More on the Heat Equation

1 Introduction

We return now to Exercise 11.2.5. The solution to that problem was apparently

$$u(x,t)=\sum_{m=1}^{\infty} a_m e^{-km^2 t}\sin(mx)$$

if the sine series of f is

$$\sum_{m=1}^{\infty} a_m \sin(mx).$$

Although u is easily seen to satisfy the differential equation and the boundary conditions, it is not at all clear that it must satisfy the initial conditions, if f is only known to be continous.

As usual, it is more convenient to phrase the problem in terms of Fourier series instead of sine series. If g is the odd 2π-periodic extension of f, then $\hat{g}(m)=a_m/2i$ if $m>0$ and $-a_m/2i$ if $m<0$. Then

$$u(x,t)=\sum_{m=-\infty}^{\infty} \hat{g}(m)e^{-km^2 t}e^{imx}, \tag{1}$$

and we are asking whether $\lim_{t\to 0^+} u(x,t)=g(x)$ for all x. Define

$$\varphi_t(x)=\sum_{m=-\infty}^{\infty} e^{-km^2 t}e^{imx}. \tag{2}$$

Then, by arguments analogous to those we used in our study of the Poisson kernel on the circle, we have

$$u(x,t)=(\varphi_t * g)(x), \tag{3}$$

Indeed, if we put $u_t(x)=u(x,t)$, u_t and $\varphi_t * g$ have the same Fourier series.

To show that $\lim_{t\to 0^+} u(x,t)=g(x)$ for all x, it would suffice to show that $\{\varphi_t\}$ is an approximate identity on the circle. But how are we to show this? We evaluated the Poisson kernel by summing the geometric series $\sum_{k=-\infty}^{\infty}(r/s)^{|k|}e^{ik\theta}$; but there is no hope of summing the series in (2) explicitly. And if we cannot do that, how can we verify the required properties of an approximate identity?

This problem is solved, as was the case for the wave equation, by looking at the problem on the whole real line instead of on an interval. We solve the heat equation with initial condtion

$g(x)$, but now for $x \in \mathbf{R}$. We expect to get a solution which is odd 2π-periodic in x, and to be able to prove that it is the same solution $u(x,t)$ of (1). The advantage of working on the whole real line is that, as was the case for the wave equation, the formulas are simpler and easier to analyze. That is to say, although we shall get a solution which is identical to $u(x,t)$, it will be written in a form which will be much easier to deal with.

How then do we work on the real line? We need an analogue of Fourier series that will work on $\mathbf{R}$. To understand what this analogue should be, let us first identify $\mathbf{T}$ with $[-\pi,\pi)$, and find the analogue of Fourier coefficients on $[-L,L)$ for any $L>0$; this analogue could then be used for solving problems like Exercise 11.2.5 if the rod stretched from 0 to L instead of from 0 to π. Then we'll take the limit, heuristically, as $L \to \infty$ to guess the analogue of Fourier coefficients on $\mathbf{R}$. In this manner we shall encounter the *Fourier transform*.

If f is in $C^1(\mathbf{T})$, we know that if

$$\hat{f}(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-im\theta}d\theta, \tag{4}$$

then

$$f(\theta) = \sum_{m=-\infty}^{\infty} \hat{f}(m)e^{im\theta}. \tag{5}$$

We rescale to $[-L,L)$. Define a function F on $[-L,L)$ by $F(x) = f(\pi x/L)$. Also, for $m \in \mathbf{Z}$, put $\xi_m = m/2L$. Finally set $(\mathcal{F}_L F)(\xi_m) = 2L\hat{f}(m)$. Substituting $\theta = \pi x/L$ in (4) and (5), we find that

$$(\mathcal{F}_L F)(\xi_m) = \int_{-L}^{L} F(x)e^{-2\pi i x\xi_m}dx, \tag{6}$$

and

$$F(x) = \sum_{m=-\infty}^{\infty} (\mathcal{F}_L F)(\xi_m)e^{2\pi i x\xi_m}\frac{1}{2L}. \tag{7}$$

In (7) we write $\frac{1}{2L} = \Delta\xi = \xi_m - \xi_{m-1}$ for any m. This spacing between the ξ_m decreases as $L \to \infty$. Finally in (6) and (7) we change i to $-i$ (this is surely allowable, since i and $-i$ are just names for the two square roots of -1). This gives

$$(\mathcal{F}_L F)(\xi_m) = \int_{-L}^{L} F(x)e^{2\pi i x\xi_m}dx, \tag{8}$$

and

$$F(x) = \sum_{m=-\infty}^{\infty} (\mathcal{F}_L F)(\xi_m)e^{-2\pi i x\xi_m}\Delta\xi. \tag{9}$$

Then, heuristically taking the limit as $L \to \infty$, we are led to believe that for some suitable class of functions F on $\mathbf{R}$, if

$$\hat{F}(\xi) = \int_{-\infty}^{\infty} F(x)e^{2\pi i x\xi}dx, \tag{10}$$

then

$$F(x) = \int_{-\infty}^{\infty} \hat{F}(\xi)e^{-2\pi i x\xi}d\xi. \tag{11}$$

This is called the *Fourier inversion formula*, and we shall discuss it in detail in the next section.

As for the heat equation, if F is as in (9), we can, analogously to (1), set

$$u(x,t) = \sum_{m=-\infty}^{\infty} (\mathcal{F}_L F)(\xi_m) e^{-4\pi^2 k \xi_m^2 t} e^{-2\pi i x \xi_m} \Delta\xi. \tag{12}$$

Then $u(x,t)$ satisfies the heat equation for $t > 0$; additionally, we would expect that $\lim_{t\to 0^+} u(x,t) = F(x)$ for $|x| < L$. Letting $L \to \infty$, we expect to obtain

$$u(x,t) = \int_{-\infty}^{\infty} \hat{F}(\xi) e^{-4\pi^2 k \xi^2 t} e^{-2\pi i x \xi} d\xi, \tag{13}$$

which appears to satisfy the heat equation for $t > 0$. Again we would expect that $\lim_{t\to 0^+} u(x,t) = F(x)$, by (11).

Since (13) appears to be analogous to (1), the discussion following (1) (in particular (2) and (3) suggest the following. Let us define

$$\Phi_t(x) = \int_{-\infty}^{\infty} e^{-4\pi^2 k \xi^2 t} e^{-2\pi i x \xi} d\xi, \tag{14}$$

It is then reasonable to expect that, in (13),

$$u(x,t) = (\Phi_t * F)(x), \tag{15}$$

To show that $\lim_{t\to 0^+} u(x,t) = F(x)$, we would only have to show that $\{\Phi_t\}$ is an approximate identity on $\mathbf{R}$. And this can be done rather easily. In fact, although we cannot compute the sum (2), it is not very difficult to compute the integral (14). It turns out that $\Phi_t(x) = (4\pi k t)^{-n/2} e^{-x^2/4kt}$., which is indeed an approximate identity on $\mathbf{R}$.

Of course, all this must be justified. If F is 2π-periodic we will also want to verify that $u(x,t)$ in (3) coincides with the solution of the heat equation, with initial value F, obtained by using Fourier series, namely $\sum_{m=-\infty}^{\infty} \hat{F}(m) e^{-km^2 t} e^{imx}$.

