iffi a certain linear combination of Pontryagin numbers of (M, f) vanishes,

ii) $n = 4i+2$ and M is smoothable iffi a certain linear combination of Stiefel-Whitney numbers of (M, f) vanishes.

<u>Proof:</u> First let $f: M \to F/PL$ correspond to the $F/PL)_k$-bundle, (E_0, t), $k \gg n$, and consider the significance of $n(M, f)$. Then

$$p_0 \times t: \; E_0 \longrightarrow M \times D^k$$

determines a PL-structure on $M \times D^k$.

Now $n(M,f)$ is defined so that $n(M,f) = 0$ iff t may be changed by a homotopy H so that it "splits". Then $p_0 \times H$ is a homotopy between $p_0 \times t$ and b say, where b is t-regular to $M \times 0$ and

$$\begin{tikzcd} L = b^{-1}(0) \subset \overset{q}{E_0} \arrow[r, "b"] \arrow[r, bend right, "g"'] & M \times D^k \end{tikzcd}$$

is a homotopy equivalence. Now it is easy using the h-cobordism theorem to alter b slightly so that (L,g,b) is a solution to the Bundle Problem for $(M{\times}D^k, M;\ p_0{\times}t)$.

Let $-E$ denote a high dimensional inverse for $E \xrightarrow{p} M$. Then $(p^*(-E),\ p^*(p))$ is a bundle over E and $(p^*(-E) = E \oplus -E, p \cdot p^*(p))$ may be identified with the trivial bundle $(M{\times}D^k, p_1)$ over M.

$$\begin{tikzcd}
 & (pG)^*(-E) \arrow[r, "b(G)"] \arrow[dl] \arrow[d] & p^*(-E) \arrow[r, "b(id)"] \arrow[dl, "p^*(p)"] \arrow[dd, "q"] & M \times D^k \arrow[dd, "p_1"] \\
W \arrow[r, "G"'] & E \arrow[d] \arrow[dr, "p"] & & \\
 & L \arrow[r, "g"'] & M \arrow[r, "id"] & M
\end{tikzcd}$$

Now $b(G)$ and $q = p \cdot p^*(p)$ are homotopy equivalences so we can find an embedding $i\colon M \to \operatorname{int}(pG)^*(-E)$ so that

$$q\ b(G)\ i \simeq \mathrm{id}\ M \quad \text{and} \quad i\ q\ b(G) \simeq \mathrm{id}(pG)^*(-E).$$

Using the h-cobordism theorem we may identify $(pG)^*(-E)$ with the normal bundle of $i(M)$ in $G^*(-E)$ (which exists if $(pG)^*(-E)$ has large enough dimension.)

Thus $(pG)^*(-E)$ is a PL-bundle over M and $t = p_2 b(id)\ b(G)$ is an F-trivialization. Let $r:(pG)^*(-E) \to M$ be the projection of this bundle. Then

$$\begin{aligned} r \times t &= r \times p_2(b(id)) \cdot b(G) \\ &\simeq qb(G) \times p_2(b(id))\ b(G) \\ &= p_1 b(id) b(G) \times p_2(b(id))\ b(G) \\ &= b(id)\ b(G), \end{aligned}$$

where the homotopy is a homotopy of maps of pairs, $(pG)^*(-E), bpG^*(-E)) \longrightarrow M \times (D^k, bD^k)$.

If we let $f:M \to F/PL$ classify $((pG)^*(-E),t)$, then the above analysis shows that $n(M,f) = 0$ iffi we can solve the bundle problem for $(M,(pG)^*(-E);\ b(G))$. Let (L,g) be such a solution.

Now consider the commutative diagram

$$\begin{tikzcd} & PL(p^*(-E)) \\ PL(M) \arrow[ur, "q^*"] \arrow[dr, "p^*"'] & \\ & PL(E) \arrow[uu, "(p^*(p))^*"'] \end{tikzcd}$$

.

Since $k \geq 3$ $(p^*(p))^*$ is bijective. Now

$$q^*(L,g) = ((pG)^*(-E),\ b(G)) = (p^*(p))^*(W,G)\ .$$

Thus $p^*(L,g)$ is concordant to (W,G).

