Contemporary Mathematics

Complex Powers of Convolution Operators on the Heisenberg Group

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Dedicated to Leon Ehrenpreis

1. Introduction

We begin with some elementary observations about homogeneous distributions on $\mathbb{R}^n$; our purpose is to prove analogues for the Heisenberg group.

On $\mathbb{R}^n$,

the Fourier transform of a homogeneous distribution K which is
smooth (resp. real analytic) away from the origin is again smooth
(1) (resp. real analytic) away from the origin.

Let the degree of homogeneity of K be k; then the degree of homogeneity of $\hat{K}$ is $j = -n - k$. The following fact follows readily from (1):

Proposition 1.1. *Suppose that $-n+j \notin \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$. Then there exists a homogeneous distribution H, which is smooth (resp. real analytic) away from 0 satisfying $H * K = \delta$, if and only if $\hat{K}$ does not vanish away from 0.*

Indeed, $\hat{H} = 1/\hat{K}$ away from 0.

For $l \in \mathbb{R}$, we now let

$$\mathcal{D}_l = \{f \in L^2(\mathbb{R}^n) : \int |\xi|^{2l} |\hat{f}(\xi)|^2 d\xi < \infty\}, \tag{2}$$

the natural domain of $\Delta^{l/2}$. We may then define a operator $S : \mathcal{D}_j \to L^2$ by $Sf = f * K$; thus S is a densely defined operator on L^2. We note the important situation in which K is supported at 0; in this case, S is a constant coefficient homogeneous differential operator of degree j, further, every constant coefficient homogeneous differential operator arises in this manner. In this situation, the differential operator S is elliptic iff $\hat{K}$ does not vanish away from 0; and then H is its homogeneous fundamental solution.

Whether or not K is supported at 0, we clearly have that $S : \mathcal{D}_j \to \mathcal{D}_{-j}$.

Now suppose that we are in the situation of Proposition 1.1. Then, clearly, $S^{-1} : \mathcal{D}_{-j} \to \mathcal{D}_j$ is the operator of convolution with H. If, *in addition*, we assume

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that S is a *positive* operator on L^2, or equivalently that $\hat{K} > 0$ away from 0, we may study its complex powers S^γ, for $\gamma \in \mathbf{C}$, as supplied by the spectral theorem. We then have:

PROPOSITION 1.2. *Suppose that $\hat{K} > 0$, and $\gamma \in \mathbf{C}$. Let $\rho = \Re\gamma$. Then the domain of S^γ is $\mathcal{D}_{\rho j}$.*

Moreover, suppose that $-n-\gamma j \notin \mathbf{Z}^+$. Then there exists a homogeneous distribution K_γ, which is smooth (resp. real analytic) away from 0, such that S^γ is the operator of convolution with K_γ.

Indeed, $\hat{K_\gamma} = (\hat{K})^\gamma$ away from 0.

Note that the restrictions on j in Propositions 1.1 and 1.2 were made only to make the statements of the propositions simpler. If, for instance, in Proposition 1.2, $m = -n - \gamma j \in \mathbf{Z}^+$, the only difference would be that K_γ might be the sum of a homogeneous distribution which is smooth (resp. real analytic) away from 0, and $p(x)\log|x|$, where p is a homogeneous polynomial of degree m.

We seek analogues of these results for the Heisenberg group $\mathbf{H}^n$, with its usual (parabolic) dilations. The (group) Fourier transform is a valuable tool in this context as well, but the situation is much more subtle, since convolution operators no longer commute, and so the group Fourier transform cannot simultaneously diagonalize them.

A homogeneous distribution on $\mathbf{H}^n$ is said to be *regular* if it is smooth away from 0. We let

$$Rhom_k = \{\text{regular homogeneous distributions of degree } k \text{ on } \mathbf{H}^n\}, \tag{3}$$

and if $k \notin \mathbf{Z}^+$, we let

$$\mathbf{K}^k = Rhom_k. \tag{4}$$

If $k \in \mathbf{Z}^+$, let $\mathcal{P}_k$ denote the space of homogeneous polynomials of degree k on $\mathbf{H}^n$, and let

$$\mathbf{K}^k = \{K = K' + p(u)\log|u| : K' \in Rhom_k, p \in \mathcal{P}_k\}. \tag{5}$$

Here we are using u to denote the coordinates on $\mathbf{H}^n$, and $|\ |$ to denote the usual homogeneous norm function on $\mathbf{H}^n$: if $u = (t,z) \in \mathbb{R} \times \mathbf{C}^n$, then

$$|u| = |(t,z)| = (|z|^4 + t^2)^{1/4}. \tag{6}$$

For any $k \in \mathbf{C}$, we let

$$\mathbf{AK}^k = \{K \in \mathbf{K}^k : K \text{ is (real) analytic away from } 0\}. \tag{7}$$

Finally, let $Q = 2n+2$. Then one has:

THEOREM 1.3. . *Suppose $k \in \mathbf{C}$; let $j = -Q - k$. Suppose $K \in \mathbf{K}^k$ (resp. $\mathbf{AK}^k$). Then there exists $H \in \mathbf{K}^{-Q+j}$ (resp. $\mathbf{AK}^{-Q+j}$), satisfying $H * K = \delta$, if and only if:*

$$\begin{aligned}&\pi_K \text{ is injective on } C^\infty \text{ vectors for every non-trivial irreducible}\\ &\text{unitary representation } \pi.\end{aligned} \tag{8}$$

This theorem is known. In the "smooth" setting, if K is supported at 0 (so that the operator of convolution with K is a left-invariant differential operator), it follows by combining results in [**20**], [**18**], [**10**], and in the more general context of graded groups, it follows by combining results in [**14**], [**10**]. The full theorem in the smooth setting was proved in [**5**], in the general context of groups with dilations. (Earlier results in this setting, including those of [**22**], are mentioned in [**5**], page 32.) In the real analytic setting, if K is supported at 0, the theorem follows by combining results in [**16**], [**10**]. The full theorem in the real analytic setting was proved in our book [**12**]. [1]

We now address the hitherto unexplored question of finding an analogue of Proposition 1.2 for $\mathbf{H}^n$.

We let $\mathcal{L}_0$ be the usual sublaplacian on $\mathbf{H}^n$, and we let $\mathcal{D}_l$ denote the natural domain of $\mathcal{L}_0^{l/2}$. In section 3, we shall explain in detail what $\mathcal{L}_0^{l/2}$ and $\mathcal{D}_l$ are. In section 4, we shall show that we may define $S : \mathcal{D}_j \to L^2$ by $Sf = f * K$. Moreover, we shall show that $S : \mathcal{D}_j \to \mathcal{D}_{-j}$.

Furthermore, in the situation of Theorem 1.3, we have that $S^{-1} : \mathcal{D}_{-j} \to \mathcal{D}_j$ is the operator of convolution with H.

Note that, if S is a positive operator on L^2, and $S \neq 0$, then a simple argument using dilations shows that k must be real. We claim:

THEOREM 1.4. *Notation as above, suppose that (8) holds, and that S is a positive operator on L^2. Say $\gamma \in \mathbf{C}$. Let $\rho = \Re\gamma$. Then the domain of S^γ is $\mathcal{D}_{\rho j}$.*

Further, there exists $K_\gamma \in \mathbf{K}^{-Q-\gamma j}$ (resp. $\mathbf{AK}^{-Q-\gamma j}$, if $j \in \mathbf{Q}$), such that S^γ is the operator of convolution with K_γ.

We have not, as yet, been able to remove the restriction that, in the real analytic setting, j be rational. In any event, Theorem 1.4 does cover the case in which S is a (left-invariant) differential operator.

In the situation of Theorem 1.4, one easily deduces that S^γ is pseudolocal (resp. analytic pseudolocal) and hypoelliptic (resp. analytic hypoelliptic).

In section 4 of this article, we shall prove Theorem 1.4 for $j \neq 0$; however, we shall need to assume certain results on resolvents of convolution operators, which, due to limitations of space here, will appear in a companion paper [**13**]. We shall state our results on resolvents in a moment.

The case $j = 0$ in Theorem 1.4 is much easier, Indeed, in this case, S is a bounded, invertible positive operator on L^2. We may thus choose a circle Γ lying entirely in the right half plane which contains the spectrum of S in its interior, and invoke the formula $S^\gamma = (2\pi i)^{-1} \int_\Gamma z^\gamma (z - S)^{-1} dz$. Again, due to space limitations here, we present the details in [**13**].

In the special case $S = \mathcal{L}_0$, we shall verify Theorem 1.4 in section 3 of this paper, without using the results of [**13**], by exhibiting an explicit formula for K_γ away from 0. In this special case, Theorem 1.4 was previously known in the smooth setting, ([**6**], [**9**]). However, the explicit form of K_γ was not known, except if $\gamma \in \mathbf{Z}$. (See [**7**] for the case $\gamma = -1$, and [**3**] for the case of γ a general negative integer). Nor was the fact that K_γ was analytic away from 0 previously known (except in the special case $\gamma \in \mathbf{Z}$).

For the rest of this introduction, we shall assume that (8) holds, that $j \neq 0$, and that S is a positive operator on L^2.

[1] In the analytic setting, if the symbol a of the operator of convolution with K happens to lie in $S_{1,0}$, the earlier methods of Métivier [**17**] produce an analytic parametrix. It seems likely that those methods would also work if a lies in any better (traditional) symbol class than $S_{1/2,1/2}$. However, unless S is a differential operator, the symbol will in general be in no better traditional symbol class than $S_{1/2,1/2}$, where there is no good theory of inversion.

In proving Theorem 1.4, we may assume that $j > 0$ (otherwise, note $(S^{-1})^{-\gamma} = S^{\gamma}$). Our strategy in proving Theorem 1.4 will be to analyze the resolvent $(rI+S)^{-1}$ for $r > 0$, and to use the formula

$$S^{\gamma} = c_{\gamma} \int_0^{\infty} r^{\gamma}(rI+S)^{-1} dr, \tag{9}$$

valid for $-1 < \Re\gamma < 0$; here $c_{\gamma} = -\sin(\pi\gamma)/\pi$. By use of dilations, we easily reduce the study of $(rI+S)^{-1}$ to the special case $(I+S)^{-1}$. Surely $(I+S)^{-1}$ is a bounded operator on L^2. It turns out that there is a locally integable function Φ, which grows no faster than a polynomial at infinity, such that if $(I+S)^{-1}f = f * \Phi$, for any Schwartz function f. The results of [**5**], [**12**] readily imply that Φ is smooth (resp. real analytic) away from 0, and its behavior near 0 is completely explained by the results in those same references. What must be carefully understood in applying (9) is the behavior of Φ near infinity.

We note the following very crude information about Φ near 0:

Proposition 1.5. *Say $l \in \mathbb{R}$. For $u \in \mathbf{H}^n$, define $B_l(u)$ to be $|u|^l$ if $l < 0$; $|(log|u|)|$ if $l = 0$; 1 if $l > 0$.*

Let D be a constant-coefficient, (parabolically) homogeneous differential operator on $\mathbf{H}^n$, of homogeneous degree m. Then for some $C_m > 0$, $|D\Phi(u)| \leq C_m B_{-Q+j-m}(u)$, whenever $|u| \leq 1$.

As we said, we need to understand the behavior of Φ near ∞. These results will be proved in [**13**]; the methods needed will be briefly sketched in section 5 of this paper.

In the smooth setting, we have the following result:

Theorem 1.6. *Notation as above, suppose that K is smooth away from 0. Let D be a constant-coefficient, homogeneous differential operator on $\mathbf{H}^n$, of homogeneous degree m. Then:*

(a) Suppose that K is supported at 0. Then for some $c, C_m > 0$, $|D\Phi(u)| \leq C_m e^{-c|u|}$, whenever $|u| \geq 1$.
(b) Suppose that K is general (not necessarily supported at 0). Suppose that $j > Q$. Then for some $C_m > 0$, $|D\Phi(u)| \leq C_m|u|^{2-j-m}$, whenever $|u| \geq 1$.