We first discuss the question of justification. To begin, we ask: can we describe a large, simple class of F for which (11) holds?

In the analogous situation on $\mathbf{T}$, we know that (5) holds for all f in the familiar space $C^\infty(\mathbf{T})$. In fact, let us make the following definition:

Definition 1.1 *Let A be a sequence in $l^\infty(\mathbf{Z})$, $l^\infty(\mathbf{N})$, or $l^\infty(\mathbf{Z}^+)$. (Here, and in what follows, $\mathbf{Z}^+ = \{0, 1, 2, \ldots\}$.) We say that A is* rapidly decreasing *if for every nonnegative integer k there exists $C_k \geq 0$ such that $(|n|+1)^k |A(n)| \leq C_k$ for all n.*

More informally, then, a sequence is rapidly decreasing if it decays faster than any negative power of n. We then have the following *isomorphism*:

Exercise 1.2 *Let* $\hat{}: L^1(\mathbf{T}) \to l^\infty(\mathbf{Z})$ *be the usual map which takes a function to its sequence of Fourier coefficients.*

Define $s(\mathbf{Z})$ *to be the following subspace of* $l^\infty(\mathbf{Z})$*:*

$$s(\mathbf{Z}) = \{\textit{sequences } A : A \textit{ is rapidly decreasing}\}.$$

Show that $\hat{}: C^\infty(\mathbf{T}) \to s(\mathbf{Z})$*, and that this map is a vector space isomorphism, with inverse* $\check{}$*, where* $\check{A}(t) = \sum_{n=-\infty}^{\infty} A_n e^{int}$*.*
(Hint: you can use Lemma 11.2.11.)

The situation on $\mathbf{R}$ is somewhat different, for two reasons. First, for (10) to make sense, it is surely not enough to require only that $F \in C^\infty(\mathbf{R})$; we need some decay on F at ∞ as well, so that the integral will converge. Moreover there is a *symmetry* between (10) and (11), which would lead us to look for a space $\mathcal{S}$ such that $\hat{}$ is an isomorphism between $\mathcal{S}$ and *itself*.

With all these clues, we are led to make the following definitions. It will not be much harder to work on $\mathbf{R}^n$ (for general n) than on $\mathbf{R}$, so we do so.

Definition 1.3 *Suppose that* $F : \mathbf{R}^n \to \mathbf{C}$*. We say that* F is rapidly decreasing *if for every nonnegative integer* k *there exists* $C_k \geq 0$ *such that* $(1 + |x|)^k |F(x)| \leq C_k$ *for all* x*.*

More informally, then, a function is rapidly decreasing if it decays faster than any negative power of $|x|$.

Note that $F : \mathbf{R}^n \to \mathbf{C}$ is rapidly decreasing if and only if $x^\alpha F(x)$ is bounded for every n-multiindex α. (Here an n-mulitindex α is defined to be an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$, with each α_j being a nonnegative integer, and by definition $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.) Indeed, this follows at once from the following simple inequality:

Proposition 1.4 *For any* $N \in \mathbf{Z}^+$*, there exists* $C_N > 0$ *such that for all* $t \in \mathbf{R}^n$*,*

$$(1 + |t|)^N \leq C_N \sum_{|\alpha| \leq N} |t^\alpha| \tag{16}$$

Proof By the binomial theorem, for some $c > 0$, $(1 + y)^N \leq c \sum_{j=0}^N y^j$ (whenever $y \geq 0$). If we let $y = |t|$, note $y^j \leq (\sum |t_k|)^j$, and expand the latter expression using the multinomial theorem, we find (16), as desired.

Definition 1.5 *The* Schwartz space *on* $\mathbf{R}^n$*,* $\mathcal{S} = \mathcal{S}(\mathbf{R}^n)$*, is the space of all* $F \in C^\infty(\mathbf{R}^n)$ *such that* $\partial^\alpha F$ *is rapidly decreasing for all* n*-multiindices* α*. (Here* $\partial = (\partial_1, \ldots, \partial_n)$ *where* $\partial_j = \partial/\partial x_j$ *for* $j = 1, \ldots, n$*, and* $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$*.)*

Familiar examples of elements of $\mathcal{S}$ are the functions $e^{-a|x|^2}$, for any $a > 0$.

We will show in the next section that $\hat{}: \mathcal{S}(\mathbf{R}^1) \to \mathcal{S}(\mathbf{R}^1)$, and that this map is an isomorphism. We'll also generalize these facts to $\mathbf{R}^n$. In this section we will study $\mathcal{S}$. We will frequently need Leibniz's rule on $\mathbf{R}^n$, which we describe now.

If β, γ are n-multiindices, we set

$$\beta + \gamma = (\beta_1 + \gamma_1, \ldots, \beta_n + \gamma_n),$$

and

$$\beta! = \beta_1! \ldots \beta_n!.$$

Exercise 1.6 *(Leibniz's rule) It is easy to show the following by induction: let α be an n-multiindex. Then there is a set of constants $\{c_{\beta,\gamma}\}$, where here $\{\beta, \gamma\}$ runs over the set of all ordered pairs of n-multiindices with $\beta + \gamma = \alpha$, such that whenever f, g are smooth on any open set U, we have* Leibniz's rule*:*

$$\partial^\alpha(fg) = \sum_{\beta+\gamma=\alpha} c_{\beta,\gamma} \partial^\beta f \partial^\gamma g. \tag{17}$$

Show that $c_{\beta,\gamma} = \alpha!/[\beta!\gamma!]$. (Hint: since the $c_{\beta,\gamma}$ do not depend on U, we may assume that $0 \in U$. Evaluate the constants by letting f, g be suitable monomials.) In particular, $c_{\alpha,0} = c_{0,\alpha} = 1$.

Returning now to the Schwartz space, note that, evidently,

$$C_c^\infty(\mathbf{R}^n) \subseteq \mathcal{S}(\mathbf{R}^n) \subseteq C^\infty(\mathbf{R}^n). \tag{18}$$

Also, evidently, a function $f \in C^\infty(\mathbf{R}^n)$ is in $\mathcal{S}$ if and only if $x^\alpha \partial^\beta f$ is a bounded function, for any n-multiindices α, β.

In this section, we let $L^1 = L^1(\mathbf{R}^n)$; if $f \in L^1$, we will denote its L^1 norm by $\|f\|_{L^1}$.

Frequently it will be useful for us to consider functions that are not quite as good as Schwartz space functions. For $N \in \mathbf{Z}^+$, we define

$$\mathcal{Z}_N(\mathbf{R}^n) = \{f \in C^N(\mathbf{R}^n) : \|f\|_{\mathcal{Z}_N} = \sum_{|\alpha|+|\beta| \leq N} \|x^\alpha \partial^\beta f\| < \infty\}. \tag{19}$$

Evidently

$$\cap_N \mathcal{Z}_N = \mathcal{S}. \tag{20}$$

We surely have that $\mathcal{Z}_N$ is a normed vector space, with norm $\| \ \|_{\mathcal{Z}_N}$. The rough analogy "$C^N(\mathbf{T})$ is to $C^\infty(\mathbf{T})$ as $\mathcal{Z}_N(\mathbf{R}^n)$ is to $\mathcal{S}(\mathbf{R}^n)$" is an appropriate way to think of $\mathcal{Z}_N(\mathbf{R}^n)$.