Let $v_{i(M)}$ = normal bundle of $M \xrightarrow{i} (pG)^*(-E)$ and $t(x)$ the tangent bundle of X. Then

$$\begin{aligned} t(M) \oplus v_{i(M)} &= i^*(t((pG)^*(-E))) \\ &= i^*((qb(G))^*(-E) \oplus \\ &\qquad (G^*(p^*(p)))^* t(W)) \\ &= (qb(G)i)^*(-E) \oplus (G'(0\text{-section}))^* t(W) \\ &= (-E) \oplus (G' \cdot 0\text{-section})^* t(W) \end{aligned}$$

So

$$\begin{aligned} p^* v_{i(M)} &= (-p^* tM) \oplus p^*(-E) \oplus G'^* t(W) \\ &= G'^*(t(W)) \oplus -(t(E))\ . \end{aligned}$$

This shows that (M,f) has almost all of the desired properties. The theorem follows since $n(M,f)$ is given by the homomorphism

$$\Omega_n^{PL}(F/PL) \xrightarrow{K} P_n\ .$$

If $n = 4i$, $P_n = Z$ and elements in $\Omega_n^{PL}(F/PL) \otimes Q$ are determined by their Pontryagin numbers. Thus K may be expressed as a rational linear combination of Pontryagin numbers. If $n = 4i + 2$, $P_n = Z_2$ and $K/\Omega_n(F/PL)$ may be factored through $\mathcal{N}_n(F/PL)$ since

a) kernel $(\Omega_n(F/PL) \longrightarrow \mathcal{N}_n(F/PL))$ is divisible by 2

b) $\mathcal{N}_n(F/PL)$ = direct sum of Z_2's .

Then the factor homomorphism

$$\mathcal{N}_n(F/PL) \longrightarrow Z_2$$

may be expressed as a linear combination of Stiefel-Whitney numbers.

<u>Addendum:</u> The form of the linear combinations appearing in Theorem 24 depends only on the dimension of M.

We remark that there is a similar solution when $n = 4i + 2$ and M is merely a PL-manifold.

In case $n = 4i$, the classes of $H^*(F/PL, Q)$ occurring in i) come from $H^*(B_{PL}, Q)$ so tangential information about G is sufficiant (and necessary) to solve the bundle problem.

It is quite an interesting and difficult problem to determine the classes of $H^*(F/PL, Z_2)$ occurring in ii) and their relationship to $H^*(B_{PL}, Z_2)$.

This has an important bearing on the obstruction theory described in Theorem 26 .

Applications to PL-Homeomorphisms

Now we may apply the results of the previous sections to study the manifolds within a given homotopy equivalence class.

Theorem 25: Let M belong to $\mathcal{M}_n$. Then there is a one to one correspondence

$$PL(M) \xrightarrow{C} (M, F/PL)$$

between the set of concordance classes of PL-structures on M and the set of free homotopy classes of maps of M into F/PL.

If M is in $\tilde{\mathcal{M}}_n$ and $bM = 0$, then $PL(M)$ is in one to one correspondence with $(M_o, F/PL)$ where $M_o = M\text{-int } D^n$. Such a correspondence is given by the composition

$$PL(M) \xrightarrow{r} PL(M_o) \xrightarrow{C} (M_o, F/PL),$$

where r is induced by $M_o \subset M$.

C is natural with respect to inclusions $M_1 \subset \text{int } M_2$ in $\tilde{\mathcal{M}}_n$ and $C(M, id) = $ pt map. In general if (L, g) is a PL-structure on M and g' is a homotopy inverse for g then

i) $C(L, g) = 0$ iffi g is homotopic to a PL-homeomorphism

ii) $b_*(C(L,g))$ classifies

$$t(M) \oplus -(g')^* t(L) \qquad \text{if } bM \neq 0$$
$$(t(M) \oplus -(g')^* t(L))/M_o \qquad \text{if } bM = 0$$

where $t(M)$ is the tangent bundle of M and $F/PL \xrightarrow{b} B_{PL}$ is the fibre of $B_{PL} \xrightarrow{J} B_F$.

Proof: This follows from Theorems 9,10,14 and Lemmas 1 and 21.

Definition 22: By a k-skeleton of M we mean a subcomplex $L \xrightarrow{i} M$ where L has the homotopy type of a k-dimensional complex and i is k-connected i.e. the relative groups $\pi_i(M,L)$ are zero for $i \leq k$.

We say that the k-skeleton $L \xrightarrow{i} M$ is a thickened k-skeleton of M if $L \xrightarrow{i} M$ is a morphism of $\bar{\mathcal{M}}_n$.

