MAT 550 – Test 2 Solutions – Spring 2009

You may use any results proved in the book. ˆ denotes the Fourier transform.

1. (a) Let $\{\psi^t\}$ be a standard approximate identity on $\mathbf{R}^n$. Show that, if $f \in L^1(\mathbf{R}^n)$, then $f * \psi^t \to f$ in L^1 as $t \to 0^+$.
(Hint: you can use the homework Exercises 11.4.8 and 11.4.9 (attached).)
(b) Say $f \in L^1(\mathbf{R}^n)$ and $\int f\varphi = 0$ for every $\varphi \in \mathcal{S}(\mathbf{R}^n)$; show that $f = 0$.
(Hint: use (a).)
(c) According to a proposition in the book, if $F, G \in L^1(\mathbf{R}^n)$, then

$\int \hat{F}\overline{G} = \int F\overline{\check{G}}$. Use this and (b) to show:
If $f \in L^1(\mathbf{R}^n)$, and also $\hat{f} \in L^1(\mathbf{R}^n)$, then $f = (\hat{f})\check{}$.
(d) Conclude that ˆ is injective on $L^1(\mathbf{R}^n)$.
Solution (a) A standard approximate identity has the form $\psi_t(x) = t^{-n}\psi(x/t)$ for some $\psi \in C_c^\infty$ with $\psi \geq 0$ and $\int \psi = 1$. Choose R such that the support of ψ is contained in $\overline{B}_R$, where B_R is the open ball of radius R centered at 0. Then for $0 < t < 1$, $\mathrm{supp}\psi_t \subseteq \overline{B}_{tR} \subseteq \overline{B}_R$. Let $K_2 = \overline{B}_R$.
Now say first $f \in C_c$, and say $\mathrm{supp} f = K_1$. Then for $0 < t < 1$, $\mathrm{supp}(f * \psi_t) \subseteq K_1 + K_2 = K$, say. Also $f * \psi_t \to f$ uniformly (as $t \to 0^+$). Now $\|f * \psi_t - f\|_1 \leq m(K)\|f * \psi_t - f\|_\infty$, so in this case, we do have $f * \psi_t \to f$ in L^1.
But C_c is dense in L^1, so by the fundamental maximal principle, we need only prove the following. Define $T_t : L^1 \to L^1$ by $T_t f = f * \psi_t$; then $\|T_t\| \leq 1$ for all t. But this is clear, since $\|T_t f\|_1 \leq \|f\|_1 \|\psi_t\|_1$, and $\int \psi_t = \int \psi = 1$ for all t.
(b) In the notation of (a), it is clearly enough to show that $f * \psi_t \equiv 0$ for all t. But $f * \psi_t(x) = \int f(y)\psi_t(x-y)dy$, and for any fixed x, the function $y \to \psi_t(x-y)$ is clearly in $\mathcal{S}(\mathbf{R}^n)$. By the hypotheses of (b), this is zero
.

(c) Actually it's a little easier to use the equivalent fact that if $F, G \in L^1(\mathbf{R}^n)$, then $\int \hat{F}G = \int F\hat{G}$.
To show that $f - (\hat{f})\check{} = 0$, it is enough to show, by (b), that for all $\varphi \in \mathcal{S}$, $\int (\hat{f})\check{}\phi = \int f\phi$.
But

$$\int (\hat{f})\check{}\phi = \int \hat{f}\hat{\phi} = \int f(\hat{\phi})\check{} = \int f\phi$$

as claimed. (We used the inversion theorem for $\mathcal{S}$.)
(d) Say $\hat{f} = 0$. Note that $0 \in L^1$. Thus, by (c), $f = (\hat{f})\check{} = \check{0} = 0$.

2. Suppose a, b, c are real numbers, and that $b^2 - ac > 0$. Find, with proof, the general solution of the partial differential equation

$$au_{xx} + 2bu_{xy} + cu_{yy} = 0.$$

Here $u(x, y)$ is to be a C^2 function on $\mathbf{R} \times \mathbf{R}$, and the subscripts denote differentiation. (Hint: what special case have we done, and how did we do it?)
Solution If $a = c = 0$, the general solution is $u = h_1(x) + h_2(y)$ for C^2 functions h_1, h_2. Otherwise assume without loss that $a \neq 0$. A special case is the wave equation, in which $b = 0$, $a = 1$, $c = -1$, and we proceed similarly. Writing the quadratic $aX^2 + 2bX + c = a(X - r_1)(X - r_2)$, where the real roots r_1, r_2 are given by the quadratic equation formula, we find

$$a(D_x - r_1 D_y)(D_x - r_2 D_y)u = 0.$$

We change coordinates to (p, q) so that $\partial/\partial p = D_x - r_1 D_y$, $\partial/\partial q = D_x - r_2 D_y$; by the chain rule we want $x = p + q$, $y = -r_1 p - r_2 q$. Inverting the matrix of this linear transformation we find $p = -D(r_2 x + y)$, $q = D(r_1 x + y)$, where $D = (r_1 - r_2)^{-1}$. The equation is now $\partial^2 u/\partial p \partial q = 0$, whose general solution is $u = h_1(p) + h_2(q) = H_1(r_2 x + y) + H_2(r_1 x + y)$ for C^2 functions H_1, H_2.
When $b^2 - ac > 0$, $ax^2 + 2bxy + cy^2$ is the equation of a hyperbola, so the PDE is called *hyperbolic*. Similarly it is called *elliptic* if $b^2 - ac < 0$.