We note the following simple facts:

Proposition 1.7 *Say $N \geq 1$.*
(a) For $1 \leq j \leq n$, $\partial_j : \mathcal{Z}_N \to \mathcal{Z}_{N-1}$.
(b) For $1 \leq j \leq n$, the map M_j of multiplication by x_j, defined by $(M_j f)(x) = x_j f(x)$, maps $\mathcal{Z}_N$ to $\mathcal{Z}_{N-1}$.

Proof (a) is evident from the definition of $\mathcal{Z}_N$ and $\mathcal{Z}_{N-1}$. (b) is an easy consequence of Leibniz's rule. This completes the proof.

Now we want to discuss a number of frequently used properties of functions in $\mathcal{Z}_N$ and in $\mathcal{S}$. The following simple propositions will frequently be of use.

Proposition 1.8 *For any $N \in \mathbf{Z}^+$, there exists C_N such that for all $f \in \mathcal{Z}_N$, we have*

$$\|(1+|x|)^N f\| \le C_N \|f\|_{\mathcal{Z}_N}. \tag{21}$$

Proof This is immediate from Proposition 1.4.

Proposition 1.9 *Say $1 \le p \le \infty$, and $k \in \mathbf{Z}^+$. Then:*
(a) $(1+|x|)^k f(x) \in L^p$ for all $f \in \mathcal{Z}_{n+k+1}$.
(b) $\mathcal{Z}_{n+1} \subseteq L^p$.
(c) If in addition $p \neq \infty$, then $\mathcal{S}$ is dense in L^p.

Proof The case $p = \infty$ is evident from Proposition 1.8, so say $p < \infty$.

By Proposition 1.8 with $N = n+k+1$, there exists $A > 0$ such that for all $f \in \mathcal{Z}_{n+k+1}$, $(1+|x|)^k |f(x)| \le A\|f\|_{\mathcal{Z}_{n+k+1}}(1+|x|)^{-(n+1)}$ for all x. Thus (a) follows from the fact that if $h(x) = (1+|x|)^{-(n+1)}$, then $h \in L^p$. (This in turn follows from the p-test in $\mathbf{R}^n$.) (b) also follows if we take $k = 0$ in (a).

For (c), it suffices to show that $C_c^\infty(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$. Let V denote the closure of $C_c^\infty(\mathbf{R}^n)$ in $L^p(\mathbf{R}^n)$. By Theorem 9.3.1, it suffices to show that $C_c(\mathbf{R}^n) \subseteq V$. Say then that $f \in C_c(\mathbf{R}^n)$, and that the support of f is contained in a compact set K. By Lemma 11.4.10, we may select a compact set $K_0 \supseteq K$ and a sequence φ_m of smooth functions with support contained in K_0, such that $\varphi_m \to f$ uniformly on $\mathbf{R}^n$. Then $\varphi_m \to f$ in L^p as well, and so (d) follows. This completes the proof.

Proposition 1.10 *Let $\mathcal{P}_0$ denote the set of functions $G \in C^\infty(\mathbf{R}^n)$ with the property that $\partial^\alpha G$ is bounded for every multiindex α. Then, if $f \in \mathcal{S}$, then $fG \in \mathcal{S}$.*

Proof This is an easy consequence of Leibniz's rule.

Proposition 1.11 *If $F \in C^\infty(\mathbf{R}^n \setminus \{0\})$, and if $k \in \mathbf{R}$, we say that F is* homogeneous of degree *k if for every $r > 0$,*

$$F(rx) = r^k F(x) \tag{22}$$

for all $x \neq 0$.
Say now that $F \in C^\infty(\mathbf{R}^n \setminus \{0\})$ is homogeneous of degree k. Then:
(a) For any multiindex α, $\partial^\alpha F$ is homogeneous of degree $k - |\alpha|$.
(b) Say $k < 0$. Then for any $r > 0$, F is bounded for $|x| > r$.

Proof (a) is immediate from (22) and the chain rule.
For (b), we define a function Ω on the unit sphere in $\mathbf{R}^n$ by $\Omega(y) = F(y)$. Since the unit sphere is compact, Ω is bounded. Finally $F(x) = F(x/|x|)|x|^k = \Omega(x/|x|)|x|^k$ is bounded for $|x| > r$. This completes the proof.

An example of a function F, which is homogeneous of degree k, is $F(x) = |x|^k$.

2 The Fourier Transform

If $f, g \in L^1(\mathbf{R}^n)$, we define $\hat{f}, \check{g} \in L^\infty(\mathbf{R}^n)$ by

$$\hat{f}(\xi) = \int e^{2\pi i x\cdot\xi} f(x)dx, \tag{23}$$

and

$$\check{g}(x) = \int e^{-2\pi i x\cdot\xi} g(\xi)d\xi. \tag{24}$$

(We are using these conventions: whenever we write $\int$ without specifying the set over which we are integrating, we mean $\int_{\mathbf{R}^n}$. Also, whenever we write L^p without specifying the underlying measure space, we mean $L^p(\mathbf{R}^n)$.)

$\hat{f}$ is called the *Fourier transform* of f. Clearly,

$$\|\hat{f}\|_\infty \leq \|f\|_1. \tag{25}$$

Recall also from Exercise 5.7.11 that, whenever $f \in L^1$, $\hat{f}$ is continuous. Similarly $\check{f}$ is continuous. In general, all results in these notes about $\hat{}$ have evident analogues for $\check{}$. Those analogues can be proved by "changing i to $-i$ in the proofs", or alternatively can be easily deduced from the results for $\hat{}$. Usually we will not bother to state these analogues explicitly. Let us make this convention: whenever we refer to any result about $\hat{}$ in the text, we are implicitly referring *either* to that result *or* to its analogue for $\check{}$.

We shall soon show that $\hat{} : \mathcal{S} \to \mathcal{S}$, with inverse $\check{}$. First, however, we prove the following lemma.

Lemma 2.1 *Say $1 \leq j \leq n$.*
(a) Suppose, that f and $x_j f$ are in L^1. (Note, in particular, that this is true if $f \in \mathcal{Z}_{n+2}$, by Proposition 1.7 (b) and Proposition 1.9 (b)). Then $\hat{f} \in C^1$. Further, for $1 \leq j \leq n$, [1]

$$\partial_j \hat{f} = (2\pi i x_j f)\hat{}. \tag{26}$$

(b) Say $f \in \mathcal{Z}_{n+2}$. Then

$$\widehat{\partial_j f}(\xi) = (-2\pi i \xi_j)\hat{f}(\xi) \tag{27}$$

Proof (a) follows at once from Exercise 5.7.12.

