Practice for Midterm I

You have 80 minutes to finish the exam. No calculators are allowed. The actual exam will be shorter.

1. Let $\mathbb{V}$ be a vector space of functions with inner product $\langle , \rangle$. Suppose a linear operator $L$ satisfies $\langle Lf, g\rangle = \langle f, Lg\rangle$ for all $f, g \in \mathbb{V}$. Prove that for two eigenfunctions of $L$ with different eigenvalues are orthogonal with respect to $\langle , \rangle$.

*Proof.* Let $\phi_1, \phi_2$ be eigenfunctions of $L$ with eigenvalues $\lambda_1$ and $\lambda_2$ respectively. This means $L\phi_1(x) = \lambda_1\phi_1(x)$ and $L\phi_2(x) = \lambda_2\phi_2(x)$. Then

$$\lambda_1\langle\phi_1,\phi_2\rangle = \langle\lambda_1\phi_1,\phi_2\rangle = \langle L\phi_1,\phi_2\rangle$$

$$= \langle\phi_1, L\phi_2\rangle = \langle\phi_1,\lambda_2\phi_2\rangle = \lambda_2\langle\phi_1,\phi_2\rangle$$

Hence we have

$$(\lambda_1-\lambda_2)\langle\phi_1,\phi_2\rangle = 0$$

and we may divide both sides by $(\lambda_1-\lambda_2)$ since $\lambda_1 \neq \lambda_2$ guarantees that $\lambda_1-\lambda_2 \neq 0$. Then we have

$$\langle\phi_1,\phi_2\rangle = 0$$

□

2. Prove that

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

Answer)

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \times \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrd\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^{\infty} d\theta = \int_0^{2\pi}\frac{1}{2}d\theta = \pi$$

Therefore,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

3. (a) The complex Fourier transform of a function $f(x)$ is defined by

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx.$$

Suppose that $\lim_{x\to\pm\infty} f(x) = 0$. Then prove that

$$\hat{f'}(\lambda) = i\lambda\hat{f}(\lambda)$$

*Proof.*

$$\hat{f'}(\lambda) = \int_{-\infty}^{\infty} f'(x) e^{-i\lambda x} dx$$

Performing integration by parts we get

$$\hat{f'}(\lambda) = f(x) e^{-i\lambda x}\Big|_{-\infty}^{\infty} + i\lambda \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx = i\lambda \hat{f}(\lambda)$$

since the boundary terms disappear because $\lim_{x\to\pm\infty} f(x) = 0$. □

(b) Using part (a), find the Fourier transform of the heat equation

$$\frac{\partial^2}{\partial x^2} u(x,t) = \frac{\partial}{\partial t} u(x,t) \qquad -\infty < x < \infty$$

$$u(x,0) = \delta(x).$$

where the delta function $\delta(x)$ is the function that satisfies

$$\int_{-\infty}^{\infty} f(x)\delta(x) dx = f(0)$$

for any continuous function $f(x)$. Also find the solution to the transformed differential equation.

Solution) Fourier Transform applied to the differential equation is

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} u(x,t) e^{-i\lambda x} dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x,t) e^{-i\lambda x} dx \qquad -\infty < x < \infty$$

$$\int_{-\infty}^{\infty} u(x,0) e^{-i\lambda x} dx = \int_{-\infty}^{\infty} \delta(x) e^{-i\lambda x} dx.$$

Part (a) says that

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} u(x,t) e^{-i\lambda x} dx = (i\lambda)^2 \hat{u}(\lambda,t)$$

and from the definition of the delta function,

$$\int_{-\infty}^{\infty} \delta(x) e^{-i\lambda x} dx = e^{-i\lambda\cdot 0} = 1.$$