3. Let $\mathbf{R}^2_+$ be the open upper half-plane, $\{(x, y) \in \mathbf{R}^2 : y > 0\}$, and let $\overline{\mathbf{R}^2_+}$ be its closure. Find a solution of the Dirichlet problem for $\mathbf{R}^2_+$, in the following sense:
Let f be a bounded continuous function on $\mathbf{R}$. Find, with proof, a function $u(x, y)$ which is continuous on $\overline{\mathbf{R}^2_+}$, and which satisfies

$$\Delta u = 0 \text{ on } \mathbf{R}^2_+, \text{ and}$$

$$u(x, 0) = f(x) \text{ for all } x \in \mathbf{R}. \tag{1}$$

Express your solution in the form $u(x, y) = (f * \varphi^y)(x)$ for a completely explicit function φ^y. (Hint: first conjecture a solution, by using the Fourier transform in the x variable.)
Solution We begin by formally seeking out a solution; then we justify it.

Proceeding formally, let us let $U(\xi, y)$ denote the Fourier transform of u in the x variable (holding y fixed); thus

$$U(\xi, y) = \int e^{2\pi i x \xi} u(x, y) dx. \tag{2}$$

In (2), we can differentiate under the integral sign to see that, at least formally, U_{yy} is the Fourier transform of u_{yy} in the x variable . We want

$$u_{xx} + u_{yy} = 0.$$

Taking the Fourier transform of this and of the boundary condition (1) in the x variable, we find formally that, if $U_y = \partial U/\partial y$, then

$$U_{yy} = -4\pi^2 |\xi|^2 U \tag{3}$$

for $\xi \in \mathbf{R}^n$, $y > 0$, with initial conditions

$$U(\xi, 0) = \hat{f}(\xi). \tag{4}$$

For fixed $\xi \in \mathbf{R}^n$, this is just a simple ODE in t, and a solution is $U(\xi, y) = e^{-2\pi|\xi|y}\hat{f}(\xi)$. Taking the inverse Fourier transform in ξ, we find that

$$u(x, t) = \int e^{-2\pi i x\cdot\xi}\hat{f}(\xi)e^{-2\pi|\xi|y}d\xi, \tag{5}$$

whence, if $f \in \mathcal{S}(\mathbf{R})$,

$$u(x, t) = (\varphi^y * f)(x), \tag{6}$$

where

$$\varphi^y \text{ is the inverse Fourier transform, in } \xi, \text{ of } e^{-2\pi|\xi|y}. \tag{7}$$

We compute

$$\varphi^y(x) = \int_{-\infty}^{0} e^{-2\pi i x\xi}e^{2\pi\xi y}d\xi + \int_{0}^{\infty} e^{-2\pi i x\xi}e^{-2\pi\xi y}d\xi.$$

Integrating the exponentials, we find that

$$\varphi^y(x) = \frac{1}{\pi}\frac{y}{x^2 + y^2}.$$

We now need to justify that u, as in (6), is indeed a solution of the problem. In fact, we do not need to assume that $f \in \mathcal{S}$.

Note that, by Problem 1(c), the Fourier transform of φ^y is $e^{-2\pi|\xi|y}$, since both φ^y and $e^{-2\pi|\xi|y}$ are clearly in L^1. This implies that $\int \varphi^y(x)dx = \widehat{\varphi^y}(0) = 1$. Moreover, a change of variables shows that if $a > 0$, then $\int_{|x|>a} \varphi^y(x)dx = \int_{|x|>a/y} \varphi^1(x)dx \to 0$ as $y \to 0$. Thus $\{\varphi^y\}$ an approximate identity on $\mathbf{R}^n$. [1] The usual arguments now show that $\lim_{y\to 0^+} u(x, y) = f(x)$ uniformly for x in any compact subset of $\mathbf{R}$. From this it follows easily that u is continuous at each point $(x, 0)$ $(x \in \mathbf{R})$.

Moreover,

$$\varphi^y(x) = \int e^{-2\pi i x\cdot\xi}e^{-2\pi|\xi|y}d\xi,$$

from which we see at once, by differentiating under the integral sign, that if $v(x, y) = \varphi^y(x)$, then v is harmonic. Finally

$$u(x, y) = \int \varphi^y(x - s)f(s)ds = \int v(x - s, y)f(s)ds.$$

By differentiating under the integral sign we see at once that u is harmonic. (Differentiating under the integral sign is easily justified, by an examination of the decay as $x \to \pm\infty$ of v and its derivatives.)

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[1] Yes, one can prove this without using the Fourier transform, but we are trying to be systematic!