We say that $g:(L,bL) \longrightarrow (M,bM)$ is homotopic to a PL-homeomorphism over the k-skeleton of M if for any thickened k-skeleton $L_k \longrightarrow L$ there is a homotopy equivalence $f:(L,bL) \to (M,bM)$ such that

1) f is homotopic to g

2) f/L_k is a PL-embedding

3) if $L'_k = L\text{-int } L_k$ and $M'_k = M\text{-int } f(L_k)$ then f induces a homotopy equivalence $(L'_k, bL'_k) \longrightarrow (M'_k, bM'_k)$.

<u>Theorem 26:</u> Let M be in $\mathcal{M}_n$ and let M_o denote M if $bM \neq 0$ and M-int D^n if $bM = 0$. Let $g:(L,bL) \to (M,bM)$ be a homotopy equivalence.

Then there is a map

$$M_o \xrightarrow{C_g} F/PL$$

with the following two properties :

i) $M_o \xrightarrow{C_g} F/PL \xrightarrow{b} B_{PL}$ classifies

$$(t(M) \ominus -(g')^* t(L))/M_o$$

ii) $C_g/(k\text{-skeleton of } M)$ is homotopic to the point map iffi g is homotopic to a PL-homeomorphism over the k-skeleton of M.

<u>Proof:</u> Let $C_g = C(L,g)$ and suppose $M_k \subset M$ is a thickened k-skeleton of M. We can change g by a homotopy so that it induces a PL-structure (L_k, g_k) on M_k. Since C is a natural transformation, $C(L_k,g_k) = C_g/M_k$. Since C is injective, $g_k:(L_k,bL_k) \to (M_k,bM_k)$ is homotopic to a PL-homeomorphism iffi C_g/M_k is homotopic to zero. Now $C_g/(k\text{-skeleton of } M_o)$ is homotopic to zero iffi C_g/M_k is homotopic to zero(by osbtruction theory) and g_k is homotopic to a PL-homeomorphism iffi g satisfies 1),2) and 3) of Definition 22 above by the homotopy extension theorem. This completes the proof.

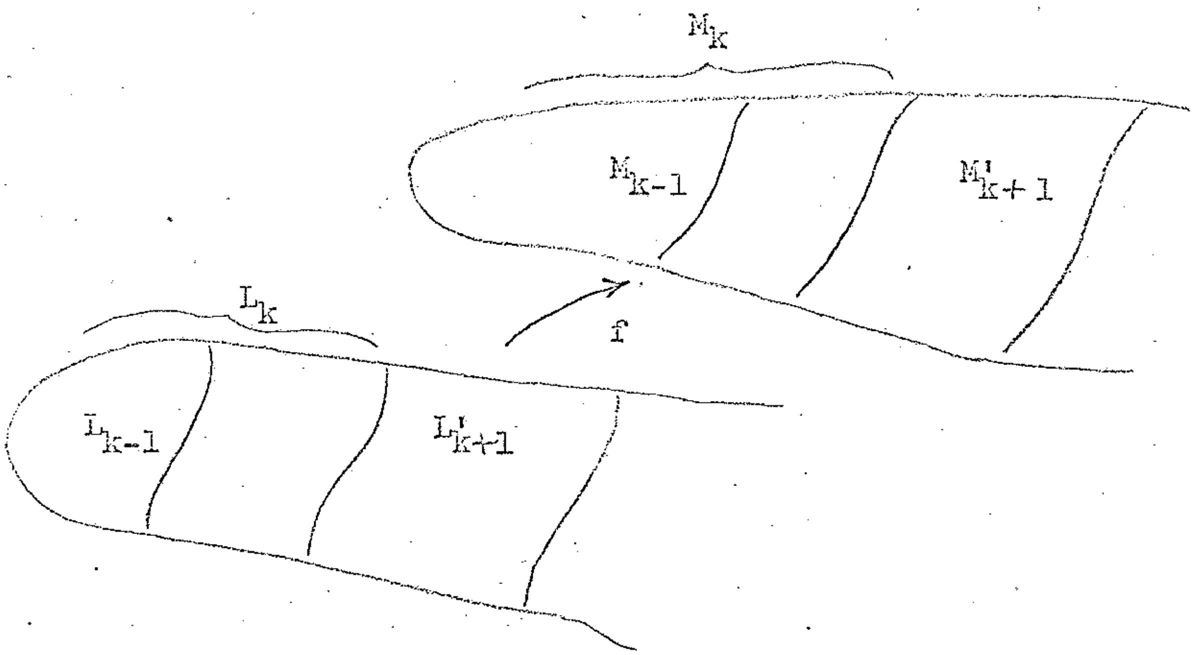
Corollary: If M belongs to $\mathcal{M}_n$ there is an obstruction theory for the problem of deforming a homotopy equivalence $(L,bL) \xrightarrow{g} (M,bM)$ into a PL-homeomorphism. The obstructions lie in