An analogous result holds on $\mathbb{R}^n$, and may be verified by use of the Fourier transform.

The exponent $2-j$ in Theorem 1.6 (b) can no doubt be reduced, but this is irrelevant for the purposes of proving Theorem 1.4.

In the real analytic setting, one has:

Theorem 1.7. *Notation as above, suppose that K is (real) analytic away from 0. Then there exists $a > 0$ such that Φ has a holomorphic extension to the "sector" $S = \{u + iv : |v| < a|u|\}$ in the complexification of $\mathbf{H}^n$. We call this extension Φ as well. Moreover:*

For some $C > 0$, $|\Phi(u+iv)| \leq CB_{-Q+j-m}(u)$, whenever $|u| \leq 1$, $u + iv \in S$. Further:

(a) Suppose that K is supported at 0. Then for some $c, C > 0$, $|\Phi(u+iv)| \leq Ce^{-c|u|}$, whenever $|u| \geq 1$, $u + iv \in S$.
(b) Suppose that K is general (not necessarily supported at 0). Suppose that $j > Q$,

and is an even integer. Then for some $C > 0$, $|\Phi(u + iv)| \leq C|u|^{3-j}$, *whenever* $|u| \geq 1$, $u + iv \in S$.

Again, an analogous theorem holds on $\mathbb{R}^n$, and again, the exponent of $3 - j$ in part (b) is not optimal.

Let us indicate how the above results on the behavior of Φ imply Theorem 1.4. We still assume that $j > 0$, and also that the other hypotheses of Theorem 1.4 hold.

If l is a positive integer, the results of [**5**], [**12**] imply that S^l is the operator of convolution with K_l, where K_l is smooth (resp. analytic) away from 0 and homogeneous of degree $-Q - jl$. We choose l so that $jl > Q + 3$ (resp. so that $jl > Q + 3$ and so that jl is an even positive integer). Since $S^\gamma = (S^l)^{\gamma/l}$, it is now clear that in proving Theorem 1.4 we may assume that $j > Q + 3$ (resp. $j > Q + 3$ and j is an even positive integer).

Say now $-1 < \Re\gamma < 0$. Substituting r^j for r in (9), we find formally that

$$K_\gamma(t, z) = jc_\gamma \int_0^\infty r^{Q+\gamma j-1}\Phi(r^2 t, rz)dr. \tag{10}$$

The estimates of Proposition 1.5, and Theorems 1.6 and 1.7, show at once that the right side of (10) converges to a smooth (resp. analytic) function of $(t, z) \in \mathbf{H}^n \setminus \{0\}$, provided that $0 < Q + j\Re\gamma$, and $Q + j\Re\gamma + 3 - j < 0$. Since $j > Q+3$, the latter condition is automatic; so we really want to require that $-Q/j < \Re\gamma < 0$. With this restriction on γ, one may prove that (10) holds. This then verifies Theorem 1.4 for $-\Re\gamma$ positive and sufficiently small. The case of general $\Re\gamma \neq 0$ is then dealt with by use of the identities $(S^\gamma)^{-1} = S^{-\gamma}$, and $S^\alpha S^\beta = S^{\alpha+\beta}$ (if $\Re\alpha, \Re\beta > 0$). A slight additional argument then deals with the case $\Re\gamma = 0$ as well.

Motivation for studying the problems addressed here comes from several sources. Once the usual calculus of pseudodifferential operators was developed, which was ideal for dealing with elliptic operators, it was soon discovered that this calculus was extremely useful in determining the behavior of functions of elliptic operators. (See especially Seeley [**21**], where formula (9) is applied in the elliptic situation.) It is only natural, then, to ask if something analogous can be done in the Heisenberg group, or strictly pseudoconvex CR manifold, settings, using the calculi of [**5**], [**12**] (see also the earlier works [**22**], [**2**]). Resolvents of elliptic operators, in particular, are extremely important in various areas of analysis, including scattering theory and the recent theory of resonances, and it is again natural to look for analogues.

I began studying these problems by calculating that Theorem 1.7(a), and consequently Theorem 1.6 (a), held in the special case $S = \mathcal{L}_0$. I presented this result in a seminar talk at MSRI, and Michael Christ then told me that Theorem 1.6 (a) (the smooth setting) followed in this case directly from the methods of Agmon. In [**13**], we follow the methods of Agmon [**1**] in our proof of Theorem 1.6 (a). I am very grateful to Michael Christ for telling me about these methods.

The proof of Theorem 1.6 (b) also relies on L^2 estimates. In contrast, our proof of Theorem 1.7 (in [**13**]) makes full use of the Fourier transform methods of [**12**].

Next, a reminiscence. Shortly after earning my Ph.D., when I first contemplated the idea of studying analyticity in subelliptic problems, I asked several people – including Leon Ehrenpreis – to show me proofs of (1) in the real analytic setting, in the hopes of learning a proof that might yield some insight when one replaced the usual dilations by parabolic dilations. Leon showed me a simple proof which used a change of contour. Although he hardly knew me, he gave me a very careful explanation. He must be a very caring advisor.

Much of the research on this paper was done at MSRI, the Fields Institute, and the Erwin Schrödinger Institute. I would like to thank these institutes for inviting me to be a member and for their hospitality.

2. Notation

We begin by explaining our notation on the Heisenberg group $\mathbf{H}^n$. For further background, see for instance Chapter 4 of [**12**].

$\mathbf{H}^n$ is the Lie group with underlying manifold $\mathbb{R} \times \mathbf{C}^n$ and multiplication

$$(t, z) \cdot (t', z') = (t + t' + 2\Im(z \cdot \bar{z}'), z + z'), \tag{11}$$

where $z \cdot \bar{z}' = \sum_{j=1}^{n} z_j \cdot \bar{z}'_j$. The parabolic dilations D_r are given by $D_r(t, z) = (r^2 t, rz)$.

The left-invariant vector fields agreeing at 0 with $\partial/\partial z_j$, $\partial/\partial \bar{z}_j$, and $\partial/\partial t$ at 0 are $Z_j = \partial/\partial z_j + i\bar{z}_j\partial/\partial t$, $\bar{Z}_j = \partial/\partial \bar{z}_j - iz_j\partial/\partial t$, and $T = \partial/\partial t$. The Kohn Laplacian, or Folland-Stein sublaplacian, is $\mathcal{L}_0 = -\frac{1}{2}\sum(Z_j\bar{Z}_j + \bar{Z}_j Z_j)$.

For each $\lambda \in \mathbb{R} \setminus \{0\}$, one has an irreducible unitary representation $u \to U_u^\lambda$ of $\mathbf{H}^n$; here U_u^λ acts on a separable Hilbert space $\mathcal{H}_\lambda$. The Fourier transform $\hat{}$ on $\mathbf{H}^n$ maps a function $f \in L^1(\mathbf{H}^n)$ to the family of bounded operators $(\hat{f}(\lambda))$, where $\hat{f}(\lambda) : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$ is given by $\hat{f}(\lambda) = \int_{\mathbf{H}^n} U_u^\lambda f(u)du$. Then, if $f, g \in L^1$, $(f * g)\hat{}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$. Moreover, let $\mathcal{S}(\mathbf{H}^n) = \mathcal{S}(\mathbb{R}^{2n+1})$ (the Schwartz space). Define $\tilde{} : \mathcal{S} \to \mathcal{S}$ by $\tilde{f}(u) = \overline{f(u^{-1})}$, and extend $\tilde{} : \mathcal{S}' \to \mathcal{S}'$. Then if $f \in L^1$, $\hat{\tilde{f}}(\lambda) = [\hat{f}(\lambda)]^*$.

Now, $\mathcal{H}_\lambda$ has an important orthonormal basis $\{E_{\alpha,\lambda}\}$ $(\alpha \in (\mathbf{Z}^+)^n)$ such that, for any $f \in \mathcal{S}(\mathbf{H}^n) = \mathcal{S}(\mathbb{R}^{2n+1})$ (the Schwartz space),

$$\widehat{\mathcal{L}_0 f}(\lambda)E_{\alpha,\lambda} = (2|\alpha| + n)|\lambda|\hat{f}(\lambda)E_{\alpha,\lambda}. \tag{12}$$

There is a standard realization of U_u^λ, called the Schrödinger representation, in which $\mathcal{H}_\lambda$ is realized as $L^2(\mathbb{R}^n)$, and the $E_{\alpha,\lambda}$ are realized as (scaled) Hermite functions.

We shall frequently drop the λ subscript when its value is clear from the context; for instance, in (12), we might write E_α in place of $E_{\alpha,\lambda}$.

In this article, an *operator family* is defined to be a collection of operators $\mathbf{R} = (R(\lambda))_{\lambda \in \mathbf{R}\setminus\{0\}}$, where $R(\lambda) : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$; where the domain of each $R(\lambda)$ contains all the E_α; and where, for all α, β, the map $\lambda \to (R(\lambda)E_{\alpha,\lambda}, E_{\beta,\lambda})$ is measurable. If $f \in \mathcal{S}(\mathbf{H}^n)$, then $\hat{f} = (\hat{f}(\lambda))$ is surely an operator family.

Say $\mathbf{R}$, $\mathbf{S}$ are operator families. When they make sense and are operator families, $\mathbf{R} + \mathbf{S}$, $\mathbf{RS}$, and $\mathbf{R}^*$ will all have the obvious meanings; for instance, $\mathbf{RS} = (R(\lambda)S(\lambda))$.

Next, let $\mathcal{R}_2 = \{$operator families $\mathbf{R}|$ for a.e. $\lambda, \|R(\lambda)\|_2 < \infty$, and $\|\mathbf{R}\|_2^2 = \int_{-\infty}^{\infty} \|R(\lambda)\|_2^2(2|\lambda|)^n d\lambda < \infty\}$. Here $\|R(\lambda)\|_2$ is the Hilbert-Schmidt norm of $R(\lambda)$. One then has the Plancherel theorem: $\hat{}$, restricted to $\mathcal{S}(\mathbf{H}^n)$, has a (unique) continuous extension to a map from $L^2(\mathbf{H}^n)$ onto $\mathcal{R}_2$; and if $f \in L^2$, then $\|f\|_2^2 = c_n\|\hat{f}\|_2^2$; here $c_n = 1/(2\pi^{n+1})$.

If $\mathbf{R}$ and $\mathbf{S}$ are operator families, and $\int_{-\infty}^{\infty}\sum_\alpha |(R(\lambda)E_\alpha, S(\lambda)E_\alpha)||\lambda|^n d\lambda < \infty$, we set

$$(\mathbf{R}|\mathbf{S}) = \int_{-\infty}^{\infty}\sum_\alpha (R(\lambda)E_\alpha, S(\lambda)E_\alpha)(2|\lambda|)^n d\lambda.$$

Thus, by polarizing Plancherel, we see that if $f, g \in L^2$, then $(f|g) = c_n(\hat{f}|\hat{g})$.

For this reason, we make the following definition. Say now $g \in \mathcal{S}'$; if $f \in \mathcal{S}$, set $(f|g) = \overline{g(\bar{f})}$. We say that $\hat{g} = \mathbf{R}$ in the $\mathcal{S}'$ sense, if $\mathbf{R}$ is an operator family with the property that for all $f \in \mathcal{S}$, $(f|g) = c_n(\hat{f}|\mathbf{R})$.

3. Powers of the Sublaplacian

In this section and the next, we will freely use some rudimentary facts from the theory of unbounded operators on Hilbert spaces. For these, see, e.g. [**19**], Chapter VIII.