Hence we end up with an ODE

$$-\lambda^2 \hat{u}(\lambda,t) = \frac{\partial}{\partial t} \hat{u}(\lambda,t)$$

$$\hat{u}(\lambda,0) = 1$$

Solving this ODE, we get

$$\hat{u}(\lambda,t) = e^{-\lambda^2 t}$$

4. Suppose $\phi_n(x)$, $n = 1,2,3,\cdots$ satisfy the orthogonality condition

$$\int_l^r \phi_n(x)\phi_m(x)\, dx = 0 \quad (n \neq m).$$

Also suppose that

$$\int_l^r \left(\phi_n(x)\right)^2 dx \neq 0.$$

Prove that the equality

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

implies that

$$a_n = \frac{\int_l^r f(x)\phi_n(x)\, dx}{\int_l^r \left(\phi_n(x)\right)^2 dx}.$$

(You do not have to talk about uniform convergence.)

*Proof.* Define the inner product $\langle , \rangle$ for any two functions $f(x)$ and $g(x)$ as

$$\langle f, g \rangle = \int_l^r f(x)g(x)\, dx.$$

Then

$$\langle f, \phi_n \rangle = \langle \sum_{k=1}^{\infty} a_k \phi_k(x)\, ,\, \phi_n(x) \rangle = \sum_{k=1}^{\infty} a_k \langle \phi_k(x), \phi_n(x) \rangle,$$

or written explicitly, this is

$$\langle f, \phi_n \rangle = a_1 \langle \phi_1(x), \phi_n(x) \rangle + a_2 \langle \phi_2(x), \phi_n(x) \rangle + a_3 \langle \phi_3(x), \phi_n(x) \rangle + \cdots .$$

The orthogonality condition implies that $\langle \phi_k(x), \phi_n(x) \rangle = 0$ unless $k = n$. This means that all the terms in the right hand side is zero accept when $n = k$, leaving us with

$$\langle f, \phi_n \rangle = a_n \langle \phi_n(x), \phi_n(x) \rangle.$$

Solving for $a_n$ we get

$$a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n(x), \phi_n(x) \rangle} = \frac{\int_l^r f(x)\phi_n(x)\, dx}{\int_l^r \left(\phi_n(x)\right)^2 dx}.$$

☐

Note: To be mathematically rigorous, one needs uniform convergence in order to have equality $\langle \sum_{k=1}^{\infty} a_k \phi_k(x)\, ,\, \phi_n(x) \rangle = \sum_{k=1}^{\infty} a_k \langle \phi_k(x), \phi_n(x) \rangle$. But we have not discussed this in class.

5. (a) Verify that the function $f(x) = x$ and $g(x) = 2x^2 - 1$ are eigenfunctions of the Sturm Liouville problem

$$\left(\sqrt{1-x^2}\, y'\right)' + \lambda \frac{1}{\sqrt{1-x^2}} y = 0 \quad -1 < x < 1$$

and find the corresponding eigenvalues.
(b) Normalize the eigenfunctions $f(x)$ and $g(x)$.

Solution) $f'(x) = 1$ and $g'(x) = 4x$. First, plug in $y = f(x)$ to get

$$\left(\sqrt{1-x^2} \cdot 1\right)' + \lambda \frac{x}{\sqrt{1-x^2}} = 0$$

which simplifies to

$$(\lambda - 1)\frac{x}{\sqrt{1-x^2}} = 0.$$

Hence $\lambda = 1$ and $y = x$ will satisfy

$$\left(\sqrt{1-x^2}y'\right)' + \lambda \frac{1}{\sqrt{1-x^2}} y = 0.$$

Which means that $f(x) = x$ is an eigenfunction of the operator $\sqrt{1-x^2}\frac{d}{dx}\sqrt{1-x^2}\frac{d}{dx}$ with eigenvalue 1.
Again plug in $y = g(x)$ to get

$$\left(\sqrt{1-x^2} \cdot 4x\right)' + \lambda \frac{2x^2-1}{\sqrt{1-x^2}} = 0$$

which simplifies to

$$(\lambda - 4)\frac{2x^2-1}{\sqrt{1-x^2}} = 0.$$

So $2x^2 - 1$ is an eigenfunction with eigenvalue 4.
To answer part b), we need to compute