$$H^2(M_o,Z_2), H^4(M_o,Z), \ldots, H^i(M_o,\pi_i(F/PL)), \ldots .$$

Suppose $M_1 \subset \text{int } M_2 \subset M_2 \subset \text{int } M_3 \subset \cdots \subset M_n = M$

form an increasing sequence of thickened skeletons of M and g is homotopic to f, a PL-homeomorphism over M_k. Then there is a cohomology class O_{k+1} in $H^{k+1}(M,\pi_{k+1}(F/PL))$ which is the precise obstruction to changing f by a homotopy on the complement of M_{k-1} so that it becomes a PL-homeomorphism over the $k+1$-skeleton, M_{k+1}.

The obstruction class O_{k+1} may be computed as follows. Let $M'_{k+1} = M_{k+1} - \text{int } M_k$ and $L'_{k+1} = L_{k+1} - \text{int } L_k$. Assume $f/L-\text{int } L_k$ induces a PL-structure on M'_{k+1}



f/bL_k is a PL-homeomorphism so let

$$M_* = (M'_{k+1} \cup_{bM_k} M'_{k+1}) \times D^r$$

$$L_* = (M'_{k+1} \cup_{f^{-1}} L'_{k+1}) \times D^r$$

and define

$$f_*: L_* \longrightarrow M_*$$

by (identity $\cup$ f)x identity D^r, r large.

Now $H_{k+1}(M'_{k+1}, bM_k)$ corresponds to the k^{th} chain group of $H_*(M_o)$ and is naturally embedded in $H_{k+1}(M_*)$. Each class u in $H_{k+1}(M'_{k+1}, bM_k)$ may be represented by a relative (k+1) disk in $(M'_{k+1}, bM_k) \times D^r$ and thus corresponds (by doubling) to an $S^{k+1} \times D^s$ in M_*, where $k+1+s = n+r$.

Now change f_* by a homotopy so that it induces a PL-structure on $S^{k+1} \times D^s$ and let

$$(E,t) = (f_*^{-1}(S^{k+1} \times D^s),\ f_*/f_*^{-1}(S^{k+1} \times D^s)) .$$

Then $(E, p_2 t)$ may be identified to an $F/PL)_r$-bundle over S^{k+1} and thus corresponds to some element $C_{k+1}(u)$ in $\pi_{k+1}(F/PL)$. The cochain $C_{k+1}(u)$

a) is a cocycle

b) vanishes iffi f can be changed by a homotopy (which is constant on M_k) to a PL-homeomorphism over M_{k+1}

c) is cohomologous to zero iffi f can be changed by a homotopy (which constant on M_{k-1}) to a PL-homeomorphism over M_{k+1}.

This may be proved using the cobordism obstruction theory of Part I.

Now we relate the obstructions to deforming $g:L \longrightarrow M$ into a PL-homeomorphism to a sequence of Bundle Problems.

Lemma (Stability) $(L,bL) \xrightarrow{g} (M,bM)$ is homotopic to a PL-homeomorphism over the k-skeleton of M iffi $g \times id:(L,bL) \times D^r \longrightarrow (M,bM) \times D^r$ is homotopic to a PL-homoeomorphism over the k-skeleton of $M \times D^r$.

Proof: This follows from Lemma 22 and Theorem 26.

The lemma shows that it suffices to consider the obstructions in dimensions which are small with respect to that of the ambient manifold.

So assume that $i << n$ and consider x in $H_{4i}(M_o,Z)$ or $H_{4i+2}(M_o,Z_2)$. Using (5) we may represent some odd multiple of x by an embedded submanifold $M_i \subset$ int M. We may assume that M_i is simply connected (since M is), smoothable, and that M_i has a normal disk bundle E .

(We may cross M with D^k to construct E if necessary.) Now change

$$g:(L,bL) \longrightarrow (M,bM)$$

by a homotopy so that it induces a PL-structure on E and let $(W,G) = (L,g)/E$. Then (W,G) defines a Bundle Problem for E. (E,M_i,G) is called a Bundle Problem associated to x by g.

<u>Definition 23:</u> We say that x in $H_{4i}(M,Z)$ or $H_{4i+2}(M,Z_2)$ "splits" the homotopy equivalence $g:(L,bL) \longrightarrow (M,bM)$ if some Bundle Problem (E,M_i,G) associated to x by g has a solution.