For $l \in \mathbb{R}$, let us set

$$\mathcal{D}_l = \{f \in L^2(\mathbf{H}^n) : \int_{-\infty}^{\infty} \sum \|\hat{f}(\lambda)E_\alpha\|^2[(2|\alpha|+n)|\lambda|]^l(2|\lambda|)^n d\lambda < \infty\}. \tag{13}$$

(Here the sum is over all $\alpha \in (\mathbf{Z}^+)^n$.) It is clear that, if $0 \leq l \leq m$, then $\mathcal{D}_l \supseteq \mathcal{D}_m$. Also, say $f \in \mathcal{S}$; then $\mathcal{L}_0^N f \in L^2$ for any N, so by (12), $\mathcal{S} \subseteq \mathcal{D}_l$ for all $l \geq 0$.

It is natural to extend $\mathcal{L}_0|_{\mathcal{S}}$ to be an operator on $\mathcal{D}_2$; here, if $f \in \mathcal{D}_2$, we define $\mathcal{L}_0 f \in L^2$ by (12). We shall see momentarily (in Proposition 3.1) that $\mathcal{L}_0$, with domain $\mathcal{D}_2$, is the in fact the closure of $\mathcal{L}_0|_{\mathcal{S}}$.

From now on, unless stated otherwise, we take $\mathcal{D}_2$ to be the domain of $\mathcal{L}_0$. It is easy to calculate that $\mathcal{L}_0$ is a self-adjoint, and moreover a positive, operator.

For $\gamma \in \mathbf{C}$, define the operator family $\mathbf{A}_\gamma$, by

$$A_\gamma(\lambda)E_\alpha = [(2|\alpha|+n)|\lambda|]^\gamma E_\alpha; \tag{14}$$

here the domain of $A_\gamma(\lambda)$ is taken to be the linear span of the E_α. Then

$$\mathcal{D}_l = \{f \in L^2(\mathbf{H}^n) : \hat{f}\mathbf{A}_{l/2} \in \mathcal{R}_2\}.$$

The spectral projectors of $\mathcal{L}_0$ are easily read off from (12) and (13) with $l = 2$. By the spectral theorem, then, if $\gamma \in \mathbf{C}$, $\mathcal{L}_0^\gamma$ is the operator with domain $\mathcal{D}_{2\Re\gamma}$, given by

$$\widehat{\mathcal{L}_0^\gamma f} = \hat{f}\mathbf{A}_\gamma. \tag{15}$$

We can obtain more insight into these operators by examining certain subspaces of $\mathcal{S}$. Let $\mathcal{Q} = \{$operator families $\mathbf{R} : (R(\lambda)E_\alpha, E_\beta) \in C_0^\infty(\mathbb{R} \setminus \{0\}$ for all α, β; and for some $N \in \mathbf{N}, (R(\lambda)E_\alpha, E_\beta) \equiv 0$ whenever $|\alpha| + |\beta| > N\}$. Then each element of $\mathcal{Q}$ is the Fourier transform of a Schwartz function f. Let $\check{\mathcal{Q}} = \{f \in \mathcal{S} : \hat{f} \in \mathcal{Q}\}$. By (12), it is apparent that if $f \in \check{\mathcal{Q}}$, then for each $M \in \mathbf{N}$, there exists $g \in \check{\mathcal{Q}}$, such that $f = \mathcal{L}_0^N g$. Integration by parts now shows that $\check{\mathcal{Q}} \subseteq \mathcal{S}_0$, where

$$\mathcal{S}_0 = \{f \in \mathcal{S} : (f|p) = 0 \text{ for all polynomials } p\}. \tag{16}$$

$\mathcal{S}_0$ is a very important space for us. When studying PDE on $\mathbb{R}^n$, behavior of the Fourier transform near 0, of any function under consideration, is usually irrelevant. Use of $\mathcal{S}_0$ enables us to make similar simplifications in the group setting.

Let $\mathbf{K}^k, \mathbf{AK}^k$ be as defined in (4), (5). Then Proposition 2.2 of [**5**] tells us:

$$\text{If } f \in \mathcal{S}_0, \text{ and } K \in \mathbf{K}^k, \text{ then } f * K \in \mathcal{S}_0. \tag{17}$$

Folland and Stein [**7**] showed that $\mathcal{L}_0$ has a fundamental solution $\Psi_{-1} \in \mathcal{K}_{-2n}$. Thus, if $f \in \mathcal{S}_0$, there exists $g \in \mathcal{S}_0$ with $f = \mathcal{L}_0 g$ (namely, $g = f * \Psi_{-1}$). Iterating this, we see that a function $f \in \mathcal{S}_0$ if, and only if, for every $N \in \mathbf{Z}^+$, $f = \mathcal{L}_0^N g_N$

for some $g_N \in \mathcal{S}_0$. If so, and if $h \in \mathcal{S}$, then $h * f = \mathcal{L}_0^N(f * g_N)$ for any N. Thus $f \in \mathcal{S}_0$, $h \in \mathcal{S}$ imply $h * f \in \mathcal{S}_0$; also $f * h = (\tilde{h} * \tilde{f})\tilde{} \in \mathcal{S}_0$.

We have explained above that $\mathcal{S} \subseteq \mathcal{D}_l$ for all $l \geq 0$. On the other hand, it is clear that if a Schwartz function $g \in \mathcal{D}_l$ for some l, then $\mathcal{L}_0 g \in \mathcal{D}_{l-2}$. Thus, $\mathcal{S}_0 \subseteq \mathcal{D}_l$ for all $l \in \mathbb{R}$.

Moreover, we claim:

Proposition 3.1. *For any $\gamma \in \mathbf{C}$, the operator $\mathcal{L}_0^\gamma$ (with domain $\mathcal{D}_{2\Re\gamma}$), is the closure of $\mathcal{L}_0^\gamma|_{\mathcal{S}_0}$.*

Proof Note first that $\mathcal{L}_0^\gamma$ is self-adjoint, hence is a closed operator.

It is clearly enough to show that $\mathcal{L}_0^\gamma$ is the closure of $\mathcal{L}_0^\gamma|_{\check{\mathcal{Q}}}$. But this is evident from (13), (15), and $\|\hat{f}(\lambda)E_\alpha\|^2 = \sum_\beta |((\hat{f}(\lambda)E_\alpha, E_\beta)|^2$.

Theorem 3.2. *For each $\gamma \in \mathbf{C}$, there exists a $\Psi_\gamma \in \mathcal{K}_{-2n-2-2\gamma}$, such that $\mathcal{L}_0^\gamma f = f * \Psi_\gamma$, for any $f \in \mathcal{S}_0$; moreover, Ψ_γ is analytic away from 0. Explicitly, say $z \neq 0$. If $\gamma, -n-1-\gamma \notin \mathbf{Z}^+$, then*

(18)
$$\Psi_\gamma(t,z) = c_n 2^{\gamma+1}\frac{\Gamma(n+1+\gamma)}{\Gamma(-\gamma)}\int_{-1}^1 (|z|^2+ist)^{-n-1-\gamma}(1-s^2)^{\frac{n}{2}-1}[h(s)]^{-\gamma-1}ds$$

where, as always, $c_n = 1/(2\pi^{n+1})$, and where

$$h(s) = \tanh^{-1}(s)/s = 1 + \frac{s^2}{3} + \frac{s^4}{5} + \ldots .$$

Of course, if $\gamma \in \mathbf{Z}^+$, then $\mathcal{L}_0^\gamma$ is a differential operator, and Ψ_γ is supported at 0. Finally, if $-n-1-\gamma = m \in \mathbf{Z}^+$, set $c_{n,m} = c_n\frac{(-1)^{m+1}}{2^{n+m}(m+n)!m!}$. Then

$$\Psi_\gamma(t,z) = c_{n,m}\int_{-1}^1 (|z|^2+ist)^m \log(|z|^2+ist)(1-s^2)^{\frac{n}{2}-1}[h(s)]^{n+m}ds \tag{19}$$

Proof In order to derive the formulae for Ψ_γ, we use the following fact, which is an immediate corollary of Theorem 6.2 of [**11**]:

Theorem 3.3. *For $u = (t,z) \in \mathbf{H}^n$, write*

$$w(u) = |z|^2 - it.$$

Suppose $0 < \Re k < n+1$. Suppose f is continuous on $(0,\infty)$, is smooth in a neighborhood of 1, and that $f_1, f_{-1} \in L^1([0,1])$, where for $s \in (0,1]$,

$$f_1(r) = f(r), \quad f_{-1}(r) = r^{k-2}f(r).$$

For $u = (t,z) \in \mathbf{H}^n$, $z \neq 0$, set $K(u) = G(w(u))$, where

(20)
$$G(w) = \int_0^\infty (rw+\bar{w})^{-k}f(r)dr = \int_0^1 (rw+\bar{w})^{-k}f_1(r)dr + \int_0^1 (r\bar{w}+w)^{-k}f_{-1}(r)dr$$

(Here we use the principal branch of the power function; note that $rw+\bar{w} = (1+r)|z|^2 + (1-r)it$ lies in the right half plane, if $r > 0$).

Then K may be extended to a function $K \in C^\infty(\mathbf{H}^n \setminus \{0\})$; consequently, $K \in Rhom_{-2k}$.

Further, define the operator family $\mathbf{J}$ *by*

$$J(\lambda)E_\alpha = c_n'|\lambda|^{-j}\Gamma(n+1-j)^{-1}\left(\int_0^1 s^N(1-s)^{j-1}f_\sigma(s)ds\right)E_\alpha, \tag{21}$$

where $N = |\alpha|$, $\sigma = \mathrm{sgn}\lambda$, *and* $c_n' = \pi^{n+1}2^{1-n}$.

Then $\hat{K} = \mathbf{J}$ *in the* $\mathcal{S}'$ *sense.*

We examine the special case of Theorem 3.3 in which $f(s) = s^{\frac{n}{2}-1}|(\log s)/(1-s)|^{j-1}$ for $s \neq 1$, $f(1) = 1$. Then f is smooth near 1, and for $0 < s < 1$, $f_1 = f_{-1} = f$. In (21), we note that for $M > 0$, $\int_0^1 s^{M-1}|\log s|^{j-1}ds = \Gamma(j)/M^j$, as the substitution $v = e^{-r}$ shows. Further, we substitute $s = (1-r)/(1+r)$ in (20), so that $r = (1-s)/(1+s)$, and note that $|\log r| = 2\tanh^{-1}s$. Finally, setting $\gamma = -j$, we see the following: If $-(n+1) < \Re\gamma < 0$, $\hat{\Psi}_\gamma = \mathbf{A}_\gamma$ in the $\mathcal{S}'$ sense. That is, for all $f \in \mathcal{S}$,

$$(f|\Psi_\gamma) = c_n(\hat{f}|\mathbf{A}_\gamma). \tag{22}$$

Now, by Theorem 3.3, if $-(n+1) < \Re\gamma < 0$, then $\Psi_\gamma(t,z)$, initially defined for $z \neq 0$, extends smoothly to $\mathbf{H}^n \setminus \{0\}$. We shall next show that for any $\gamma \notin \mathbf{Z}^+$, this extension may in fact be done real analytically. This is a special case of the following proposition:

Proposition 3.4. *Suppose that* $k \in \mathbf{C}$, $g \in L^1([-1,1])$ *and that* g *is real analytic near* 0. *For* $(t,z) \in \mathbf{H}^n$ *with* $z \neq 0$, *set*

$$G(t,z) = \int_{-1}^1 (|z|^2 + ist)^{-k}g(s)ds. \tag{23}$$

Then G *may be extended to be a real analytic function on* $\mathbf{H}^n \setminus \{0\}$.

Proof It is evident that G is real analytic in the region $U_1 = \{(t,z) : z \neq 0\}$, where it is defined. Let us show that G may be extended real analytically to the region $U_2 = \{(t,z) : t > 0\}$; a similar argument will then show that it may be extended real analytically to the region $U_3 = \{(t,z) : t < 0\}$. Since these three regions cover $\mathbf{H}^n \setminus \{0\}$, this will prove the proposition.