$$\frac{f(x)}{\left(\int_{-1}^{1} \left(f(x)\right)^2 p(x) dx\right)^{\frac{1}{2}}}$$

where $p(x)$ can be read off from the original differential equation as $\frac{1}{\sqrt{1-x^2}}$. We compute

$$\int_{-1}^{1} \left(f(x)\right)^2 p(x) dx = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} dx = 2\int_{0}^{1} \frac{x^2}{\sqrt{1-x^2}} dx$$

$$= 2\int_{0}^{\frac{\pi}{2}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta = 2\int_{0}^{\frac{\pi}{2}} \sin^2\theta \, d\theta = \int_{0}^{\frac{\pi}{2}} 1 - \cos 2\theta \, d\theta = \frac{\pi}{2}$$

Hence

$$\frac{f(x)}{\left(\int_{-1}^{1} \left(f(x)\right)^2 p(x) dx\right)^{\frac{1}{2}}} = \sqrt{\frac{2}{\pi}} x.$$

Like wise, we need to compute

$$\int_{-1}^{1} \left(g(x)\right)^2 p(x) dx = \int_{-1}^{1} \frac{4x^4 - 4x^2 + 1}{\sqrt{1-x^2}} dx = 2\int_{0}^{1} \frac{4x^4 - 4x^2 + 1}{\sqrt{1-x^2}} dx$$

$$= 2\int_{0}^{\frac{\pi}{2}} \frac{4\sin^4\theta - 4\sin^2\theta + 1}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta = 2\int_{0}^{\frac{\pi}{2}} 4\sin^4\theta - 4\sin^2\theta + 1 \, d\theta$$

$$= 2\int_{0}^{\frac{\pi}{2}} 1 + 4\sin^2\theta(\sin^2\theta - 1) \, d\theta = 2\int_{0}^{\frac{\pi}{2}} 1 - 4\sin^2\theta\cos^2\theta \, d\theta = 2\int_{0}^{\frac{\pi}{2}} 1 - \cos^2 2\theta \, d\theta = \int_{0}^{\frac{\pi}{2}} 1 - \cos 4\theta \, d\theta = \frac{\pi}{2}$$

(There are several different ways to use trigonometric identities.)
Therefore the normalization of the eigenfunction $g(x)$ is $\sqrt{\frac{2}{\pi}}(2x^2-1)$.

6. (a) Define

$$u(x,t)=\begin{cases}2-|x+t-2| & 0\le x+t<4.\\ 0 & \text{otherwise}\end{cases} \tag{1}$$

Show that this is a solution to the wave equation

$$\frac{\partial^2}{\partial x^2}u=\frac{\partial^2}{\partial t^2}u \quad (-\infty<x<\infty)$$

What are the appropriate initial conditions?
(b) With the initial condition obtained from part (a), solve the wave equation

$$\frac{\partial^2}{\partial x^2}u=\frac{\partial^2}{\partial t^2}u \quad (0<x<\infty)$$

$$u(0,t)=0$$

for time $t=1$ using the d'Alambert solution.

Solution) Notice that $u(x,t)=\phi(x+t)$ where

$$\phi(x)=\begin{cases}2-|x-2| & 0\le x<4.\\ 0 & \text{otherwise}\end{cases}$$

Then

$$\frac{\partial}{\partial x}u(x,t)=\phi'(x+t),\ \ \frac{\partial^2}{\partial x^2}u(x,t)=\phi''(x+t),\ \ \frac{\partial}{\partial t}u(x,t)=\phi'(x+t),\ \ \frac{\partial^2}{\partial t^2}u(x,t)=\phi''(x+t).$$

So (1) holds. Note that

$$|x+t-2|=\begin{cases}x+t-2 & x+t-2\ge 0\\ -(x+t-2) & x+t-2<0\end{cases}$$

So $u(x,t)$ can be rewritten as

$$u(x,t)=\begin{cases}2+(x+t-2) & 0\le x+t<2.\\ 2-(x+t-2) & 2\le x+t<4.\\ 0 & \text{otherwise}\end{cases} \tag{2}$$