<u>Theorem 27:</u> Let M be in $\mathcal{M}_n$ and $g:(L,bL) \to (M,bM)$ be a homotopy equivalence. Let $C_g: M_o \to F/PL$ represent $C(L,g)$ and suppose C_g is homotopic to the point map over the k-skeleton of M_o, k odd, $k << n$. Let O_{k+1} in $H^{k+1}(M_o, \pi_{k+1}(F/PL))$ be the obstruction to extending the homotopy of C_g with the point map over the $k+1$ skeleton of M_o. Then if x is in $H_{k+1}(M_o, \pi_{k+1}(F/PL))$,

$$O_{k+1}(x) = 0$$

iffi x "splits" g.

Proof: Let $L' \subset \text{int } M$ represent some odd multiple of x in $H_{k+1}(M_0, \pi_{k+1}(F/PL))$. Suppose E is a normal disk bundle of $L' \subset M$ and $(W,G) = (L,g)/E$. Let $C_G = C(W,G)$. By the naturality of C, $C_G/(E/L'\text{-pt}) = C_g/(E/L'\text{-pt}) = 0$ by hypothesis. The obstruction to deforming $C_G = C_g/E$ to a point then it just $O_{k+1}\cdot(\text{gen } H_{k+1}(E))$ by the naturality of the obstruction. Thus G is homotopic to a PL-homeomorphism iffi $O_{k+1}(x)=0$. Thus if $O_{k+1}(x)=0$ it is clear that x "splits" g. In fact, we have shown that in this case any Bundle Problem associated to x is solvable.

Now suppose that x"splits" g. Let

$$\begin{array}{ccc} W & \xrightarrow{G} & E \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & L' \end{array}$$

be the solution of some Bundle Problem associated to x. Now G is homotopic to a PL-homeomorphism over the k-skeleton of E so it follows from Lemma 22 and Theorem 26 that f is homotopic to a PL-homeomorphism over the k-skeleton of L. But then Lemma 21 implies f is homotopic to a PL-homeomorphism. Thus the same is true of G, so $O_{k+1}(x)=0$.

The rationale behind Theorem 27 is the following. It is possible to "compute" the homomorphisms determined by obstructions to deforming $g:L \longrightarrow M$ to a PL-homeomorphism "before" we begin the deformation. We merely cross M with a high-dimensional disk, represent odd multiples of all the pertinent homology classes by embedded submanifolds and look at the associated Bundle Problems.

We state three Corollaries which follow from the existence of the obstruction theory. Let M belong to $\mathcal{M}_n$ and let $f:(L,bL) \longrightarrow (M,bM)$ be a homotopy equivalence.

Corollary 1 Suppose

$$\begin{aligned} H^{4i+2}(M_o,Z_2) &= 0 \\ H^{4i}(M_o,Z) &= 0 \qquad i>0 . \end{aligned}$$

Then f is homotopic to a PL-homeomorphism.

Corollary 2 Suppose $f^*(p_i(M,Q)) = p_i(L,Q)$ for $i \leq \left[\frac{\dim M}{4}\right]$, and

$$\begin{aligned} H^{4i+2}(M_o,Z_2) &= 0 \\ H^{4i}(M_o,Z) &\quad \text{is free} \end{aligned}$$

then f is homotopic to a PL-homeomorphism. (Note that Corollary 2 applies if f is a homeomorphism.)

Corollary 3 Suppose f is a stable PL-tangential equivalence and that

$$H^{4i+2}(M_o,Z_2) = 0 \qquad \text{and}$$

$(\text{Torsion } H^{4i}(M_o,Z)) \otimes Z_{S_i}) = 0$, where S_i is defined in Corollary 2 of Theorem 20. Then f is homotopic to a PL-homeomorphism.

Proof: These results follow from Theorem 25 and the Corollaries to Theorem 20.

Corollaries 1 and 2 are positive results towards the Hurewicz Conjecture and the Houptvermutung. There are many examples showing why the conditions of Corollary 1 are necessary. (See below).

Definition 23: Let (M) denote the set of PL-homeomorphism classes of manifolds which are homotopically equivalent to M. Let L represent an element of (M) and let $g:L \to M$ be a homotopy equivalence. Let $P_*(M)$, $H(M)$, and $T_*(M)$ be the subsets of (M) defined respectively by those L such that g can be chosen to be

$P_*(M)$: a correspondence of rational Pontryagin classes

$H(M)$: a homeomorphism

$T_*(M)$: a stable PL-tangential equivalence.