For some $\epsilon > 0$, g may be extended from the interval $(-2\epsilon, 2\epsilon)$ to be a holomorphic function on $\{z \in \mathbf{C} : |z| < 2\epsilon\}$; we call this extension g also.

Let Γ denote the curve in $\mathbf{C}$ consisting of the straight line segment from -1 to $-\epsilon$, followed by the semicircle $\epsilon e^{i\theta}$ $(-\pi < \theta < 0)$; followed by the straight line segment from ϵ to 1. Say first that $(t,z) \in U_1 \cap U_2$, and that $|z|^2 = a > 0$. Then if $s \in \mathbf{C}$, $\Im s \leq 0$, then $\zeta = |z|^2 + ist$ lies in the half plane $\{\zeta : \Re\zeta \geq a\}$. Thus we may deform the contour in (23), and write

$$G(t,z) = \int_\Gamma (|z|^2 + ist)^{-k}g(s)ds. \tag{24}$$

for $(t,z) \in U_1 \cap U_2$. Note, however, that if $s \in \Gamma$ and $(t,z) \in U_2$, then $(|z|^2 + ist)$ lies in the right half plane and has absolute value at least ϵ. Thus (24) enables us to extend G real analytically to all of U_2, as desired.

Proposition 3.4 shows that Ψ_γ has a real analytic extension to $\mathbf{H}^n \setminus \{0\}$, provided $\gamma \in \Omega = \{\gamma \in \mathbf{C} : \gamma, -n-1-\gamma \notin \mathbf{Z}^+\}$. Moreover – by replacing the contour of integration in (18) by a contour like Γ in the proof of Proposition 3.4, where necessary – we see that $\Psi_\gamma(t,z)$ is a continuous function of $((t,z),\gamma) \in (\mathbf{H}^n \setminus \{0\}) \times \Omega$. In addition, for fixed $(t,z) \in \mathbf{H}^n \setminus \{0\}$, $\Psi_\gamma(t,z)$ is holomorphic in $\gamma \in \Omega$.

Assume, in this paragraph, that $-n-1-\gamma = m \in \mathbf{Z}^+$. Since the proof of Proposition 3.4 could be carried out equally well if, in (23), we replaced $(|z|^2+ist)^{-k}$ by $(|z|^2+ist)^m \log(|z|^2+ist)$, we see that Ψ_γ has a real analytic extension to $\mathbf{H}^n \setminus \{0\}$. Moreover, since the proof of Proposition 3.4 could be carried out equally well if, in (23), we replaced $(|z|^2+ist)^{-k}$ by $(|z|^2+ist)^m \log(\frac{|z|^2+ist}{|(t,z)|})$, and since, in (19), $\log(|z|^2+ist)$ is the sum of the homogeneous function $\log(\frac{|z|^2+ist}{|(t,z)|})$ and $\log|(t,z)|$, we see that $\Psi_\gamma \in \mathbf{K}^{2m}$.

To complete the proof of Theorem 3.2, we need to obtain an analogue of (22) for all $\gamma \in \mathbf{C} \setminus \mathbf{Z}^+$. In order to do this, we assume, for the rest of the proof that $f \in \mathcal{S}_0$; we shall analyically continue (22) from the region $-(n+1) < \Re\gamma < 0$.

First, we claim that the right side of (22) defines a holomorphic function of $\gamma \in \mathbf{C}$. Indeed, if $F \in \mathcal{S}$, we surely have that $\|\hat{F}(\lambda)E_\alpha\| \leq \|F\|_{L^1}$. Since, for any $N \in \mathbf{Z}^+$, $\mathcal{L}_0^N f \in \mathcal{S}$, and there exists $g_N \in \mathcal{S}$ with $f = \mathcal{L}_0^N g_N$, it follows readily that for any $M \in \mathbb{R}$, $|(\hat{f}(\lambda)E_\alpha, A_\gamma(\lambda)E_\alpha)| \leq C_{\gamma,M}[(2|\alpha|+n)|\lambda|]^{-M}$, where $C_{\gamma,M}$ remains bounded for γ in a compact subset of $\mathbf{C}$. It now follows readily that the right side of (22), which is $c_n \int_{-\infty}^{\infty} \sum_\alpha (\hat{f}(\lambda)E_\alpha, A_\gamma(\lambda)E_\alpha)(2|\lambda|)^n d\lambda$, is absolutely convergent, and defines a holomorphic function of $\gamma \in \mathbf{C}$. (For this, see Proposition 5.2 of [**11**], and its proof.)

Let us now explain how to continue the left side of (22) from the region $-(n+1) < \Re\gamma < 0$ into the region $D = \{\gamma \in \mathbf{C} : -(n+1) < \Re\gamma, \gamma \notin \mathbf{Z}^+\}$. Note first that, for any $\gamma \in D$, $\Psi_\gamma(t,z) = \Psi_\gamma(t,-z)$. Therefore, if p is a homogeneous monomial in $z, \bar{z}, t$ of odd homogeneous degree, we must have $\int_{a<|u|<b} p(t,z)\Psi_\gamma(t,z)dzd\bar{z}dt = 0$ for any $0 < a < b$. (For we must have $p(t,z) = z^\alpha \bar{z}^\beta t^m$ where $|\alpha|+|\beta|$ is odd.) In particular, this holds if 2γ is an odd positive integer. The discussion on page 40 of [**5**] now implies that there exists $\Lambda_\gamma \in \mathcal{K}_{-2n-2-2\gamma}$, such that $\Lambda_\gamma = \Psi_\gamma$ away from 0. Further, Λ_γ and Ψ_γ are equal as elements of $\mathcal{S}'$ if $-(n+1) < \Re\gamma < 0$. Moreover the explicit formula for Λ_γ (a special case of the formula in the same reference), together with the remarks made four paragraphs ago, show that (f, Λ_γ) is holomorphic in $\gamma \in D$. Accordingly,

$$(f|\Lambda_\gamma) = (\hat{f}|\mathbf{A}_\gamma) \tag{25}$$

for all $\gamma \in D$.

Analytic continuation further shows, directly, that (22) holds in the region $D' = \{\gamma : \Re\gamma < 0, -n-1-\gamma \notin \mathbf{Z}^+\}$. Finally, to obtain (22) for $\gamma = -n-1-m$, $m \in \mathbf{Z}^+$, we write (22) for $\gamma \in D'$, divide both sides by $\Gamma(n+1+\gamma)$, differentiate both sides with respect to γ, and take the limit as γ approaches $-n-1-m$. The result is that (22) holds for $\gamma = -n-1-m$ as well, since elements of $\mathcal{S}_0$ are orthogonal to polynomials. (Note that the derivative of $1/\Gamma(z)$ at $z = -m$ is $(-1)^m m!$, for instance since $1/\Gamma(z) = [\sin(\pi z)\Gamma(1-z)]/\pi$.)

Thus (25) holds for all $\gamma \in D$, and (22) holds for all γ with $\Re\gamma < 0$. Set $\Lambda_\gamma = \Psi_\gamma$ for any γ with $\Re\gamma < 0$ or with $\gamma \in \mathbf{Z}^+$; then (25) holds for all $\gamma \in \mathbf{C}$. Finally, say $f \in \mathcal{S}_0, g \in \mathcal{S}$. Then for any $\gamma \in \mathbf{C}$,

$$(g|f * \Lambda_\gamma) = (\tilde{f} * g|\Lambda_\gamma) = c_n(\hat{f}^* \hat{g}|\mathbf{A}_\gamma) = c_n(\hat{g}|\hat{f}\mathbf{A}_\gamma) = (g|\mathcal{L}_0^\gamma f)$$

by Plancherel; so $\mathcal{L}_0^\gamma f = f * \Lambda_\gamma$. This completes the proof of Theorem 3.2.

4. Powers of Convolution Operators

In this section, we discuss convolution with general elements of $\mathbf{K}^k$. As we observed in (17), if $K \in \mathbf{K}^k$, then there is an operator $\mathcal{K} : \mathcal{S}_0 \to \mathcal{S}_0$, given by $\mathcal{K}f = f * K$. We write $\mathcal{K} = \mathcal{O}_0(K)$ for this operator. Let $\mathcal{P}_k$ be as before (5) if $k \in \mathbf{Z}^+$, 0 if $k \notin \mathbf{Z}^+$. Note that $\mathcal{O}_0(K) = 0$ if and only if $K \in \mathcal{P}_k$. (Indeed, if a tempered distribution annihilates $\mathcal{S}_0$, its Euclidean Fourier transform must be supported at 0, so it must be a polynomial.) If $K_1, K_2 \in \mathbf{K}^k$, let us say $K_1 = K_2$ mod $\mathcal{P}_k$ if $K_1 - K_2 \in \mathcal{P}_k$.

Recall that we have permanently set $Q = 2n + 2$. Proposition 2.3 of [**5**] tells us this: say $K_i \in \mathbf{K}^{k_i}, \mathcal{K}_i = \mathcal{O}_0(K_i)$, $i = 1, 2$. Let $k = k_1 + k_2 + Q$. Then $\mathcal{K}_1\mathcal{K}_2 = \mathcal{O}_0(K)$ for some $K \in \mathbf{K}^k$. Evidently, K is uniquely determined mod $\mathcal{P}_k$. In [**5**], a procedure is given for singling out a particular K, even if $k \in \mathbf{Z}^+$, and this K is denoted by $K_2\underline{*}K_1$.

If $\Re k < 0$, there is a more direct way of defining $K_2\underline{*}K_1$:

PROPOSITION 4.1. *Say $K_i \in \mathbf{K}^{k_i}$, $i = 1, 2$. Let $k = k_1 + k_2 + Q$; suppose $\Re k < 0$. For $i = 1, 2$, write $K_i = K_i' + K_i''$, where $K_i' \in \mathcal{E}'(\mathbf{H}^n)$, the distributions of compact support, and where $K_i'' \in C^\infty$. Set*

$$K_2 * K_1 == K_2' * K_1' + K_2' * K_1'' + K_2'' * K_1' + K_2'' * K_1''. \tag{26}$$

Then $K_2\underline{*}K_1 = K_2 * K_1 = K$, say; and for all $f \in \mathcal{S}(\mathbf{H}^n)$,

$$f * K = (f * K_2) * K_1' + (f * K_2) * K_1''. \tag{27}$$

Proof (27), and the fact that $K_2 * K_1 \in \mathbf{K}^k$, were established in Lemma 9.5 (a) and (b) of [**4**] and their proofs. Moreover, if $f \in \mathcal{S}_0$, it follows immediately from (27) that $f * K = (f * K_2) * K_1$. Thus $K_2 * K_1 = K_2\underline{*}K_1$ mod $\mathcal{P}_k$; but $\mathcal{P}_k = \{0\}$, so we are done.

Moreover, by Theorem 7.5 of [**12**], if $K_i \in \mathbf{AK}^{k_i}$, then $\mathcal{K}_1\mathcal{K}_2 \in \mathbf{AK}^k$. [2]

If now $K_i \in \mathbf{K}^{k_i}, \mathcal{K}_i = \mathcal{O}_0(K_i)$, $i = 1, 2, 3$, and $k = \sum_{i=1}^3 k_i + 2Q$, then evidently $K_3\underline{*}(K_2\underline{*}K_1) = (K_3\underline{*}K_2)\underline{*}K_1$ mod $\mathcal{P}_k$. Moreover, evidently, $(K_2\underline{*}K_1)\tilde{} = \tilde{K}_1\underline{*}\tilde{K}_2$, mod $\mathcal{P}_k$.