To find the initial conditions,

$$u(x,0)=f(x)=\begin{cases}2-|x-2| & 0\le x<4.\\ 0 & \text{otherwise}\end{cases}$$

and

$$\frac{\partial}{\partial t}u(x,0)=g(x)=\begin{cases}1 & 0\le x<2.\\ -1 & 2\le x<4.\\ 0 & \text{otherwise}\end{cases} \tag{3}$$

This is obtained by differentiating equation (2) by $t$ and setting $t = 0$.
b) The d'Alambert solution for infinite half line is

$$u(x,t) = \frac{1}{2}\left(f_o(x+t) + f_o(x-t)\right) + \frac{1}{2}\left(G_e(x+t) - G_e(x-t)\right)$$

where $G(x)$ is the antiderivative of $g(x)$, which from equation (3) we have

$$G(x) = \int_0^x g(s)ds = \begin{cases} x+a & 0 \le x < 2. \\ -x+b & 2 \le x < 4. \\ c & \text{otherwise} \end{cases}$$

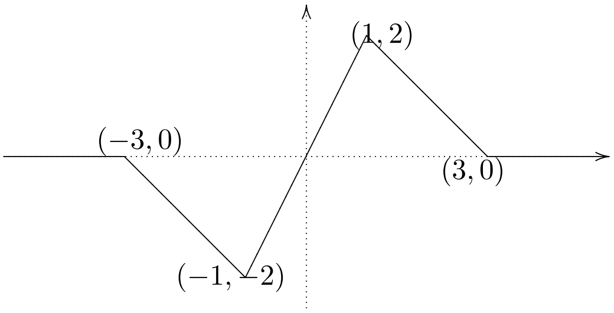
for some constants $a$,$b$ and $c$ to be determined as follows. First, $a = G(0) = \int_0^0 g(s)\,ds = 0$. Now, since $G(x)$ should be continuous, $x\Big|_{x=2} = -x+b\Big|_{x=2}$. We then obtain $b = 4$. Likewise, considering continuity at $x = 4$ will give us $c = 0$. So

$$G(x) = \int_0^x g(s)ds = \begin{cases} x & 0 \le x < 2. \\ -x+4 & 2 \le x < 4. \\ 0 & \text{otherwise} \end{cases}$$

The solution at $t = 1$ is therefore,

$$u(x,1) = \frac{1}{2}\left(f_o(x+1) + f_o(x-1)\right) + \frac{1}{2}\left(G_e(x+1) - G_e(x-1)\right)$$

where $f_o(x)$ and $G_e(x)$ are odd extensions and even extensions of $f(x)$ and $G(x)$. Its graph is (after drawing graph of $\frac{1}{2}\left(f_o(x+1) + f_o(x-1)\right)$ and $\frac{1}{2}\left(G_e(x+1) - G_e(x-1)\right)$ by method of averaging and then adding the two graphs → too involved to write it here.)



7. Solve the wave equation

$$\frac{\partial^2}{\partial x^2}u = \frac{\partial^2}{\partial t^2}u \quad (0 < x < 2)$$

$$u(0,t) = u(2,t) = 0$$

$$u(x,0) = 1 - |x-1|$$

$$\frac{\partial}{\partial t}u(x,0) = 0$$

using separation of variables.