For (b), we may assume without loss of generality that $j = 1$. We first make the following observations. Say $\xi \in \mathbf{R}^n$ and $x_2, \ldots, x_n \in \mathbf{R}$ are fixed. By Proposition 1.8, for some $C_1 > 0$,

$$|f(x)| \leq C_1(1 + |x|)^{-n-2} \leq C_1(1 + |x_1|)^{-n-2}, \tag{28}$$

[1] We are abusing notation in (26). We ought to define, for $1 \leq j \leq n$, an operator M_j mapping measurable functions to measurable functions, by $(M_j f)(x) = x_j f(x)$. Then in place of (26), we would correctly write $\partial_j \hat{f} = (2\pi i M_j f)\hat{}$. Having said this, however, our intended meaning in writing (26) is unambiguous, and we shall continue to abuse notation in this manner, since it is more natural and suggestive to do so.

while by Proposition 1.8, and Proposition 1.7 (a), for some $C_2 > 0$,

$$|\partial_1 f(x)| \leq C_2(1+|x|)^{-n-1} \leq C_2(1+|x_1|)^{-n-1}. \tag{29}$$

These inequalities show that it is valid to integrate by parts to find

$$\int_{-\infty}^{\infty} e^{2\pi i x\cdot\xi}(\partial_1 f)(x)dx_1 = -(2\pi i\xi_1)\int_{-\infty}^{\infty} e^{2\pi i x\cdot\xi} f(x)dx_1. \tag{30}$$

(b) now follows, since by Fubini's theorem, when computing $\widehat{\partial_1 f}$, we may integrate with respect to x_1 first. This completes the proof.

We may in fact generalize Lemma 2.1 as follows:

Proposition 2.2 *Say $k \in \mathbf{N}$, $f \in L^1$, and that $x^\alpha f \in L^1$ for all α with $|\alpha| \leq k$. Then $\hat{f}$ is a C^k function.*

Morover, say that $|\beta| \leq k$. Then

$$\partial^\beta \hat{f} = (2\pi i)^{|\beta|}(x^\beta f)\hat{}. \tag{31}$$

(Similarly for $\check{}$.)

In particular, this is true for all $f \in \mathcal{Z}_{k+n+1}$.

Proof To see this, one need only differentiate (23) repeatedly under the integral sign (using Exercise 5.7.12). For the last statement, just note that if $f \in \mathcal{Z}_{k+n+1}$, then $x^\alpha f \in \mathcal{Z}_{n+1} \subseteq L^1$ whenever $|\alpha| \leq k$. This completes the proof.

Corollary 2.3 *For any $N \geq 0$,*

$$\hat{} : \mathcal{Z}_{N+n+1}(\mathbf{R}^n) \to \mathcal{Z}_N(\mathbf{R}^n).$$

(Note $\mathcal{Z}_{N+n+1}(\mathbf{R}^n) \subseteq L^1(\mathbf{R}^n)$, since $N+n+1 \geq n+1$.) In fact, if $f \in \mathcal{Z}_{N+n+1}$, and $|\alpha|+|\beta| \leq N$, then

$$\xi^\alpha \partial^\beta \hat{f} = (-1)^{|\alpha|}(2\pi i)^{|\beta|-|\alpha|}[\partial^\alpha x^\beta f]\hat{} \tag{32}$$

(Note that, by Proposition 1.7, we have here that $\partial^\alpha x^\beta f \in \mathcal{Z}_{n+1} \subseteq L^1$, so that $[\partial^\alpha x^\beta f]\hat{}$ is a bounded continuous function.)

Consequently, $\hat{} : \mathcal{S} \to \mathcal{S}$.

Proof Say $f \in \mathcal{Z}_{N+n+1}$. By Proposition 2.2, $\hat{f} \in C^N$, and (32) holds if $\alpha = 0$.

In proving (32) we may therefore assume $\alpha \neq 0$. Say $|\alpha| = M$. We may write $\partial^\alpha = \partial_{j_1} \ldots \partial_{j_M}$ for certain numbers $j_1, \ldots, j_M$ with $1 \leq j_1, \ldots, j_M \leq n$. Here $M + |\beta| = |\alpha| + |\beta| \leq N$. Thus

(*) Whenever $2 \leq l \leq M$, $\partial_{j_l} \ldots \partial_{j_M} x^\beta f \in \mathcal{Z}_{n+2}$.

Indeed, the number of ∂'s and x's appearing here is fewer than $M + |\beta| \leq N$.

(32) is now immediate, from iterating Lemma 2.1 (b). In (32), $\xi^\alpha \partial^\beta \hat{f}$ is bounded, so $\hat{f} \in \mathcal{Z}_N$, as desired.

Using Corollary 2.3, we can easily prove the analogue of the Riemann-Lebesgue lemma for the Fourier transform:

Proposition 2.4 Riemann-Lebesgue Lemma *$\hat{}$: $L^1(\mathbf{R}^n) \to C_0(\mathbf{R}^n)$ continuously. (Here $C_0(\mathbf{R}^n)$ is the space of continuous functions on $\mathbf{R}^n$ which vanish at ∞.)*

Proof We already know that if $f \in L^1$, then $\hat{f}$ is bounded and continuous; further, we have (25). We need to show that $\hat{f}$ vanishes at infinity.

Say $\epsilon > 0$. By Proposition 1.9 (c), we may select $g \in \mathcal{S}$ with $\|f - g\|_1 < \epsilon/2$. Also, since $\hat{g} \in \mathcal{S}$ (by Corollary 2.3), we may select $N > 0$ such that $|\hat{g}(\xi)| < \epsilon/2$ whenever $|\xi| > N$. Thus, if $|\xi| > N$,

$$\begin{aligned} |\hat{f}(\xi)| &\leq |(f-g)\hat{}(\xi)| + |\hat{g}(\xi)| \\ &\leq \|f-g\|_1 + \epsilon/2 \\ &< \epsilon \end{aligned}$$

as desired. (In the second line we have used (25).) This completes the proof.

3 The Fourier Inversion Theorem for $\mathcal{S}$

We are almost ready to prove that $\hat{}$ and $\check{}$ are inverse maps on $\mathcal{S}$. As in our proofs of Theorems 10.2.1 and 10.2.4, it will be important to have one explicit example.

Proposition 3.1 *On $\mathbf{R}^n$, let $F(x) = e^{-\pi|x|^2}$. Then $\hat{F} = F$. In particular, $(\hat{F})\check{} = F$.*

Proof Surely $F \in \mathcal{S}$. It suffices to prove the proposition when $n = 1$, since $\int_{\mathbf{R}^n} e^{2\pi i x\cdot\xi} e^{-\pi|x|^2} dx = \prod_{j=1}^n \int_{-\infty}^{\infty} e^{2\pi i x_j\cdot\xi_j} e^{-\pi x_j^2} dx_j$.

So say $n = 1$. We define an operator $A_0 : \mathcal{S} \to \mathcal{S}$ by

$$A_0 f = (2\pi x + \frac{d}{dx})f. \tag{33}$$

This operator has the following two evident properties:

(i) $A_0 f = 0 \Leftrightarrow f = cF$ for some constant c;
(ii) For all $f \in \mathcal{S}$, $\widehat{A_0 f} = -iA_0\hat{f}$.