Say now that $K \in \mathbf{K}^k$, and let $\mathcal{K} = \mathcal{O}_0(K)$. Then $\mathcal{K}$ is a densely defined operator on L^2, which is closable, since its adjoint is densely defined. Indeed, its adjoint clearly contains $\mathcal{O}_0(\tilde{K})$, which is defined on the L^2-dense subspace $\mathcal{S}_0$. Let $\bar{\mathcal{K}}$ denote the closure of $\mathcal{K}$.

LEMMA 4.2. *Suppose $K \in \mathbf{K}^k$, and let $\mathcal{K} = \mathcal{O}_0(K)$. Let $j = -Q - k$. Then $\mathcal{D}_{\Re j}$ is contained in the domain of $\bar{\mathcal{K}}$. Moreover, if $f \in \mathcal{D}_{\Re j}$, then*

$$\bar{\mathcal{K}}f = [\mathcal{L}_0^{j/2} f] * (\Psi_{-j/2}\underline{*}K). \tag{28}$$

Proof Note first that the right side of (28) makes sense, and defines an L^2 function, if $f \in \mathcal{D}_{\Re j}$. Indeed, $\mathcal{L}_0^{j/2} f \in L^2$, and $\Psi_{-j/2}\underline{*}K$ is a PV (principal value) distribution (that is, an element of $\mathbf{K}^{-Q}$). Moreover, if $f \in \mathcal{S}_0$, then the right side of (28) equals $[\mathcal{L}_0^{-j/2}\mathcal{L}_0^{j/2} f] * K = f * K = \mathcal{K}f$.

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[2] Actually, $K_2\underline{*}K_1$ is defined differently in [**12**], so we need to show that the definitions coincide mod $\mathcal{P}_k$. Let $K = K_2\underline{*}K_1$, as defined in [**12**], and suppose that $f \in \mathcal{S}_0$; we need to show that $f * K = (f * K_1) * K_2$. Choose $N \in \mathbf{Z}^+$ with $2N > \Re k$, pick $g \in \mathcal{S}_0$ with $f = \mathcal{L}_0^N g$, and use equation (7.22) of [**12**].

To prove the lemma, suppose that $f \in \mathcal{D}_{\Re j}$. We need to show that there exists a sequence $\{f_m\} \subseteq \mathcal{S}_0$, such that $f_m \to f$ in L^2, and such that $\mathcal{K}f_m$ approaches the right side of (28) in L^2. But using Proposition 3.1, we may find a sequence $\{f_m\} \subseteq \mathcal{S}_0$, such that $f_m \to f$ in L^2, and such that $\mathcal{L}_0^{j/2} f_m \to \mathcal{L}_0^{j/2} f$ in L^2. But then, since $\Psi_{-j/2}\underline{*}K$ is a PV distribution, we have that $\mathcal{K}f_m = [\mathcal{L}_0^{j/2} f_m] * (\Psi_{-j/2}\underline{*}K) \to [\mathcal{L}_0^{j/2} f] * (\Psi_{-j/2}\underline{*}K)$ as well, as desired.

In the above proof, we used the familiar fact [**15**] that convolution with a PV distribution is a bounded map on L^2. In fact, we have:

Lemma 4.3. *Let K be a PV distribution. Say $l \in \mathbb{R}$. Define $\mathcal{T} : \mathcal{D}_l \to L^2$ by $\mathcal{T}f = f * K$. Then, in fact, $\mathcal{T} : \mathcal{D}_l \to \mathcal{D}_l$.*

Proof Say $f \in \mathcal{D}_l$. In order to show that $f * K \in \mathcal{D}_l$, it suffices, by Proposition 3.1, to show that there is a sequence $\{g_m\} \subseteq \mathcal{S}_0$, such that $g_m \to f * K$ in L^2, and such that $\{\mathcal{L}_0^{l/2} g_m\}$ is a convergent sequence in L^2. But, by Proposition 3.1, there is a sequence $\{f_m\} \subseteq \mathcal{S}_0$, such that $f_m \to f$ in L^2, and such that $\mathcal{L}_0^{l/2} f_m \to \mathcal{L}_0^{l/2} f$ in L^2. Set $g_m = f_m * K$. Then $g_m \to f * K$ in L^2, while $\mathcal{L}_0^{l/2} g_m = (f_m * K) * \Psi_{l/2} = ([\mathcal{L}_0^{l/2} f_m * \Psi_{-l/2}] * K) * \Psi_{l/2} = \mathcal{L}_0^{l/2} f_m * K_1$, where K_1 is the PV distribution $(\Psi_{-l/2}\underline{*}K)\underline{*}\Psi_{l/2}$. Therefore $\mathcal{L}_0^{l/2} g_m \to \mathcal{L}_0^{l/2} f * K_1$ in L^2, and the proof is complete.

Corollary 4.4. *In the situation of Lemma 4.2, $\bar{\mathcal{K}} : \mathcal{D}_{\Re j} \to \mathcal{D}_{-\Re j}$.*

Proof It is evident that $\mathcal{L}_0^{j/2} : \mathcal{D}_{\Re j} \to \mathcal{D}_{-\Re j}$. Thus, the corollary follows from (28) and Lemma 4.3.

One might ask if, in the situation of Lemma 4.2, the domain of $\bar{\mathcal{K}}$ in fact equals $\mathcal{D}_{\Re j}$. However, it is clear that the analogous fact on Euclidean space is not necessarily true – unless we are in the situation of Proposition 1.1, in which case it clearly is true. Similarly, on $\mathbf{H}^n$, the domain of $\bar{\mathcal{K}}$ equals $\mathcal{D}_{\Re j}$, under an "invertibility" hypothesis, namely that there exists $H \in \mathbf{K}^{-Q+j}$ with $H\underline{*}K = K\underline{*}H = \delta$. There are several equivalent, and – in practice – more easily checked ways, of formulating this invertibility hypothesis. Before going further, we wish to describe these equivalent formulations (Theorem 4.6 below).

One of these formulations has the appeal that it is directly analogous to the hypothesis in the Euclidean situation that $\hat{K}$ not vanish away from 0. To describe this formulation, we shall need some facts about the group Fourier transform of K.

We define the *Schwartz vectors* in $\mathcal{H}_\lambda$ to be $\mathcal{H}_\lambda^\infty = \{v = \sum_\alpha v_\alpha E_{\alpha,\lambda} \in \mathcal{H}_\lambda :$ for any $N \in \mathbf{N}, \|v\|_N^2 = \sum_\alpha (|\alpha|+1)^N |v_\alpha|^2 < \infty\}$. We topologize $\mathcal{H}_\lambda^\infty$ as a Frechet space. It is clear that the operator $A_\gamma(\lambda)$, defined in (14), extends to a continuous operator from $\mathcal{H}^\infty$ to itself; we call this operator again $A_\gamma(\lambda)$.

In the propositions which follow we shall identify all $\mathcal{H}_\lambda$, so that $E_{\alpha,\lambda}$ is identified with $E_{\alpha,\mu}$ for all α, μ. It is then a fundamental fact ([**8**] page 211, line 7) that if $f \in L^1(\mathbf{H}^n)$ and $v \in \mathcal{H}$, then $\hat{f}(\lambda)v$ is continuous in $\lambda \in \mathbb{R} \setminus \{0\}$. In particular, say $\gamma \in \mathbf{C}$, $f \in \mathcal{S}_0$, $v \in \mathcal{H}^\infty$. It follows from Theorem 3.2, that $(f * \Psi_\gamma)\hat{}(\lambda)v = \hat{f}(\lambda)A_\gamma(\lambda)v$, for a.e. λ. In fact, this is true for all λ, since both sides are continuous in λ.

We assert:

PROPOSITION 4.5. *Suppose $k \in \mathbf{C}$, $K \in \mathcal{K}^k$; let $j = -Q - k$. Then:*
(a) There exists an operator family $\mathbf{J} = (J(\lambda))$, with the domain of each $J(\lambda)$ being $\mathcal{H}^\infty$, such that $(f|K) = c_n(\hat{f}|\mathbf{J})$ for all $f \in \mathcal{S}_0$. Set $J_+ = J(1)$, $J_- = J(-1)$. We have $J_+, J_- : \mathcal{H}^\infty \to \mathcal{H}^\infty$ continuously. For any $\lambda > 0$, $J(\lambda) = |\lambda|^{j/2} J_+$; for any $\lambda < 0$, $J(\lambda) = |\lambda|^{j/2} J_-$. $\mathbf{J}$, with these properties, is unique; we say $\hat{K} = \mathbf{J}$ in the $\mathcal{S}_0'$ sense. If $k = -Q$, then J_+ and J_- extend to bounded operators on $\mathcal{H}$. Moreover, if $f \in \mathcal{S}_0$, then $(f * K)\hat{} = \hat{f}\hat{K}$, in the sense that for any λ, $(f * K)\hat{}(\lambda)$ and $\hat{f}(\lambda)\hat{K}(\lambda)$ agree when tested on $\mathcal{H}^\infty$.
(b) Suppose also $(\tilde{K})\hat{} = \tilde{\mathbf{J}}$ in the $\mathcal{S}_0'$ sense. Then for any λ, $[J(\lambda)]^* = \tilde{J}(\lambda)$ on $\mathcal{H}^\infty$. By this we mean that $(J(\lambda)v, w) = (v, \tilde{J}(\lambda)w)$ for all $v, w \in \mathcal{H}_\infty$.

Let $\mathcal{F}$ be the Euclidean Fourier transform on $\mathbf{H}^n$, thought of as $\mathbb{R}^{2n+1}$. We let (λ, ξ) be the dual variables to $(t, (x, y))$ on the Fourier transform side. (Here $z = x + iy$.)

We may now present the equivalent formulations of our "invertibility hypothesis":

THEOREM 4.6. *Suppose $k \in \mathbf{C}$, $j = -Q - k$, and $K \in \mathbf{K}^k$ (resp. $\mathbf{AK}^k$). Let $\mathcal{K} = \mathcal{O}_0(K)$, $\tilde{\mathcal{K}} = \mathcal{O}_0(\tilde{K})$. Suppose also that $\hat{K} = \mathbf{J}$ and $(\tilde{K})\hat{} = \tilde{\mathbf{J}}$ in the $\mathcal{S}_0'$ sense. Also, let $K_1 = \Psi_{-j/2} \underline{*} K \in \mathbf{K}^{-Q}$. Then the following are equivalent:*
(a) There exists $H \in \mathbf{K}^{-Q+j}$ (resp. $\mathbf{AK}^{-Q+j}$) with $H \underline{*} K = K \underline{*} H = \delta$.
(b) $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are invertible on $\mathcal{S}_0$.
(c) The operator $f \to f * K_1$ is invertible on L^2.
(d) $\mathcal{F}K(0, \xi) \neq 0$ whenever $\xi \neq 0$; and $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are injective on $\mathcal{S}_0$.
(e) $\mathcal{F}K(0, \xi) \neq 0$ whenever $\xi \neq 0$; and $J_+, J_-, \tilde{J}_+$ and $\tilde{J}_-$ are injective on $\mathcal{H}^\infty$.

We delay the proofs of Proposition 4.5 and Theorem 4.6 to an appendix at the end of this section, so as not to interrupt the flow of the presentation.

Next we remark the following simple abstract Hilbert space lemma.

LEMMA 4.7. *Let $\mathcal{H}$ be a Hilbert space. Suppose that A is a self-adjoint operator on $\mathcal{H}$ with domain $\mathcal{D}$. Suppose also that B is a bounded invertible operator on $\mathcal{H}$, such that B^* and $(B^*)^{-1}$ map $\mathcal{D}$ to $\mathcal{D}$. Then the adjoint of BA (with domain $\mathcal{D}$) is AB^* (again with domain $\mathcal{D}$).*

Proof We need to show that if $v, w \in \mathcal{H}$, and if $(BAu, v) = (u, w)$ for all $u \in \mathcal{D}$, then $v \in \mathcal{D}$, and $AB^*v = w$. But $(Au, B^*v) = (u, w)$ for all $u \in \mathcal{D}$, so $B^*v \in \mathcal{D}$ and $AB^*v = w$. Finally – since $(B^*)^{-1}$ maps $\mathcal{D}$ to $\mathcal{D}$ – we see that $v \in \mathcal{D}$.