Solution) First, let $u(x,t) = \phi(x)T(t)$. Then $\frac{\partial^2}{\partial x^2}u = \frac{\partial^2}{\partial t^2}u$ implies that

$$\frac{\phi''(x)}{\phi(x)} = \frac{T''(t)}{T(t)} = p \tag{4}$$

where constant $p$ is introduced because the first equality implies that both sides are independent of $x$ and $t$, yielding that it is equal to some constant. The boundary condition $u(0,t) = u(2,t) = 0$ translates to

$$\phi(0)T(t) = \phi(2)T(t) = 0.$$

Since we want nontrivial solutions, $T(t) \not\equiv 0$ and therefore

$$\phi(0) = \phi(2) = 0 \tag{5}$$

Our goal is to find nontrivial solution satisfying this boundary condition and equation (4) which implies that

$$\phi''(x) - p\phi(x) = 0.$$

For this boundary value problem to have nontrivial solutions, $\phi(x)$ needs to be trigonometric functions rather than hyperbolic functions. So $p \leq 0$. case 1) $p = 0$
We get

$$\phi''(x) = 0$$

which has the general solution $\phi(x) = ax + b$. But again the boundary conditions force $a = 0$ and $b = 0$.
case 2) $p < 0$
Let $p = -\lambda$. We have

$$\phi''(x) + \lambda^2\phi(x)$$

which has the general solution $\phi(x) = A\cos\lambda x + B\sin\lambda x$. The boundary condition (5) forces $A = 0$ and $B\sin 2\lambda = 0$. We can't let $B = 0$ lest we get trivial solutions. So $\sin 2\lambda = 0$ and hence

$$\lambda_n = \frac{n\pi}{2}.$$

For this value of $\lambda_n$,

$$T''(t) + \lambda_n^2 T(t) = 0$$

so that $T(t) = a_n\cos\lambda_n t + b_n\sin\lambda_n t$. Hence we have found all nontrivial solutions of the form $\phi(x)T(t)$, namely

$$u(x,t) = \sin\frac{n\pi}{2}x\left(a_n\cos\frac{n\pi}{2}t + b_n\sin\frac{n\pi}{2}t\right)$$

By the principle of superposition, we now know that

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{2}t + b_n\sin\frac{n\pi}{2}t\right)\sin\frac{n\pi}{2}x$$

satisfies the wave equation and the boundary conditions. Now we impose the initial conditions. First, $u(x,0) = 1 - |x-1|$ tells us that

$$1 - |1-x| = \sum_{n=1}^{\infty} a_n\sin\frac{n\pi}{2}x$$

So $a_n$'s are obtained by the Fourier sine expansion. Namely

$$a_n = \frac{2}{2}\int_0^2 (1-|1-x|)\sin\frac{n\pi}{2}x\,dx$$

$$= \int_0^1 x\sin\frac{n\pi}{2}x\,dx + \int_1^2 (2-x)\sin\frac{n\pi}{2}x\,dx = \frac{8\sin\frac{n\pi}{2}}{n^2\pi^2}$$

Second, we use $\frac{\partial}{\partial t}u(x,0)=0$ to get

$$0 = \sum_{n=1}^{\infty} b_n \frac{n\pi}{2}\sin\frac{n\pi}{2}x$$

This Fourier sine series leads to the trivial values $b_n\frac{n\pi}{2}=0$. So all $b_n$'s are zero. Hence the answer is

$$u(x,t)\sum_{n=1}^{\infty}\left(\frac{8\sin\frac{n\pi}{2}}{n^2\pi^2}\right)\cos\frac{n\pi}{2}t\sin\frac{n\pi}{2}x$$

8. Solve the Laplace equation

$$\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u = 0 \quad (0<x<1, 0<y<2) \tag{6}$$

$$\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,2) = 0 \tag{7}$$

$$u(0,y)=y, \;\; u(1,y)=1 \tag{8}$$

Solution) By separation of variables method, we first find all nontrivial solutions of the form $u(x,y)=X(x)Y(y)$ satisfying the Laplace equation (6) and the boundary condition (7). The Laplace equation implies that