(Here (ii) follows from Lemma 2.1.) Since $A_0 F = 0$, we find $A_0\hat{F} = 0$, so $\hat{F} = CF$ for some constant C. In fact $C = \hat{F}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$, by Exercise 8.3.5. This completes the proof. (The operator $A = \frac{1}{2\sqrt{\pi}}A_0$ is the *annihilation operator* of mathematical physics.)

We shall also need some important facts about smooth functions that vanish at $0 \in \mathbf{R}^n$. To begin, we give an n-dimensional version of the Fundamental Theorem of Calculus:

Proposition 3.2 *Suppose $a, y \in \mathbf{R}^n$, and that F is a C^1 real-valued function on an open neighborhood of the line segment joining a to $a + y$. Then*

$$F(a+y) = F(a) + \sum_{j=1}^n y_j \int_0^1 \frac{\partial F}{\partial x_j}(a+ty)dt. \tag{34}$$

Proof There exists an open interval I containing $[0,1]$, such that $g(t) := F(a+ty)$ exists for all $t \in I$, and such that g is C^1 on I. We have

$$g'(t) = \sum_{j=1}^{n} y_j \frac{\partial F}{\partial x_j}(a+ty) \tag{35}$$

Apply the 1-dimensional Fundamental Theorem of Calculus to g, in the form

$$g(1) = g(0) + \int_0^1 g'(t)dt. \tag{36}$$

Combining (35) and (36), we find (34), as desired.

In particular, we have the following very useful lemma:

Lemma 3.3 *Suppose $f \in C^\infty(\mathbf{R}^n)$.*
(a) Then

$$f(0) = 0 \tag{37}$$

if and only if there is a set of functions $\{F_1, \dots, F_n\} \subseteq C^\infty(\mathbf{R}^n)$ such that

$$f(x) = \sum_{j=1}^{n} x_j F_j(x) \tag{38}$$

for all x.
(b) If in fact $f \in C_c^\infty(\mathbf{R}^n)$, and (37) holds, then we may choose $\{F_1, \dots, F_n\} \subseteq C_c^\infty(\mathbf{R}^n)$ such that (38) holds.

Proof (a) It is evident from Leibniz's rule that (38) implies (37). Conversely, if (37) holds, we have (38), with

$$F_j(x) = \int_0^1 \frac{\partial f}{\partial x_j}(tx)dt. \tag{39}$$

This is a smooth funcion of x since we may differentiate under the integral sign. This proves (a).

For (b), choose $\{F_j\}$ as in (38), and also choose $\zeta \in C_c^\infty(\mathbf{R}^n)$ such that $\zeta = 1$ on supp$f$. If we multiply (38) through by $\zeta(x)$, we see that for all x,

$$f(x) = \zeta(x)f(x) = \sum_{j=1}^{n} x_j (\zeta F_j)(x).$$

That is, (38) still holds if the functions $\{F_j\}$ are replaced by the compactly supported functions $\{\zeta F_j\}$. This completes the proof.

For $\mathcal{S}(\mathbf{R}^n)$, we have the following analogue of Lemma 3.3.

Lemma 3.4 *Suppose $f \in \mathcal{S}$, and $f(0) = 0$. Then there are functions $f_1, \dots, f_n \in \mathcal{S}$ such that $f = \sum_{j=1}^n x_j f_j$.*

Proof Select $\zeta \in C_c^\infty(\mathbf{R}^n)$ such that $\zeta = 1$ in a neighborhood of 0. By Lemma 3.3 there are functions $g_1, \ldots, g_n \in C_c^\infty(\mathbf{R}^n)$ such that $\zeta f = \sum_{j=1}^n x_j g_j$. If we can find functions $h_1, \ldots, h_n \in \mathcal{S}$ such that $(1-\zeta)f = \sum_{j=1}^n x_j h_j$, then we can set $f_j = g_j + h_j$. In short, in proving the lemma, we may assume f vanishes in a bounded neighborhood of 0 (otherwise, replace f by $(1-\zeta)f$).

But if f vanishes in a bounded neighborhood U of 0, to prove the lemma it suffices to set $f_j(x) = x_j f(x)/|x|^2$. Indeed clearly $f_j \in C^\infty(\mathbf{R}^n)$, and $f = \sum_{j=1}^n x_j f_j$. Moreover, let us choose $\zeta' \in C_c^\infty(U)$ with $\zeta' \equiv 1$ in a neighborhood of 0. Then $f_j = gG$, where $g(x) = x_j f(x)$, and $G(x) = (1-\zeta')(x)/|x|^2$. Surely $g \in \mathcal{S}$, so to prove that $f_j \in \mathcal{S}$, we need only show, by Proposition 1.10, that $G \in \mathcal{P}_0$. To see this we need only note that G is smooth and that, outside U, $G(x)$ agrees with the function $1/|x|^2$, which is homogeneous of degree -2. Thus, by Proposition 1.11, for any multiindex α, $\partial^\alpha G$ agrees, outside U, with a function G_α which is homogeneous of degree $-2-|\alpha|$, and is hence bounded outside U. This completes the proof.

With this preparation, we are now in a position to prove the Fourier Inversion Theorem for $\mathcal{S}$, by means of arguments quite analogous to those we used to prove Theorem 10.2.1.

Theorem 3.5 Fourier Inversion Theorem for $\mathcal{S}$ *The map* $\hat{}: \mathcal{S} \to \mathcal{S}$ *is a bijection, with inverse* $\check{}$.

Proof It is enough to show that:

(*) For all $f \in \mathcal{S}$, and all $x \in \mathbf{R}^n$, $(\hat{f})\check{}(x) = f(x)$.

For then, by taking complex conjugates of both sides of this equation, we will know that $(\check{\overline{f}})\hat{} = \overline{f}$ for all $\overline{f} \in \mathcal{S}$ as well.

We first make several reductions.

1. We claim that in proving (*), we may assume $x = 0$. That is, we claim that we need only show:

(**) For all $f \in \mathcal{S}$, $(\hat{f})\check{}(0) = f(0)$.

For say (**) were known, and let us prove (*). The function $g(t) = f(t+x)$ is in $\mathcal{S}$. So by (**) we would know

$$g(0) = \int \hat{g}(\xi) d\xi. \tag{40}$$

But

$$\begin{aligned} \hat{g}(\xi) &= \int f(t+x) e^{2\pi i t\cdot\xi} dt \\ &= \int f(t) e^{2\pi i (t-x)\cdot\xi} dt \\ &= [\int f(t) e^{2\pi i t\cdot\xi} dt] e^{-2\pi i x\cdot\xi} \\ &= \hat{f}(\xi) e^{-2\pi i x\cdot\xi} \end{aligned}$$

(We used the change of variable $t \to t - x$.) So (40) is the same as (*).

2. Let $F(x) = e^{-\pi|x|^2}$. If $f = cF$ (any constant) then f satisfies (**), by Proposition 3.1. Also. if two functions satisfy (**), so does their sum.

3. We claim that we need only show:

(***) Suppose $f \in \mathcal{S}$, and $f(0) = 0$. Then $\int \hat{f}(\xi) d\xi = 0$.