From this we now deduce:

THEOREM 4.8. *Suppose $K \in \mathbf{K}^k$, and let $\mathcal{K} = \mathcal{O}_0(K)$, $\tilde{\mathcal{K}} = \mathcal{O}_0(\tilde{K})$ Let $j = -Q - k$. Suppose that $\mathcal{K}, \tilde{\mathcal{K}} : \mathcal{S}_0 \to \mathcal{S}_0$ are both invertible. Then:*
(i) The domains of $\bar{\mathcal{K}}$ and $\bar{\tilde{\mathcal{K}}}$ equal $\mathcal{D}_{\Re j}$.
(ii) $\bar{\mathcal{K}}^* = \bar{\tilde{\mathcal{K}}}$.

Proof Select $H \in \mathbf{K}^{-Q+j}$ with $H \underline{*} K = K \underline{*} H = \delta$. Let $B : L^2 \to L^2$ be the invertible operator defined by $Bf = f * (\Psi_{-j/2} \underline{*} K)$.

Let $\mathcal{K}'$ denote the restriction of $\bar{\mathcal{K}}$ to $\mathcal{D}_{\Re j}$. Then $\mathcal{K} \subseteq \mathcal{K}' \subseteq \bar{\mathcal{K}}$, so $\bar{\mathcal{K}}^* = (\mathcal{K}')^* = \mathcal{K}^*$. Since $\bar{\mathcal{K}}^* = (\mathcal{K}')^*$, it follows that the domain of $\bar{\mathcal{K}}^*$ is $\mathcal{D}_{\Re j}$, by (28), Lemma 4.7 (applied with $A = \mathcal{L}_0^{j/2}$ and $\mathcal{D} = \mathcal{D}_{\Re j}$), and Lemma 4.3. Since $\bar{\mathcal{K}}^* = \mathcal{K}^*$, we clearly have that $\bar{\mathcal{K}}^* \supseteq \tilde{\mathcal{K}}$. Since the adjoint of any densely-defined operator is closed,

$$\bar{\mathcal{K}}^* \supseteq \bar{\tilde{\mathcal{K}}} \supseteq \bar{\tilde{\mathcal{K}}}|_{\mathcal{D}_{\Re j}}. \tag{29}$$

Since the leftmost and rightmost operators in (29) have the same domains, all three operators in (29) must be equal. This proves (ii), and also (i) for $\bar{\tilde{\mathcal{K}}}$; (i) for $\bar{\mathcal{K}}$ follows, since all the hypotheses of the theorem hold with $\tilde{K}$ in place of K, and since $K = (\tilde{K})\tilde{}$.

Now, again suppose $K \in \mathbf{K}^k$, $\mathcal{K} = \mathcal{O}_0(K)$, $j = -Q - k$. Note that $\mathcal{K}$ is symmetric if and only if $\tilde{K} = K$ modulo a polynomial. If this is true, then, since $\tilde{K} \in \mathbf{K}^{\bar{k}}$, k must be real.

Corollary 4.9. *Suppose $K \in \mathbf{K}^k$, and that $\mathcal{K} = \mathcal{O}_0(K) : \mathcal{S}_0 \to \mathcal{S}_0$ is invertible. Let $j = -Q - k$. Suppose also $\tilde{K} = K$ (mod $\mathcal{P}_k$). Then:*
(i) $\bar{\mathcal{K}}$ is a self-adjoint operator, with domain $\mathcal{D}_j$.
(ii) Let $\mathcal{T}$ be any self-adjoint operator which contains $\mathcal{K}$. Then $\mathcal{T} = \bar{\mathcal{K}}$.

Proof (i) is an immediate corollary of Theorem 4.8. For (ii), simply note that $\mathcal{T}$, being self-adjoint, is closed, hence contains the self-adjoint operator $\bar{\mathcal{K}}$. But a self-adjoint operator cannot have any proper self-adjoint extensions, so these operators must be equal.

Suppose again $K \in \mathbf{K}^k$, $\mathcal{K} = \mathcal{O}_0(K)$, We shall say that $\mathcal{K}$ is *positive* if $(\mathcal{K}f, f) > 0$ for all nonzero $f \in \mathcal{S}_0$. Recall that this condition implies, by polarization, that $\mathcal{K}$ is symmetric on $\mathcal{S}_0$, i.e., that $\tilde{K} = K$ mod $\mathcal{P}_k$. Note also that if $\mathcal{K}$ is positive and invertible (on $\mathcal{S}_0$, of course), then by Corollary 4.9, $\bar{\mathcal{K}}$ is a positive self-adjoint operator. In the companion paper [**13**], we shall show that, in the notation of Theorem 4.6 (e), $\mathcal{K}$ is positive and invertible if and only if: $\mathcal{F}K(0,\xi) > 0$ whenever $\xi \neq 0$, and $(J_+v, v) > 0$, $(J_-v, v) > 0$ for all $v \in \mathcal{H}^\infty$. We do not use this fact here.

At this point we are able to prove Theorem 1.4 for $j \neq 0$, assuming that Proposition 1.5, and Theorems 1.6 and 1.7 are known. Precisely, we shall show:

Theorem 4.10. *Suppose $K \in \mathbf{K}^k$ (resp. $\mathbf{AK}^k$), $\mathcal{K} = \mathcal{O}_0(K)$ is positive and invertible. Let $j - Q - k$; suppose $j \neq 0$ (resp. $j \neq 0$ and $j \in \mathbf{Q}$).*

Then, for any $\gamma \in \mathbf{C}$, there exists $K_\gamma \in \mathbf{K}^{-Q-\gamma j}$ (resp. $\mathbf{AK}^{-Q-\gamma j}$), such that if we write $\mathcal{K}_\gamma = \mathcal{O}_0(K_\gamma)$, then

$$(\bar{\mathcal{K}})^\gamma = \overline{\mathcal{K}_\gamma}. \tag{30}$$

In particular,

$$\text{if } f \in \mathcal{S}_0, \text{ then } (\bar{\mathcal{K}})^\gamma f = f * K_\gamma. \tag{31}$$

Moreover, for any γ, $\mathcal{K}_\gamma$ is invertible, with inverse $\mathcal{K}_{-\gamma}$; and $\tilde{K}_\gamma = K_{\bar{\gamma}}$ mod $\mathcal{P}_{-Q-\bar{\gamma}j}$. Thus, by Theorem 4.8, the domain of $(\bar{\mathcal{K}})^\gamma$ is $\mathcal{D}_{j\Re\gamma}$.

Finally, if γ is real, then $\mathcal{K}_\gamma$ is positive.

Proof, assuming Proposition 1.5, Theorem 1.6 and Theorem 1.7 Clearly, for any $l \in \mathbf{Z}$, there exists $K_l \in \mathbf{K}^{-Q-lj}$ (resp. $\mathbf{AK}^{-Q-lj}$), such that if we set $\mathcal{K}_l = \mathcal{O}_0(K_l)$, then $(\mathcal{K})^l = \mathcal{K}_l$. Surely $\mathcal{K}_l$ is invertible and positive.

We claim that $(\bar{\mathcal{K}})^l = \overline{\mathcal{K}_l}$. Indeed, $(\bar{\mathcal{K}})^l$ clearly contains $(\mathcal{K})^l = \mathcal{K}_l$, since $\bar{\mathcal{K}} \supseteq \mathcal{K}$, $(\bar{\mathcal{K}})^{-1} \supseteq \mathcal{K}^{-1}$, and $\mathcal{K}$ and $\mathcal{K}^{-1}$ both map $\mathcal{S}_0$ to $\mathcal{S}_0$. But $(\bar{\mathcal{K}})^l$ is self-adjoint, so by Corollary 4.9 (b), $(\bar{\mathcal{K}})^l = \overline{\mathcal{K}_l}$, as claimed.

This implies that, for any $\gamma \in \mathbf{C}$, $(\bar{\mathcal{K}})^\gamma = (\overline{\mathcal{K}_l})^{\gamma/l}$. Thus, in proving the theorem, we may, if we wish, replace K by K_l; and therefore, we may assume that $j > Q+3$ (resp. $j > Q+3$, and j is an even integer).

We now claim that it is enough to show the following: If $-Q/(2j) < \Re\gamma < 0$, then:

(*) $\mathcal{S}_0$ is contained in the domain of $(\bar{\mathcal{K}})^\gamma$; and there exists $K_\gamma \in \mathbf{K}^{-Q-\gamma j}$ (resp. $\mathbf{AK}^{-Q-\gamma j}$), such that (31) holds.

Indeed, say we knew (*) for $-Q/(2j) < \Re\gamma < 0$. Then, since $(\bar{\mathcal{K}})^\alpha(\bar{\mathcal{K}})^\beta = (\bar{\mathcal{K}})^{\alpha+\beta}$ whenever $\Re\alpha, \Re\beta < 0$, (*) would follow for all $\gamma \in \mathbf{C}$ with $\Re\gamma < 0$. However, we also know that (*) holds whenever $\gamma \in \mathbf{N}$. Since $(\bar{\mathcal{K}})^\alpha(\bar{\mathcal{K}})^N \subseteq (\bar{\mathcal{K}})^{\alpha+N}$ for all $\alpha, N \in \mathbf{C}$, it now follows that (*) holds for all $\gamma \in \mathbf{C}$. All the assertions of the theorem then follow at once, except (30), and its consequence, that the domain of $(\bar{\mathcal{K}})^\gamma$ is $\mathcal{D}_{j\Re\gamma}$. To prove (30), first observe that by (*), $(\bar{\mathcal{K}})^\gamma \supseteq \mathcal{K}_\gamma$. Since $(\bar{\mathcal{K}})^\gamma$ is closed, we see that $(\bar{\mathcal{K}})^\gamma \supseteq \overline{\mathcal{K}_\gamma}$ for any γ. In particular we also have $(\bar{\mathcal{K}})^{\bar{\gamma}} \supseteq \overline{\mathcal{K}_{\bar{\gamma}}}$. Taking adjoints of the latter containment, and using Theorem 4.8, we see that $\overline{\mathcal{K}_\gamma} \supseteq (\bar{\mathcal{K}})^\gamma$, and (30) now follows.

We therefore now assume $-Q/(2j) < \Re\gamma < 0$; to complete the proof of the theorem, it suffices to verify (*). Let $S = \bar{\mathcal{K}}$. Let $\Phi \in C^\infty(\mathbf{H}^n \setminus \{0\})$ be the function satisfying $(I+S)^{-1}f = f * \Phi$ for any $f \in \mathcal{S}$, and which satisfies the conclusions of Proposition 1.5, Theorem 1.6 and Theorem 1.7. Define K_γ by (10); then surely $K_\gamma \in \mathbf{K}^{-Q-\gamma j}$ (resp. $\mathbf{AK}^{-Q-\gamma j}$), and it will suffice to verify that (*) holds for this K_γ.

Dilating the equation $(I+S)^{-1}f = f * \Phi$ ($f \in \mathcal{S}_0$), we see that, whenever $r > 0$, $(r^j I + S)^{-1} f = r^{Q-j} f * \Phi_r$, where $\Phi_r = \Phi \circ D_r$.

Now let $S = \int_0^\infty \lambda dP_\lambda$ be the spectral decomposition of S. Say $f \in \mathcal{S}_0$. We shall show:

(**) Whenever $\eta \in \mathbf{C}$, $-Q/j < \Re\eta < 0$, we have $\int_0^\infty \lambda^\eta d(P_\lambda f, f) d\lambda = (\mathcal{K}_\eta f, f)$.