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = p$$

where constant $p$ is introduced because the first equality implies that both sides are independent of $x$ and $t$, yielding that it is equal to some constant. The boundary condition (7) translates to

$$Y'(0)=0, \quad Y'(2)=0.$$

For this boundary value problem to have nontrivial solutions, $Y(y)$ needs to be trigonometric functions rather than hyperbolic functions. So $p\le 0$. case 1) $p=0$. Then

$$Y''=0, \quad Y'(0)=0, \quad Y'(2)=0.$$

implies that $Y=c$ for some constant $c$. And

$$X''=0$$

implies that $X=bx+a$. So $u(x,y)=bx+a$. case 2) $p<0$. Then let $p=-\lambda^2$ and we get

$$Y''+\lambda^2 Y=0, \quad Y'(0)=0, \quad Y'(2)=0.$$

This has the general solution $Y(y) = C_1 \cos \lambda y + C_2 \sin \lambda y$. Differentiating $Y'(y) = -C_1\lambda \sin \lambda y + C_2\lambda \cos \lambda y$ and imposing the boundary conditions, one obtains $C_2 = 0$ and $\sin 2\lambda = 0$. Which implies that

$$\lambda_n = \frac{n\pi}{2}$$

and $Y = C_1 \cos \lambda_n y$.
Then for $X(x)$,

$$X'' - \lambda_n^2 X = 0$$

and the general solution is $X(x) = a_n \cosh \lambda_n x + b_n \sinh \lambda_n x$. Hence we have $u(x, y) = (a_n \cosh \lambda_n x + b_n \sinh \lambda_n x) \cos \lambda_n y$.
By the principle of superposition,

$$u(x, y) = a_0 + b_0 x + \sum_1^{\infty} (a_n \cosh \lambda_n x + b_n \sinh \lambda_n x) \cos \lambda_n y.$$

We now use (8). $u(0, y) = y$ implies that

$$y = a_0 + \sum_1^{\infty} a_n \cos \frac{n\pi}{2} y$$

hence

$$a_0 = \frac{2}{4} \int_0^2 y dy = 1$$

and

$$a_n = \frac{2}{2} \int_0^2 y \cos \frac{n\pi}{2} y \, dy = \frac{4}{(n\pi)^2} \left((-1)^n - 1\right)$$

Now using $u(1, y) = 1$ implies that

$$1 = 1 + b_0 + \sum_1^{\infty} (a_n \cosh \frac{n\pi}{2} + b_n \sinh \frac{n\pi}{2}) \cos \frac{n\pi}{2} y.$$

Hence $b_0 = 0$ and $a_n \cosh \frac{n\pi}{2} + b_n \sinh \frac{n\pi}{2} = 0$. Solving the last one for $b_n$ we get

$$b_n = \frac{4}{(n\pi)^2} \left(1 - (-1)^n\right) \coth \frac{n\pi}{2}$$

Hence we have the solution as

$$u(x, y) = 1 + \frac{4}{(n\pi)^2} \left((-1)^n - 1\right) \sum_1^{\infty} (\cosh \frac{n\pi}{2} x - \coth \frac{n\pi}{2} \sinh \frac{n\pi}{2} x) \cos \frac{n\pi}{2} y.$$

9. Suppose you want to solve the Laplace equation

$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0 \quad (0 < x < 2, 0 < y < 1)$$

$$u(0, y) = u(2, y) = y$$

$$u(x,0) = u(x,1) = 1$$

using separation of variables. Then you have to write $u(x,y) = u_1(x,y) + u_2(x,y)$. Write down the appropriate boundary conditions for $u_1$ and $u_2$.

Answer)

$$\frac{\partial^2}{\partial x^2}u_1 + \frac{\partial^2}{\partial y^2}u_1 = 0 \quad (0 < x < 2, 0 < y < 1)$$

$$u_1(0,y) = u_1(2,y) = 0$$

$$u_1(x,0) = u_1(x,1) = 1$$

and

$$\frac{\partial^2}{\partial x^2}u_2 + \frac{\partial^2}{\partial y^2}u_2 = 0 \quad (0 < x < 2, 0 < y < 1)$$

$$u_2(0,y) = u_2(2,y) = y$$

$$u_2(x,0) = u_2(x,1) = 0$$

($u_1$ and $u_2$ could be interchanged.)