MAT 550 – Test 1 – Solutions

You may use any results proved in chapters 1-10 of the book. If you quote theorems, please be sure to state them precisely. In questions 3 and 4, for $x \in \mathbf{R}$, we let $e_n(x) = e^{inx}$. Each question is worth 20 points.

1. We use Lebesgue measure in this problem. Suppose that E is a measurable subset of the unit square $[0,1] \times [0,1] \subseteq \mathbf{R}^2$. For $0 \leq x \leq 1$, let E_x be the cross-section $\{y \in \mathbf{R} : (x,y) \in E\}$, and for $0 \leq y \leq 1$, let E^y be the cross-section $\{x \in \mathbf{R} : (x,y) \in E\}$; then E_x is measurable for almost every $x \in [0,1]$, and E^y is measurable for almost every $y \in [0,1]$. Let T be the set of $y \in [0,1]$ for which E^y is measurable.

   Suppose that for almost every $x \in [0,1]$, the measure of E_x is less than or equal to 1/4. Let $S = \{y \in T : m(E^y) \geq 1/2\}$. Show that S is measurable, and $m(S) \leq 1/2$.

   Solution: By Tonelli as applied to the function χ_E, the functions $g(x) := m(E_x)$ and $h(y) := m(E^y)$ are measurable. Thus S is measurable. Moreover, by Tonelli, the measure of E equals $\int_0^1 m(E_x)dx \leq 1/4$. But by Tonelli that measure also equals $\int_0^1 m(E^y)dy$, which is therefore $\leq 1/4$. By Tchebyshev, $m(S) \leq \int m(E^y)/(1/2) \leq 1/4$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $E, E_1, E_2, \ldots \in \mathcal{M}$, and that $\lim_{n\to\infty} \mu(E \Delta E_n) = 0$. (Here Δ means symmetric difference.) Show that $\{E_n\}$ has a subsequence $\{E_{n_k}\}$ such that both of the following hold:
   (i) For almost every $x \in E$, $x \in E_{n_k}$ for all but finitely many k; and
   (ii) For almost every $x \in E^c$, $x \in E_{n_k}^c$ for all but finitely many k.

   Solution: The hypotheses are equivalent to saying that $\chi_{E_n} \to \chi_E$ in measure. So a subsequence $\chi_{E_{n_k}} \to \chi_{E_n}$ a.e, which is equivalent to the conclusion.

   Alternatively, choose a subsequence E_{n_k} with $\sum \mu(E \Delta E_{n_k}) < \infty$, so that almost every x is in only finitely many of the sets $E \Delta E_{n_k}$, which is again equivalent to the conclusion.

3. (a) Suppose $\mathbf{A} = (\ldots, A_{-1}, A_0, A_1, \ldots) \in \ell^1(\mathbf{Z})$. Show that $\sum_{n=-\infty}^{\infty} A_n e_n$ converges absolutely and uniformly to a function in $C(\mathbf{T})$.
(b) In (a), set $\check{\mathbf{A}} = \sum_{n=-\infty}^{\infty} A_n e_n$, so that $\check{} : l^1(\mathbf{Z}) \to C(\mathbf{T})$. Find, with proof, $(\check{\mathbf{A}})\hat{}$.
(c) Suppose $f \in L^1(\mathbf{T})$ and that $\hat{f} \in \ell^1(\mathbf{Z})$. Let $g = (\hat{f})\check{}$. Show that $f = g$ (so that, by (a), f is continuous).

Solution: (a) follows from Weierstrass's M-test, with $M_n = A_n$.
(b) follows from writing $(\check{\mathbf{A}})\hat{}(k) = (\sum_n A_n e_n, e_k) = \sum_n (A_n e_n, e_k) = A_k$, the interchange of summation and integration being justified by the uniform convergence of the integrand $\sum A_n e_n e_{-k}$.
For (c), we have from (b) that $\hat{f} = \hat{g}$. By injectivity of $\hat{}$, $f = g$ a.e., so f agrees a.e. with the continuous function g.

4. Say $f \in L^2(\mathbf{T})$. For any $N \in \mathbf{Z}$, let $S_N = \sum_{n=-N}^{N} \hat{f}(n) e_n$. Suppose that for all N, $|S_N| \leq 1$ a.e. Show that $|f| \leq 1$ a.e.

Solution: By Parseval, $S_N \to f$ in L^2, so a subsequence of the S_N approaches f almost everywhere, which gives the conclusion at once.
It is true that $S_N \to f$ a.e., but we haven't proved that, and in fact, it is very difficult to do so.

5. Say $f \in L^1(\mathbf{T})$, and that $|f(\theta)| \geq 1/\sqrt{|\theta|}$ for $-\pi < \theta < \pi$. Show that $\sup_n |n\hat{f}(n)| = \infty$.

Solution: Assume, to get a contradiction, that the sup were finite. Then, for some finite C, $|n\hat{f}(n)| \leq C$ for every n. So for $n \neq 0$, $|\hat{f}(n)| \leq C/n$, so $\hat{f} \in \ell^2(\mathbf{Z})$. By Parseval there is a $g \in L^2(\mathbf{T})$ with $\hat{g} = \hat{f}$; by injectivity of $\hat{}$ on $L^1(\mathbf{T})$, $f = g$ a.e. So $f \in L^2(\mathbf{T})$; but this is impossible, since $\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta \geq \int_{-\pi}^{\pi} d\theta/|\theta| = \infty$.