Suppose (**) is known. Taking $\eta = 2\Re\gamma$, we see that f is in the domain of S^γ. Then, taking $\eta = \gamma$, we see that $(S^\gamma f, f) = (\mathcal{K}_\gamma f, f)$. Polarizing, we see that for all $f, g \in \mathcal{S}_0$, $(S^\gamma f, g) = (\mathcal{K}_\gamma f, g)$, and (*) follows.

To prove (**), write $\mathcal{I} = \int_0^\infty \lambda^\eta d(P_\lambda f, f) d\lambda$. First assume that η is real. Note that $\lambda^\eta = jc_\eta \int_0^\infty r^{j+j\eta-1}(r^j + \lambda)^{-1} dr$, as the substitution $s = r^j$ shows. Thus, by Tonelli's theorem,

$$\begin{aligned}
\mathcal{I} &= jc_\eta \int_0^\infty r^{j+j\eta-1} \int_0^\infty (r^j+\lambda)^{-1} d(P_\lambda f, f) d\lambda dr \\
&= jc_\eta \int_0^\infty r^{j+j\eta-1} ((r^j + S)^{-1} f, f) dr \\
&= jc_\eta \int_0^\infty r^{Q+j\eta-1} (f * \Phi_r, f) dr. \qquad (32)
\end{aligned}$$

Note that $\Phi \in L^\infty$; so $|\Phi_r|, |f| * |\Phi_r| \in L^\infty$, with L^∞ norms uniformly bounded for $0 < r < \infty$. Thus, if we write $\int_0^\infty = \lim_{\epsilon \to 0^+, N \to \infty} \int_\epsilon^\infty$ in the last expression in

(32), we see by Fubini's theorem that

$$\mathcal{I} = \lim_{\epsilon \to 0^+, N \to \infty} (f * [jc_\eta \int_\epsilon^N r^{Q+j\eta-1} \Phi_r], f) dr = (\mathcal{K}_\eta f, f) \tag{33}$$

as required. The last equality in (33) follows from the dominated convergence theorem. Here the dominating function is $g = (|f| * J)|f|$, where $J(u) = jc_\gamma \int_0^\infty r^{Q+j\eta-1} |\Phi_r|(u) dr$. We have $g \in L^1$, since J is homogeneous of degree $-Q - j\eta$, hence is in $L^1 + L^\infty$, which implies $|f| * J \in L^\infty$.

If η is not real, we now know that the integrals in (32) converge if η is replaced there by $\Re\eta$; so (32) holds by Fubini's theorem. Then (33) holds as well, with a similar justification as in the case of η real. This completes the proof.

Remarks 1. Of course Φ depends on K; we write $\Phi = \Phi_K$ when necessary.
2. Notation as in the first paragraph of the proof, what we have in fact shown is the following. Suppose that Proposition 1.5, and Theorem 1.6 are known, with K_l, Φ_{K_l} in place of K, Φ_K, for some $l \in \mathbf{Z}$. Then, in the smooth setting, Theorem 4.10 holds for K. (Similarly in the analytic setting.)

We see at once from (31) that $(\bar{\mathcal{K}})^\gamma : \mathcal{S}_0 \to \mathcal{S}_0$ continuously. Since $(\bar{\mathcal{K}})^{-\gamma}$ is the formal adjoint of $(\bar{\mathcal{K}})^\gamma$ on $\mathcal{S}_0$, we may, by duality, extend $(\bar{\mathcal{K}})^\gamma : \mathcal{S}_0 \to \mathcal{S}_0$ to a map $(\bar{\mathcal{K}})^\gamma : (\mathcal{S}_0)' \to (\mathcal{S}_0)'$. Now, there is a mapping $r : \mathcal{S}' \to (\mathcal{S}_0)'$, where, if $\Lambda \in \mathcal{S}'$, then $r(\Lambda)$ is the restriction of Λ to $\mathcal{S}_0$. It is not hard to show, using standard methods of distribution theory and the Euclidean Fourier transform, that this mapping r gives rise to an isomorphism $(\mathcal{S}_0)' \sim \mathcal{S}'/\mathcal{P}$, where $\mathcal{P} = \{\text{polynomials}\}$. (The only thing that one needs to prove here is that each element of $(\mathcal{S}_0)'$ may be extended to be an element of $\mathcal{S}'$.)

Here we only want to observe the evident fact that $\mathcal{E}'$, the distributions of compact support, may be viewed as a subspace of $(\mathcal{S}_0)'$, and thus, if $f \in \mathcal{E}'$, we may consider $(\bar{\mathcal{K}})^\gamma f \in (\mathcal{S}_0)'$; here $(\bar{\mathcal{K}})^\gamma f = f * K_\gamma$.

We then have the following result:

Corollary 4.11. *In the situation of Theorem 4.10, for any γ, $(\bar{\mathcal{K}})^\gamma : \mathcal{E}' \to (\mathcal{S}_0)'$ is pseudolocal and hypoelliptic (resp. analytic pseudolocal and analytic hypoelliptic).*

Proof If $f \in \mathcal{E}'$, then $(\bar{\mathcal{K}})^\gamma f = f * K_\gamma$. In the real analytic setting, the analytic pseudolocality of $(\bar{\mathcal{K}})^\gamma$ follows from Lemma 5.3 (a) and (c) of [**12**]; the pseudolocality in the smooth setting is routine.

Suppose now that $f * K_\gamma$ is smooth (resp. analytic) on an open set V; we need to show that f is smooth (resp. analytic) on V. We may clearly assume V is relatively compact. Select $\phi \in C_c^\infty(\mathbf{H}^n)$ with $\phi = 1$ in a neighborhood of 0 which is sufficiently large that $f * [(1-\phi)K_\gamma] = 0$ on V. We use Proposition 4.1. We change notation to the notation of that proposition, setting $K_2 = K_\gamma$, $K_1 = K_{-\gamma}$, $K_i' = \phi K_i$ $(i = 1, 2)$. Then $f * K_2'' = 0$ on V, so that $f * K_2'$ is smooth (resp. analytic) on V. Since $K_2 * K_1 = \delta$, (26) implies that $f = (f * K_2') * K_1 + (f * K_2'') * K_1$. The smoothness of f on V is now routine. If we are in the real analytic setting, the analyticity of f on V now follows from Lemma 5.3 of [**12**].

4.1. Appendix to Section 4. Here we prove Proposition 4.5 and Theorem 4.6.

Proof of Proposition 4.5 Here we draw upon the elementary discussion in Section 7 of [**4**], where part (a) was proved for $k = -Q$. For general k, let $K_1 = K\underline{*}\Psi_{-j/2} \in \mathcal{K}^{-Q}$, let $\mathbf{J_1} = \hat{K}_1$, and let $\mathbf{J} = \mathbf{J_1}\mathbf{A}_{j/2}$. If $f \in \mathcal{S}_0$, $(f|K) = (f * \Psi_{j/2}|K_1)$, and $f * K = (f * K_1) * \Psi_{j/2}$, so (a) follows. Note that the uniqueness of $\mathbf{J}$ in (a) follows from considering $(f|K)$ for $f \in \check{\mathcal{Q}}$. Similarly for (b): by continuity, we may assume $v = E_\alpha$, $w = E_\beta$ for some α, β; and we need only remark that $(f|K) = \overline{(\tilde{f}|\tilde{K})}$ for $f \in \check{\mathcal{Q}}$.

It readily follows from (b) that, if $f \in L^2$, then $(K * f)\hat{} = \mathbf{J}\hat{f}$.

Before proving Theorem 4.6, we need some preparation.

If $v, w \in \mathcal{H}^\infty$, define the finite rank operator $W[v,w] : \mathcal{H} \to \mathcal{H}$ by $W[v,w](v) = w$, $W[v,w](u) = 0$ if $(v,u) = 0$. The following fact follows at once from Proposition 7.8 of [**4**]:

Proposition 4.12. *Say $v_+, w_+, v_-, w_- \in \mathcal{H}_\infty$. Then there exists $K \in \mathcal{K}^{-Q}$ such that if $\hat{K} = \mathbf{J}$, then $J_+ = W[v_+, w_+]$, and $J_- = W[v_-, w_-]$.*

From this we infer:

Proposition 4.13. *Suppose $\varphi \in C_c^\infty(\mathbb{R}\setminus\{0\})$. Say $w_+, w_- \in \mathcal{H}_\infty$. Then there exists $f \in \mathcal{S}_0$ such that $\hat{f}(\lambda) = \varphi(\lambda)W[E_0, w_+]$ if $\lambda > 0$, and $\hat{f}(\lambda) = \varphi(\lambda)W[E_0, w_-]$ if $\lambda < 0$.*

Proof Choose K as in Proposition 4.12, with $v_+ = v_- = E_0$. Then choose $f_1 \in \check{\mathcal{Q}} \subseteq \mathcal{S}_0$ with $\hat{f}_1(\lambda) = \varphi(\lambda)W[E_0, E_0]$ for all λ. Then put $f = K * f_1$.

Corollary 4.14. *Let f be as in Proposition 4.13, and let $K, \mathbf{J}$ be as in Proposition 4.5. Then for $\lambda > 0$, $\widehat{K * f}(\lambda) = \varphi(\lambda)|\lambda|^{j/2}W[E_0, J_+w_+]$ if $\lambda > 0$, and $\widehat{K * f}(\lambda) = \varphi(\lambda)|\lambda|^{j/2}W[E_0, J_-w_-]$ if $\lambda < 0$.*

Proof For $\lambda > 0$, say, we have $(\tilde{f}*\tilde{K})\hat{}(\lambda) = \overline{\varphi(\lambda)}|\lambda|^{j/2}W[J_+v, E_0]$, and $\widehat{K * f}(\lambda)$ is the adjoint of this operator.

Proof of Theorem 4.6 First we prove the equivalences in the smooth setting; we discuss the analytic setting at the end of the proof.

$(a) \Rightarrow (b)$ is evident. For $(b) \Rightarrow (a)$, note that $(\tilde{\mathcal{K}})^{-1}$ is clearly the formal adjoint of $\mathcal{K}^{-1}$ on $\mathcal{S}_0$. Clearly $\mathcal{K}^{-1}$ is linear, and commutes with left translations. For $g \in \mathcal{S}_0$, $r > 0$, write $g_r = r^{-Q} g \circ D_{1/r}$, where $D_{1/r}(t,z) = (r^{-2}t, r^{-1}z)$. Then we clearly have that for any $f \in \mathcal{S}_0$, $((\tilde{\mathcal{K}})^{-1}f)_r = r^{-\bar{j}}(\tilde{\mathcal{K}})^{-1}f_r$. Proposition 3.2 of [**5**] now implies that $\mathcal{K}^{-1} = \mathcal{O}_0(H)$ for some $H \in \mathbf{K}^{-Q+j}$, so (a) follows.

$(a) \Rightarrow (c)$ is evident; the inverse of $f \to f * K_1$ is $f \to f * [H\underline{*}\Psi_{j/2}]$. For $(c) \Rightarrow (a)$, we note that, by Theorem 5.1 of [**5**], there exists $H_1 \in \mathbf{K}^{-Q}$ such that $H_1\underline{*}K_1 = K_1\underline{*}H_1 = \delta$; so we have (a), with $H = H_1\underline{*}\Psi_{-j/2}$.