Indeed, say that (***) were known, and let us prove (**). Say $f(0) = K$; then $f = [f - KF] + KF$. (**) holds for $h = KF$ and $g(x) = f - KF$, since $g(0) = 0$; so it holds for f too.

4. Finally we prove (***). To see this, use Lemma 3.4 to write $f = \sum_{j=1}^n x_j f_j$, so that $\hat{f} = (2\pi i)^{-1} \sum_{j=1}^n \partial_j \hat{f}_j$. Since all $\hat{f}_j \in \mathcal{S}$, (***) is evident. This completes the proof.

4 Plancherel's Theorem

In this section we will prove *Plancherel's Theorem*, which is an analogue for the Fourier transform of Parseval's Theorem.

We first note the following very useful elementary fact:

Proposition 4.1 *If $f, g \in L^1$, then*

$$\int f\hat{g} = \int \hat{f} g. \tag{41}$$

Proof We have

$$\int f\hat{g} = \int f(x)[\int e^{2\pi i x\cdot\xi} g(\xi) d\xi] dx = \int g(\xi)[\int e^{2\pi i x\cdot\xi} f(x) dx] d\xi = \int g\hat{f},$$

as desired.

Proposition 4.2 *If $f, G \in L^1$, then*

$$\int f\overline{\check{G}} = \int \hat{f}\overline{G}. \tag{42}$$

Proof This is immediate from Proposition 4.1, if in that Proposition we take $g = \overline{G}$, and note that, by (23) and (24), $\hat{\overline{G}} = \overline{\check{G}}$.

We then have the following analogue of Parseval's theorem:

Theorem 4.3 Plancherel's Theorem *The map* $\hat{}: \mathcal{S} \to \mathcal{S}$ *is an isometric isomorphism, if we consider* $\mathcal{S}$ *as a subspace of* L^2*. Since* $\mathcal{S}$ *is dense in* L^2*, we may extend* $\hat{}$ *uniquely to be an isometric isomorphism of* L^2 *with itself. This extension agrees with* $\hat{}$ *on* $L^2 \cap L^1$*, so it is consistent to call this extension* $\hat{}$ *as well. Accordingly, if* $f \in L^2$*, we have*

$$\|f\|_2^2 = \|\hat{f}\|_2^2. \tag{43}$$

Proof Taking $G = \hat{f}$ in (42), we see that (43) holds for all $f \in \mathcal{S}$. Thus $\hat{}: \mathcal{S} \to \mathcal{S}$ is an isometric isomorphism if $\mathcal{S}$ is topologized as a subspace of L^2. By Lemma 10.3.5 (b), $\hat{}$ extends uniquely to an isometric isomorphism, which we shall call $\mathcal{F}$, from L^2 onto L^2. Thus $\|\mathcal{F}f\|_2^2 = \|f\|_2^2$ for all $f \in L^2$. Polarizing this identity, we see that for all $f, g \in L^2$,

$$(\mathcal{F}f, \mathcal{F}g) = (f, g). \tag{44}$$

All that remains to be shown is that $\mathcal{F}f = \hat{f}$ for all $f \in L^1 \cap L^2$. For this, say $\phi \in \mathcal{S}$, and in (44), put $g = \check{\phi}$. Then $g \in \mathcal{S}$, so that $\mathcal{F}(g) = \hat{g} = \phi$. We see that for any $f \in L^2$,

$$(\mathcal{F}f, \phi) = (f, \check{\phi}) = (\hat{f}, \phi).$$

Thus $\mathcal{F}f - \hat{f}$ is orthogonal, in L^2, to $\mathcal{S}$, which is dense in L^2. Accordingly $\mathcal{F}f - \hat{f} = 0$, as desired.

5 The Fourier Transform and Convolution

In order to carry out the plan indicated in the Introduction, and in particular to obtain (15) from (13), we need to discuss the relation between the Fourier transform and convolution on $\mathbf{R}^n$.

If f, g are measurable functions on $\mathbf{R}^n$, we have, in (11.79), defined their convolution $f * g$ by the formula

$$f * g(x) = \int_{\mathbf{R}^n} f(y)g(x-y)dy \tag{45}$$

whenever this makes sense. Evidently, to say that $f * g(x)$ "exists" or "makes sense" is to say that the function h defined by

$$h(y) = f(y)g(x-y) \tag{46}$$

is integrable on $\mathbf{R}^n$. In the next proposition, we summarize some basic properties of convolution of L^1 functions.

Proposition 5.1 *Write* $L^1 = L^1(\mathbf{R}^n)$*. Say* $f, g \in L^1$*. Then:*
(a) $f * g(x)$ *exists for a.e.* x*, and* $f * g \in L^1$*; further* $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$*, and* $\int f * g(x)dx = \int f(x)dx \int g(x)dx$ *(all integrals over* $\mathbf{R}^n$*);*
(b) $f * g = g * f$ *(a.e.).*

Proof The proof of (a) is completely analogous to the proof of Proposition 11.2.9, its analogue for the circle $\mathbf{T}$; we leave the details to the reader.

We prove (b). For a.e. x, we know that the function h defined by (46) is in L^1. For such x, it is then permissible to make the change of variables $y' = x - y$ in (45), to see that $f * g(x) = g * f(x)$; this proves (b).

Proposition 5.2 *Suppose that $f, g \in L^1$. Then*

$$\widehat{f * g} = \hat{f}\hat{g}, \tag{47}$$

and

$$(f * g)\check{} = \check{f}\check{g}. \tag{48}$$

As a consequence, if $F, G \in \mathcal{S}$, then

$$\hat{FG} = \hat{F} * \hat{G}. \tag{49}$$

Proof As usual we write $e_\xi(x) = e^{2\pi i x \cdot \xi}$. Just as in the proof of Proposition 11.2.10, to prove (48) one observes that if $x, y \in \mathbf{R}^n$ then $e_\xi(x) = e_\xi(x-y)e_\xi(y)$. From this it is easy to see that

$$\widehat{f * g}(\xi) = \int e_\xi(x)(f * g)(x)dx = \int [(e_\xi f) * (e_\xi g)](x)dx = \hat{f}(\xi)\hat{g}(\xi),$$

by Proposition 5.1 (a). This proves (47). The proof of (48) is similar (use $e_{-\xi}$ in place of e_ξ). (49) is an immediate consequence of (48) and the inversion theorem. This completes the proof.

We have now seen many analogues between Fourier series on $\mathbf{T}$ and the Fourier transform on Euclidean space $\mathbf{R}^n$. There is, however, one great advantage of working on Euclidean space, which is the presence of *dilations* (the maps $x \to rx$ for $r > 0$). Dilations induce the following important maps on functions:

Definition 5.3 *If f is a function on $\mathbf{R}^n$, and $r > 0$, we let $\sigma_r f$ and $\sigma^r f$ be the functions given by*

$$\sigma_r f(x) = f(rx), \;\; \sigma^r f(x) = r^{-n} f(\frac{x}{r}).$$

We then have the following important fact, which has no analogue for Fourier series.

Proposition 5.4 *If $u \in L^1$ and $r > 0$, then $(\sigma_r f)\hat{} = \sigma^r \hat{f}$.*

Proof This follows from the definition of the Fourier transform and from a simple change of variable of integration.