Remark: It is clear that $T_*(M) \subset P_*(M)$. It follows from Novikov (20) that $H(M) \subset P_*(M)$.

Recall that $P_*(X,F/PL)$ denotes the subset of $(X,F/PL)$ consisting of those maps which induce trivial cohomology homomorphisms with Q-coefficients in positive dimensions and $T_*(X,F/PL)$ denotes the kernel of

$$(X,F/PL) \xrightarrow{b_*} (X,B_{PL}).$$

Let M belong to $\overline{\mathcal{M}}_n$ and let M_o denote M-int D^n if $bM = 0$ and $M_o = M$ if $bM \neq 0$. Then as corollaries to Theorem 25 we obtain

<u>Corollary 4</u> There is a natural projection of $(M_o,F/PL)$ <u>onto</u> (M). $P_*(M_o,F/PL)$ is carried <u>onto</u> $P_*(M)$ and the image of $T_*(M_o,F/PL)$ contains $T_*(M)$.

<u>Proof:</u> The first statement follows from the fact that there is a natural projection of $PL(M)$ onto (M). The map $(L,g) \to (L)$ is well-defined because of the h-corbordism theorem. So

$$(M_o,F/PL) \xrightarrow{C^{-1}} PL(M) \longrightarrow (M)$$

is the desired projection.

The second statement follows from b) of Theorem 25. The fact that $P_*(M_o,F/PL)$ goes "into" $P_*(M)$ is clear when $bM \neq 0$ or $\dim M \neq 4i$. In the excluded case it is still true because of the Hirzebruch Index Formula.

It is not so clear that $T_*(M_o, F/PL)$ is carried into $T_*(M)$ if $bM = 0$ and $n = 8k+2$.

<u>Corollary 5</u> $P_*(M)$ is finite. In fact if $d = \left[\frac{\dim M}{4}\right]$,

$$\operatorname{card} P_*(M) \leq \operatorname{ord}\left(\bigoplus_{i=0}^{d}\left(H^{4i+2}(M_o, Z_2) \oplus \text{Torsion } H^{4i}(M_o, Z)\right)\right).$$

<u>Corollary 6</u> (M) is finite if $H^{4i}(M_o, Q) = 0$ $i > 0$. $H(M)$ is finite. In fact if $H_i = (E_i/D_i)S_i$ where E_i is the index of

$$H^{4i}(B_{Top}) \longrightarrow H^{4i}(B_o) \xrightarrow{\text{coef. } p_i} Z.$$

then

$$\operatorname{card} H(M) \leq \operatorname{ord}\left(\bigoplus_{i=0}^{d}\left(H^{4i+2}(M_o, Z_2) \oplus (\text{Torsion } H^{4i}(M_o, Z) \otimes Z_{H_i})\right)\right)$$

<u>Corollary 7</u>

$$\operatorname{card} T_*(M) \leq \operatorname{ord}\left(\bigoplus_{i=0}^{d}\left(H^{4i+2}(M_o, Z_2) \oplus (\text{Torsion } H^{4i}(M_o) \otimes Z_{S_i})\right)\right).$$

<u>Proof:</u> 5) and 7) follow from Corollaries 3) and 4) of Theorem 20. 6) follows by a completely analogous argument.

<u>Examples:</u>

1) We give some examples which illustrate the conditions in Corollary 1.

These examples arise from the maps

$$S^{2k} \xrightarrow{\text{gen } \pi_{2k}} F/PL$$

$$f$$

and

$$\begin{tikzcd} S^{2k-1} \cup_d e^{2k} \arrow[r, "1"'] \arrow[rr, bend right, "f_d"'] & S^{2k} \arrow[r, "\text{gen } \pi_{2k}"] & F/PL \end{tikzcd} .$$

In the first case if we let $M = S^{2k} \times D^r$, r large, and (L,g) in PL(M) correspond to f, then for $k \geqslant 2$ and $\equiv 0,1,2 \pmod 4$ L is not even homeomorphic to M. This follows from the fact that $\pi_* B_{PL} \to \pi_* B_{TOP}$ is a monomorphism (24).

In the second case if we let M (regular neighborhood of $S^{2k-1} \cup_d e^{2k}$ in S^r), r large, and (L,g) correspond to f_d then for certain d, M and L are not tangentially equivalent.

If $k = 2i$, then L is not a π-manifold if d does not divide S_i.

For any k L is not smoothable if d does not divide the generator of $\theta_{2k-1}(\partial\pi)$ in θ_{2k-1}. (e.g. if $k = 8m+2$, $m > 0$, and d is some large power of 2 then L is not smoothable.)