We conclude the proof in the smooth setting by showing $(a) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a)$. For $(a) \Rightarrow (d)$, note that, since $(a) \Leftrightarrow (b)$, we need only demonstrate that $\mathcal{F}K(0,\xi) \neq 0$ whenever $\xi \neq 0$. Let $k_1 = k$, $k_2 = -Q + j$. If $\Re k_1, \Re k_2 < -2$, (d) is automatic from Proposition 7.2 (b) of [**4**], since that proposition implies that $1 = \mathcal{F}\delta(0,\xi) = \mathcal{F}K(0,\xi)\mathcal{F}H(0,\xi)$ for $\xi \neq 0$. But we can always reduce to the case $\Re k_1, \Re k_2 < -2$. Indeed, since $\Re(k_1 + k_2) = -2Q \leq -8$, we may select $N \in \mathbf{Z}$ with $\Re k_1 - 2N < -2$, $\Re k_2 + 2N < -2$. Put $H_0 = \Psi_{-N}\underline{*}H$, $K_0 = K\underline{*}\Psi_N$. Then

$K_0\underline{*}H_0 = \delta$, $K_0 \in \mathbf{K}^{k_1-2N}$, $H_0 \in \mathbf{K}^{k_2+2N}$, so $\mathcal{F}K_0(0,\xi) \neq 0$ for $\xi \neq 0$. However, if $N \geq 0$, we have $K_0 = \mathcal{L}_0^N K$; and if $N < 0$, we have we have $K = \mathcal{L}_0^{-N} K_0$. In either case, $\mathcal{F}K(0,\xi) = (|\xi|^2/4)^{-N}\mathcal{F}K_0(0,\xi)$, and (d) follows.

For $(d) \Rightarrow (e)$, let us show, for instance, if $0 \neq v \in \mathcal{H}^\infty$, then $J_+v = 0$. Select $f \in \mathcal{S}_0$ as in Proposition 4.13, with $w_+ = v$, $w_- = 0$, $\varphi \neq 0$. Since $f * K \neq 0$, it follows from Corollary 4.14 that $J_+v \neq 0$.

For $(e) \Rightarrow (a)$, we need only quote Theorem 6.2 of [**5**], but we must reconcile concepts with those in that reference. We need to show that, for any non-trivial irreducible unitary representation π of $\mathbf{H}^n$, π_K (notation as in [**5**]) is injective on C^∞ vectors. Say first π is the representation of $\mathbf{H}^n$ on $\mathcal{H}_\lambda$, and $v \in \mathcal{H}^\infty$; it is enough to show that show that $\pi_K v$ (notation as in [**5**]) agrees with our $\hat{K}(\lambda)v$. Say, for example, $\lambda > 0$, and choose f as in Proposition 4.13, with $\varphi(\lambda) = 1$. Then $\pi_K v = \pi_K \hat{f}(\lambda)E_0 = (K * f)\hat{}(\lambda)E_0 = \hat{K}(\lambda)v$ as claimed; similarly for $\lambda < 0$. The other non-trivial representations π are one-dimensional, indexed by $\xi \in \mathbb{R}^{2n} \setminus 0$. Choosing $f \in \mathcal{S}_0(\mathbf{H}^n)$ with $\mathcal{F}f(0,\xi) = 1$, we have $\pi_K = \pi_K[\mathcal{F}f(0,\xi)] = \mathcal{F}(K * f)(0,\xi) = \mathcal{F}K(0,\xi)\mathcal{F}f(0,\xi) \neq 0$, as desired. The last equality is immediate from Proposition 7.2 (a) of [**4**] if $\Re k < -2$; if not, we reduce to this case by writing $\tilde{f} = \mathcal{L}_0^N \tilde{g}$ with $g \in \mathcal{S}_0$, where $\Re k - 2N < -2$.

Finally we prove the theorem in the real analytic setting. It suffices to show this: say $K \in \mathbf{AK}^k$ and (a) holds; then $H \in \mathbf{AK}^{-Q+j}$. But $(a) \Rightarrow (e)$, so the conclusion follows from Theorem 5.1 of [**12**].

5. Results on resolvents – a sketch

In this section, we briefly sketch the proofs of Proposition 1.5, Theorem 1.6 and Theorem 1.7.

Proposition 1.5 follows from the results of [**5**]. In fact, near 0, Φ, which is the convolution kernel of $(I + \mathcal{K})^{-1}$, has an asymptotic expansion $\Phi \sim \sum_{m=0}^\infty H_m + q$, where $H_m \in \mathbf{K}^{-Q+j-m}$, and q is smooth. Here $H_0 = H$ is as in Theorem 4.6 (a). Similarly, the real analytic analogue of Proposition 1.5, which is the first part of Theorem 1.7, follows from the results of [**12**]. In fact, for some $\epsilon > 0$, there is a "sector" S in the complexification of $\mathbf{H}^n$, such that in $S \cap \{u + iv : |u| < \epsilon\}$, we have $\Phi = \sum_{m=0}^\infty H_m + q$, where H_m is the holomorphic extension to S of an element of $\mathbf{AK}^{-Q+j-m}$, and q is holomorphic.

In proving Theorem 1.6, we claim that we may make the extra assumption that $\mathcal{K} = (\mathcal{K}_1)^2$, where $\mathcal{K}_1 = \mathcal{O}_0(K_1)$, for some $K_1 \in \mathbf{K}^{-Q-j/2}$. Indeed, if Theorem 1.6 were known under this "extra assumption", then by Remark 2 after the proof of Theorem 4.10, Theorem 4.10 would follow in full generality in the smooth setting. (Indeed, in the notation of that remark, we would know Theorem 1.6 with K_l, Φ_{K_l} in place of K, Φ_K, provided l was an even integer, for in that case $\mathcal{O}_0(K_l) = (\mathcal{O}_0(K_{l/2}))^2$, so that the "extra assumption" would be automatically satisfied.) But Theorem 4.10 (in the smooth setting) implies that the "extra assumption" is in fact *always* satisfied.

For $s \in \mathbb{R}$ we define the Folland-Stein "Sobolev space" S_s^2 to be the space of all $f \in \mathcal{S}'(\mathbf{H}^n)$ for which $\hat{f}$ is an operator family, and for which

$$\|f\|_s^2 = c_n \int_{-\infty}^{\infty} \sum \|\hat{f}(\lambda)E_\alpha\|^2[1 + (2|\alpha| + n)|\lambda|]^s(2|\lambda|)^n d\lambda < \infty.$$

We note the following facts about S_s^2, which are analogues of familiar facts about Sobolev spaces on $\mathbb{R}^n$:
1. If $s \in \mathbf{Z}^+$, then S_s^2 is the space of $f \in L^2$ all of whose left-invariant derivatives, up to (parabolic) degree s, are in L^2.
2. $|(f|g)| \leq \|f\|_s \|g\|_{-s}$ (for $f, g \in \mathcal{S}$, for instance).
3. Let $B = \{g \in C^\infty(\mathbf{H}^n) : Dg \in L^\infty$ for any left-invariant differential operator $D\}$. Say $g \in B$. Then, for any s, multiplication by g is a bounded operator on S_s^2. (This follows from [**6**], Theorem 4.7.)
4. If $s < -Q/2$, then $\delta \in S_s^2$.
5. If $s > Q/2$, then $S_s^2 \subseteq L^\infty$.

In order to prove Theorem 1.6, we need an a priori estimate. We assume that $j > Q$; let $s = j/2$. Using our "extra assumption", we have that $\|\mathcal{L}_0^{s/2} f\|_{L^2} < C\|\mathcal{K}_1 f\|_{L^2}$, for $f \in \mathcal{S}$, since the operator of convolution with $\Psi_{-s/2} \underline{*} K_1$ is invertible on L^2. For $f \in \mathcal{S}$, we have $\|f\|_s \leq C(\|\mathcal{L}_0^{s/2} f\|_{L^2} + \|f\|_{L^2})$. Squaring this, and noting that $|\mathcal{K}_1 f\|_{L^2}^2 = (\mathcal{K} f, f)$, we see that $\|f\|_s^2 \leq C((\mathcal{K}+I)f, f)$. Using fact #2 above, we obtain the a priori estimate

$$\|f\|_s \leq C\|(\mathcal{K}+I)f\|_{-s}. \tag{34}$$

Let us then sketch how one proves Theorem 1.6 when $D = I$. In part (a) of the theorem, we have that $\mathcal{K}$ is a left-invariant differential operator L. We use the methods of Agmon [**1**]. Let φ be a smooth function on $\mathbf{H}^n$ such that $\varphi(0) = 0$ and such that $\varphi(u) = |u|$ for $|u| > 1$; since $s > Q/2$, by fact #5, it will suffice to show that for some $c > 0$, $e^{c\varphi}(L+I)^{-1}\delta = e^{c\varphi}(L+I)^{-1}e^{-c\varphi}\delta \in H^s$. Since $-s < -Q/2$, by fact #4 it will suffice to show that for some $c > 0$, there is an a priori estimate

$$\|f\|_s \leq C\|e^{c\varphi}(L+I)e^{-c\varphi} f\|_{-s}. \tag{35}$$

Let $L_c + I = e^{c\varphi}(L+I)e^{-c\varphi} = (L+I) + D_c$, say. Since $Z_m\varphi$ and $\bar{Z}_m\varphi$ are in B for any m, fact #3 and a brief calculation show that for $c < 1$, we have an a priori estimate $\|D_c f\|_{-s} \leq C_0 c\|f\|_{s-1}$, with C_0 independent of c. By (34), we see that $\|f\|_s \leq C(\|(L_c+I)f\|_{-s} + \|D_c f\|_{-s}) \leq C(\|(L_c+I)f\|_{-s} + C_0 c\|f\|_{s-1})$. Choosing c sufficiently small that $CC_0 c < 1/2$, we obtain (35).

For part (b) of the theorem, one wants to show that $p(\mathcal{K}+I)^{-1}\delta$ is bounded for any monomial $p(t,z)$ of (parabolic) degree less than j, and for this it is enough to show that $p(\mathcal{K}+I)^{-1}\delta \in S_s^2$. This is clear from (34) if $p = 1$; let us verify it, for example, for $p(t,z) = z_m$. Since $z_m\delta = 0$, we have $z_m(\mathcal{K}+I)^{-1}\delta = [z_m, (\mathcal{K}+I)^{-1}]\delta = -(\mathcal{K}+I)^{-1}[z_m, \mathcal{K}](\mathcal{K}+I)^{-1}\delta$, which, by (34), will be in S_s^2 provided we can show that $[z_m, \mathcal{K}]$ is a bounded operator from S_s^2 to S_{-s}^2. Note that $[z_m, \mathcal{K}]$ is just the operator of convolution with $z_m K$. If F is the convolution kernel of $(\mathcal{L}_0 + I)^{-s/2}$, we need to show that convolution with $G = F * (z_m K) * F$ is a bounded operator on L^2. In fact, $G \in L^1$. Here is a brief explanation as to why. Near 0, it may be shown that F looks basically like $\Psi_{-s/2}$, so the singularity of G at 0 is of the order $(-Q+s)+(k+1)+(-Q+s)+2Q = -Q+1$, so G is integrable there. At ∞, it may be shown by use of part (a) of the theorem that F decays exponentially. $z_m K$ is homogeneous of degree $k+1 = -Q-j+1 < -Q$; so $z_m K$ is also integrable at ∞; thus G is integrable there as well.

Finally, we very briefly explain the method of proof of Theorem 1.7. Say we are in the real analytic setting, and let H be as in Theorem 4.6. Then H, being analytic away from 0 and in $\mathbf{K}^{-Q+j}$, has a holomorphic extension H to a "sector" in which we have $|H(u+iv)| \leq C|u|^{-Q+j+1}$ for $|u| \geq 1$. Morally speaking, Φ, the convolution kernel of $(I+\bar{\mathcal{K}})^{-1}$, should be "no worse" than H. Thus it is reasonable to conjecture, though difficult to prove, that, at the very least:

(*) There exist $C, M > 0$ and a "sector" S_1 such that Φ has a holomorphic extension to S_1, in which we have $|\Phi(u+iv)| \leq C|u|^M$ for $|u| \geq 1$.

Once (*) is proved, it is possible to use a version of the Phragmen-Lindelöf Theorem for "sectors" like S_1, together with Theorem 1.6, in order to establish the much better bounds for Φ of Theorem 1.7, in a smaller "sector".

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