In particular, we find:

Proposition 5.5 *Say $c > 0$, and let $f(x) = e^{-c\pi|x|^2}$. Then $\hat{f}(\xi) = c^{-n/2}e^{-\pi|\xi|^2/c}$.*

Proof If $F(x) = e^{-\pi|x|^2}$, then by Proposition 3.1, $\hat{F} = F$. But $f = \sigma_r F$, where $r = \sqrt{c}$, so the Proposition follows from Proposition 5.4.

6 The Heat Equation on $\mathbf{R}^n$

Let us now attempt to carry out the plan in the Introduction. Let us in fact work on $\mathbf{R}^n$. We shall find a solution $u(x,t)$ of the following problem:

Heat Equation Problem on $\mathbf{R}^n$ with initial data F Suppose that F is a bounded continuous function on $\mathbf{R}^n$. Find a function $u(x,t)$ which is continuous for $(x,t) \in \mathbf{R}^n \times [0,\infty)$, and C^2 for $(x,t) \in \mathbf{R}^n \times (0,\infty)$, satisfying

$$u_t = \Delta u \tag{50}$$

for $x \in \mathbf{R}^n$, $t > 0$, with initial conditions

$$u(x,0) = F(x). \tag{51}$$

(In (50), $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$.)

Of course we expect that, if $n = 1$, a solution should be given by (13) or (15).

As usual, one begins by formally seeking out a solution, then by justifying it.

Proceeding formally, let us let $U(\xi,t)$ denote the Fourier transform of u in the x variable (holding t fixed); thus

$$U(\xi,t) = \int e^{2\pi i x\cdot\xi} u(x,t)dx. \tag{52}$$

In (52), we can differentiate under the integral sign to see that, at least formally, U_t is the Fourier transform of u_t in the x variable. Taking the Fourier transform of (50) and of (51) in the x variable, and using (32), we find formally that, if $U_t = \partial U/\partial t$, then

$$U_t = -4\pi^2|\xi|^2 U \tag{53}$$

for $\xi \in \mathbf{R}^n$, $t > 0$, with initial conditions

$$U(\xi,0) = \hat{F}(\xi). \tag{54}$$

For fixed $\xi \in \mathbf{R}^n$, this is just a simple ODE in t, and a solution is $U(\xi,t) = e^{-4\pi^2|\xi|^2 t}\hat{F}(\xi)$. Taking the inverse Fourier transform in ξ, we do find that, (as in (13) and (15) if $n = 1$),

$$u(x,t) = \int e^{-2\pi i x\cdot\xi}\hat{F}(\xi)e^{-4\pi^2 k|\xi|^2 t}d\xi, \tag{55}$$

whence, if $F \in \mathcal{S}$, by Proposition 5.2,

$$u(x,t) = (\Phi_t * F)(x), \tag{56}$$

where

$$\Phi_t \text{ is the inverse Fourier transform, in } \xi, \text{ of } e^{-4\pi^2 k|\xi|^2 t}. \tag{57}$$

By Proposition 5.5 with $c = \sqrt{4\pi kt}$, we see that

$$\Phi_t(x) = (4\pi kt)^{-n/2}e^{-|x|^2/4kt}. \tag{58}$$

We now need to justify that u, as in (56), is indeed a solution of the problem. In fact, we do not need to assume that $F \in \mathcal{S}$.

Theorem 6.1 *Suppose F is a bounded continuous function on $\mathbf{R}^n$. Then the function u defined by $u(x,t) = \Phi_t * F(x)$ for $t > 0$ and $u(x,t) = F(x)$ for $t = 0$ is a solution of the heat equation problem with initial data F.*

Proof Note first that $\int \Phi_t(x)dx = \hat{\Phi}_t(0) = 1$. For $t > 0$, set $\psi_t = \Phi_{t^2}$. Then $\{\psi_t\}$ is evidently an approximate identity on $\mathbf{R}^n$. The usual arguments now show that $\lim_{t\to 0^+} u(x,t) = F(x)$ uniformly for x in any compact subset of $\mathbf{R}^n$. From this it follows easily that u is continuous at each point $(x,0)$ $(x \in \mathbf{R}^n)$.

Moreover,

$$\Phi_t(x) = \int e^{-2\pi i x\cdot\xi} e^{-4\pi^2 k|\xi|^2 t} d\xi,$$

from which we see that at once that if $v(t,x) = \Phi_t(x)$, then v satisfies the heat equation $v_t - k\Delta v = 0$ for $t > 0$. Finally

$$u(x,t) = \int \Phi_t(x-y)F(y)dy = \int v(x-y,t)F(y)dy.$$

By differentiating under the integral sign we see at once that u also satisfies the heat equation for $t > 0$. (Differentiating under the integral is surely justified, because of the very rapid decay at infinity of Φ_t and its derivatives.) This completes the proof.

Now suppose that $n = 1$ and, as in the Introduction, F is 2π-periodic and continuous. Then, in Theorem 6.1,

$$u(x,t) = \int_{-\infty}^{\infty} F(x-y)\Phi_t(y)dy, \tag{59}$$

which is again 2π-periodic in x.

On the other hand, arguing as in the Introduction, we expected the solution to be

$$u_1(x,t) = (F * \varphi_t)(x), \tag{60}$$

where $*$ now denotes convolution on $\mathbf{T}$, and

$$\varphi_t(x) = \sum_{m=-\infty}^{\infty} e^{-km^2 t} e^{imx}. \tag{61}$$

Although u and u_1 don't appear to be equal, we are going to prove that they in fact are equal. First let us note the following simple proposition.

Proposition 6.2 *Suppose F is a measurable, 2π-periodic function on $\mathbf{R}$, that F is integrable on $[0, 2\pi]$, and that $g \in \mathcal{S}(\mathbf{R})$. For $x \in \mathbf{R}$, put*

$$G(x) = \sum_{k=-\infty}^{\infty} g(x + 2k\pi). \tag{62}$$

Then G is a smooth 2π-periodic function on $\mathbf{R}$, and

$$\int_{-\infty}^{\infty} F(x)g(x)dx = \int_0^{2\pi} F(x)G(x)dx. \tag{63}$$

Proof Since $g \in \mathcal{S}$, there is a constant $C > 0$ with $|g(x)| \leq C(1+|x|)^{-2}$ for all x. Thus, if $x \in [0, 2\pi)$,

$$|g(x+2k\pi)| \leq C(1+|x+2k\pi|)^{-2} \leq C(1+|2j\pi|)^{-2}$$

where $j = k-1$ if $k \geq 1$, $j = 0$ if $k = 0$, and $j = k+1$ if $k < 0$. Altogether, for some $C' > 0$, if $x \in [0, 2\pi)$,

$$|g(x+2k\pi)| \leq C'(1+|k|)^{-2}. \tag{64}$$

Thus the series in (62) converges absolutely and uniformly on $[0, 2\pi)$. It is clear from (62) that g is 2π-periodic. Since also $g^{(m)} \in \mathcal{S}$ for any m, $\sum_{k=-\infty}^{\infty} g^{(m)}(x+2k\pi)$ also converges absolutely and uniformly on $[0, 2\pi)$ and equals $G^{(m)}(x)$. In particular G is smooth. Finally

$$\begin{aligned}
&\int_{-\infty}^{\infty} F(x)g(x)dx \\
= &\lim_{M\to\infty} \sum_{k=-M}^{M} \int_{2\pi k}^{2\pi(k+1)} F(x)g(x)dx \\
= &\lim_{M\to\infty} \sum_{k=-M}^{M} \int_{0}^{2\pi} F(x)g(x+2k\pi))dx \\
= &\int_{0}^{2\pi} F(x)G(x)dx
\end{aligned}$$

as desired. (In the third line, we have used the periodicity of F.)