The first set of examples illustrates the Hom conditions in Corollary 1; the second set the Ext conditions.

The phenomenon of L above not being smoothable can be explained by a Bockstein operation relating the F/PL obstructions to F/O obstructions in an analogous smooth theory.

2) We construct a closed smooth 8-manifold M such that

i) there is homotopy equivalence $g: M \to \mathbb{C}P^4$ such that $g^* t(\mathbb{C}P^4) = t(M)$, where $t(X)$ denotes the smooth stable tangent bundle of X

ii) M is not PL-homeomorphic to $\mathbb{C}P^4$.

Consider the map

$$\begin{tikzcd} \mathbb{C}P^4\text{-pt} \arrow[r, "h"] \arrow[rrr, bend right, "f"] & \mathbb{C}P^3 \arrow[r, "\deg 1"'] & S^6 \arrow[r, "\text{gen } \pi_6"] & F/PL \end{tikzcd},$$

where h is some homotopy equivalence. Let (M_o, g_o) correspond to f in $PL(\mathbb{C}P^4_o)$. Let (M,g) = cone on (M_o, g_o) in $PL(\mathbb{C}P^4)$.

i) Now g_o is a stable PL tangential equivalence because $\pi_6(F/PL) \xrightarrow{b} \pi_6(B_{PL})$ is the zero map.

Thus g is a stable PL tangential equivalence because any stable fibre homotopically trivial PL bundle over S^8 with zero p_2 is trivial.

Thus we can smooth M so that i) holds.

ii) (M_o, g_o) is not concordant to zero because $f^*(Z_2 \text{coeff}) \neq 0$ in dimension 6, by Theorem 20.

Thus (M,g) is not concordant to zero.

Now $\mathbb{C}P^4$ has two homotopy classes of homotopy equivalences and each of these is represented by a

PL-homeomorphism. Thus M is not PL-homeomorphic to $\mathbb{C}P^4$.

The same arguement works for $\mathbb{C}P^8$ using

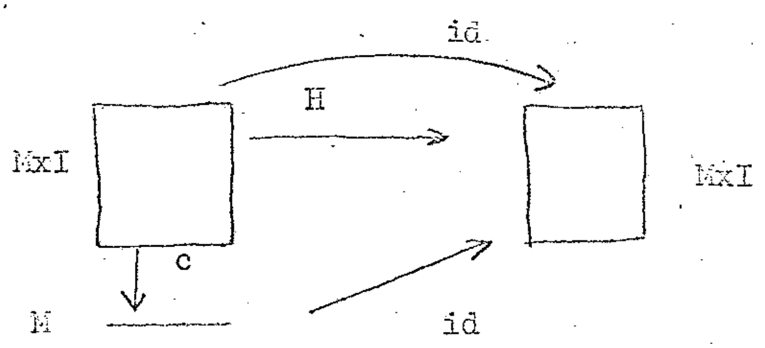
$$\mathbb{C}P^7 \longrightarrow S^{14} \longrightarrow F/PL \quad .$$

Now we briefly describe how Theorem 9 may be used to study the pseudo-isotopy classes of PL-homeomorphisms within a given homotopy class. If suffices to consider the homotopy class of the identity.

Let (c,H) be a pair consisting of a PL-homeomorphism $M \xrightarrow{c} M$ and a homotopy

$$(M,bM)xI \xrightarrow{H} (M,bM)xI$$

between c and the identity. The pair (c,H) determines a PL-structure on MxI, $(MxI \cup_c M,\ H \cup id)$ which restricts to $0 = (MxbI,\ id)$ on $MxbI$. Thus (c,H) determines an element in $PL(Mx(I,bI);0)$.



If M is in $\bar{\mathcal{M}}_n$ we can apply the relative h-cobordism theorem to show that if (c_1, H_1) and (c_2, H_2) <u>determine the same element in</u> $PL(M\times(I, bI), 0)$ <u>then</u> c_1 <u>and</u> c_2 <u>are pseudo-isotopic</u>. Furthermore this pseudo-isotopy extends to a homotopy H' between H_1 and H_2. The converse is true if H' preserves appropriate subspaces.

If M is in $\mathcal{M}_n$, Theorem 9 implies $PL(M\times(I, b\dot{I}); 0)$ is in one-to-one correspondence with $B_k(M\times(I, bI); v)$. By Theorems 12 and 14 $B_k(M\times I, bI; v)$ is in one-to-one correspondence with (susp. M, F/PL).