We apply Proposition 6.2 to (59). We note that, in (59), the function F_x defined by $F_x(y) = F(x-y)$ is continuous and 2π-periodic in y. Thus, by Proposition 6.2,

$$u(x,t) = \int_0^{2\pi} F(x-y)G_t(y)dy, \tag{65}$$

where now G_t is the smooth 2π-periodic function defined by

$$G_t(y) = \sum_{k=-\infty}^{\infty} \Phi_t(y+2k\pi). \tag{66}$$

In other words, $u(x,t) = 2\pi(F * G_t)(x)$, where now $*$ denotes convolution on the *circle* $\mathbf{T}$. If u is to equal u_1 (in (60)), we expect that G_t must equal $\varphi_t/2\pi$, or more suggestively, that $\varphi_t/2\pi$ (as in (61)) *is the Fourier series of* G_t. This is a consequence of the following fact:

Theorem 6.3 *In the situation of Proposition 6.2, the Fourier series of G is*

$$G(x) = \sum_{k=-\infty}^{\infty} g(x+2k\pi) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \hat{g}(-\frac{m}{2\pi})e^{imx}, \tag{67}$$

where here $\hat{}$ *denotes the* Fourier transform*.*

Proof Taking $F(x) = e^{-imx}$ in Proposition 6.2, we see that the mth Fourier coefficient of G is

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-imx}G(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-imx}g(x)dx = \frac{1}{2\pi}\hat{g}(-\frac{m}{2\pi})$$

as desired.

Applying Theorem 6.3 to $g(x) = \Phi_t(x)$, by (57), we see that

$$G_t(x) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\hat{g}(-\frac{m}{2\pi})e^{imx} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-km^2t}e^{imx} = \frac{\varphi_t(x)}{2\pi},$$

as claimed.

Exercise 6.4 *Show that $\{\varphi_t\}$ is an approximate identity on* $\mathbf{T}$.
(Hint: knowing that $\varphi_t = 2\pi G_t$ certainly helps!)

Remarks 1. One can rescale in order to put (67) into a more elegant form. Changing x to $2\pi x$, we find

$$\sum_{k=-\infty}^{\infty} g(2\pi[x+k]) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\hat{g}(-\frac{m}{2\pi})e^{2\pi imx}.$$

Setting $f(x) = g(2\pi x)$, we see by Proposition 5.4 that

$$\sum_{k=-\infty}^{\infty} f(x+k) = \sum_{m=-\infty}^{\infty}\hat{f}(-m)e^{2\pi imx}.$$

This holds for any $f \in \mathcal{S}(\mathbf{R})$. In particular, by setting $x = 0$, we find the elegant *Poisson summation formula*:

$$\sum_{k=-\infty}^{\infty} f(k) = \sum_{k=-\infty}^{\infty}\hat{f}(k),$$

again for any $f \in \mathcal{S}(\mathbf{R})$.
2. We have now seen that, if F is 2π-periodic, then u_1 is a solution of the heat equation problem with initial data F, since it coincides with u. Could there however be other 2π-periodic solutions of the problem? The following theorem shows that there are no others:

Theorem 6.5 *Say F is 2π-periodic and continuous on* $\mathbf{R}$. *Then there is a unique solution to the heat equation problem, with initial data F, which is also 2π-periodic in x.*

Proof Let $v(x,t)$ be such a solution. Let $\hat{F}(m)$ denote the mth Fourier coefficient of F, $\frac{1}{2\pi}\int_0^{2\pi} F(x)e^{-imx}dx$. By the injectivity of $\hat{}$ on $L^1(\mathbf{T})$, it suffices to show that, for any $m, t > 0$, if we set $V(m,t) = \frac{1}{2\pi}\int_0^{2\pi} v(x,t)e^{-imx}dx$, then $V(m,t) = e^{-km^2t}\hat{F}(m)$. But if we differentiate under the integral sign, we see that

$$\frac{dV}{dt}(m,t) = \frac{1}{2\pi}\int_0^{2\pi} v_t(x,t)e^{-imx}dx = \frac{1}{2\pi}\int_0^{2\pi} kv_{xx}(x,t)e^{-imx}dx = -km^2V(m,t),$$

since we may integrate by parts twice in the last integral. Also $V(m,t)$ is continuous for $t \geq 0$ and C^1 for $t > 0$, and $V(m,0) = \hat{F}(m)$. Consequently $V(m,t) = e^{-km^2t}\hat{F}(n)$, as claimed.

Note that the proof of this theorem is very similar to the formal arguments that led up to (53). Note also that it would be wrong to try to prove this result by writing $v(x,t) = \sum_{m=-\infty}^{\infty} V(m,t)e^{imx}$ and differentiating the sum, formally obtaining $\sum V_t(m,t)e^{imx} = -k\sum m^2V(m,t)e^{imx}$; we don't know that the convergence of the series is fast enough to guarantee that this can be carried out.

Uniqueness theorems for the heat equation may be shown in other contexts, as a consequence of the *maximum principle for the heat equation*. Say, in a bounded domain $U \subseteq \mathbf{R}^n$, the temperature at position x and time t is $u(x,t)$, for times $0 \leq t \leq T$. As time goes by, we expect the temperature to become more evenly dispersed, and so we would not expect u to have an interior maximum at any time $0 < t < T$, or for that matter, at time T either. In fact, by using arguments similar to the proof of the maximum principle for harmonic functions, one can show:

Exercise 6.6 *(a) Suppose $U \subseteq \mathbf{R}^n$ is a bounded open set, and that $T > 0$. Suppose that $u(x,t)$ is continuous on $\overline{U} \times [0,T]$, is C^2 on $U \times (0,T)$, and satisfies the heat equation $u_t - u_{xx} = 0$ in $U \times (0,T)$. Let S be the boundary of $\overline{U} \times [0,T]$, and let $P = \overline{U} \times \{T\}$ Show that the maximum of u on $\overline{U} \times [0,T]$, is achieved at a point in $S \setminus P$.*
(Hint: first carry out the argument under the hypothesis that u is actually C^2 on $U \times (0,T_0)$ for some $T_0 > T$.)
(b) Conclude that if u_1 and u_2 are two functions on $\overline{U} \times [0,T]$, which agree on $S \setminus P$, and such that the hypotheses of (a) are satisfied if either $u = u_1$ or u_2, then $u_1 \equiv u_2$ on $\overline{U} \times [0,T]$.