Thus we can classify pairs (c, H) for M in $\bar{\mathcal{M}}_n$ by elements of

$$(\text{susp } M_0, F/PL).$$

$$F/PL \subseteq B_{PL} \to B_F$$

$$0\ Z_2\ 0\ Z\ 0\ Z_2\ 0\ Z \cdots$$

1) $F/PL = K(Z_2, 2) \underset{\delta Sq^2}{\times} K(Z, 4) \times K(Z_2, 6) \times K(Z, 8) \times \cdots$

mod class of odd torsion groups.

2) $F/PL \Rightarrow B_O$ mod class of 2-torsion groups

L

$\shortparallel$

$(1 + L_1 + L_2 + \cdots) \otimes (1 + L_1 + L_2 + \cdots)$ $\qquad \Omega X = B_0$

Bibliography

1. Browder, W. and M.W.Hirsch, Surgery on PL-manifolds and applications, (to appear) .

2. Browder, W., Embedding 1-connected manifolds, (to appear).

3. __________., Manifolds with $\pi_1 = Z$, (to appear).

4. Brown, E. H., Cohomology theories, Annals of Math. 75 (1962), 467-484.

5. Conner, P. and E. Floyd, Differentiable periodic maps, Springer-Verlag, (1964).

6. Haefliger, A. and C. T. C. Wall, Piecewise linear bundles in the stable range, (to appear).

7. Hilton, P. J. and S. Wylie, Homology theory, Cambridge University Press, (1962), p. 428, Theorem 10.6.2.

8. Hirsch, M. W., Obstruction theories for smoothing manifolds and maps, Bull. Amer. Math. Soc., 69 (1963), 352-356.

9. Hirsch, M. W. and W. Browder, Surgery on PL-manifolds and applications, (to appear).

10. Hirsch, M. W. and B. Mazur, Smoothings of piecewise linear manifolds, mimeographed, Cambridge University Topology Symposium 1964.

11. Kervaire, M. and J. Milnor, Groups of homotopy spheres:I. Annals of Math. 77 (1963), 504-537.

12. Magur, B., The method of infinite repetition in pure topology:I, Annals of Math. 80 (1964) 201-226.

13. __________, Morse Theory, Morse symposium, (1963) 145-166.

14. Milnor, J., On simply connected 4-manifolds, Symposium Internacional de Topologia Algebrica Mexico (1958), 122-128.

15. __________, Microbundles and differentiable structures, mimeographed, Princeton University, (1961).

16. __________, A procedure for killing the homotopy groups of a differentiable manifold, Symposia in Pure Math. A.M.S. vol. III (1961), 39-55.

17. Milnor, J., On spaces having the homotopy type of a CW-complex, Trans. Amer. Math. Soc. 90 (1959), 272-280.

18. Milnor, J. and M. Kervaire, Groups of homotopy spheres: I Annals of Math. 77 (1963), 504-537.

19. Milnor, J., (to appear)

20. Novikov, S. P. , Topological invariance of rational Pontryagin classes, Doklady 1965, Vol. 163, 298-300.

21. Smale, S. On the structure of manifolds, Amer. J. Math. 84 (1962), 387-399.

22. Stasheff, J., A classification theorem for fibre spaces, Topology 2 (1963) 239-246.

23. Steenrod, N., Cohomology operations and obstructions to extending continuous functions, mimeographed, Princeton University, (1957).

33. Spivak, M., On spaces satisfying Poincaré duality, Thesis, Princeton University, (1964).

24. Sullivan, D., (to appear)

25. Wagoner, J., Thesis, Princeton University, (1966).

26. Wall, C. T. C. , An extension of results of Novikov and Browder, Am. Jour. of Math., (to appear).

27. ______________, Killing the middle homotopy group of odd dimensional manifolds, Trans. Amer. Math. Soc. 103 (1962).

28. ______________, On simply connected 4-manifolds, Jour. Lon. Math. Soc. 39 (1964) 141-149.

29. Wall, C. T. C. and A. Haefliger, Piecewise linear bundles in the stable range, (to appear).

30. Wall, C. T. C., Surgery on non-simply connected manifolds, mimeographed, Cambridge University (1964).

31. Zeeman, E. C. and J. F. P. Hudson, On combinatorial isotopy, Institute des Hautes Etudes Scientifiques 19 (1963), 69-94.

32. Zeeman, E. C., Seminar on combinatorial topology, Institute des Hautes Etudes Scientifiques (1963) (revised 1965).

33. Spivak, M., On spaces satisfying Poincare duality, Thesis, Princeton University, (1964).