MAT341 Practice Final

1. Let D be a bounded domain in $\mathbb{R}^3$ with a simply connected piecewise smooth boundary ∂D.

a) Prove the Green's identity

$$\int_D f\triangle g - g\triangle f dV = \int_{\partial D} f\frac{\partial}{\partial \mathbf{n}}g - g\frac{\partial}{\partial \mathbf{n}}f d\sigma$$

using the Gauss divergence theorem

$$\int_D \nabla\cdot\mathbf{F}dV = \int_{\partial D}\mathbf{F}\cdot\mathbf{n}d\sigma.$$

Here, $\mathbf{n}$ is the outward unit normal vector to the boundary ∂D and f and g are functions from $\mathbb{R}^3$ to $\mathbb{R}$. Also, $\mathbf{F}$ is a vector field on $\mathbb{R}^3$.

Proof. Apply the divergence theorem with $\mathbf{F} = f\nabla g - g\nabla f$ to get

$$\int_D \nabla\cdot(f\nabla g - g\nabla f)dV = \int_{\partial D}(f\nabla g - g\nabla f)\cdot\mathbf{n}d\sigma.$$

Now,

$$\begin{aligned}\nabla\cdot(f\nabla g - g\nabla f) &= \nabla f\cdot\nabla g + f\nabla^2 g - (\nabla g\cdot\nabla f + g\nabla^2 f)\\ &= f\nabla^2 g - g\nabla^2 f = f\triangle g - g\triangle f.\end{aligned}$$

Hence

$$L.H.S. = \int_D f\triangle g - g\triangle f dV$$

while using the identity $\nabla f\mathbf{n} = \frac{\partial}{\partial \mathbf{n}}f$, we have

$$R.H.S. = \int_{\partial D} f(\nabla g\cdot\mathbf{n}) - g(\nabla f\cdot\mathbf{n})d\sigma. = \int_{\partial D} f\frac{\partial}{\partial \mathbf{n}}g - g\frac{\partial}{\partial \mathbf{n}}f d\sigma$$

□

b) Prove that if f is a solution to the partial differential equation

$$\triangle f = f \quad (\text{ in } D)$$

$$f = 0 \quad (\text{ on } \partial D)$$

and g is a solution to the partial differential equation

$$\triangle g = 2g \quad (\text{ in } D\ )$$

$$g = 0 \quad (\text{ on } \partial D\ )$$

then

$$\int_D fgdV = 0.$$

Proof. Using the Green's identity of part a) gives us

$$\int_D f\triangle g - g\triangle f dV = \int_{\partial D} 0 \times \frac{\partial}{\partial \mathbf{n}} g - 0 \times \frac{\partial}{\partial \mathbf{n}} f d\sigma = 0$$

since both f and g vanish on ∂D. Now, using $\triangle f = f$ and $\triangle g = 2g$, we have

$$L.H.S. = \int_D 2fg - gfdV = \int_D fgdV$$

□

2. a) Prove that the only solution to the following Dirichlet problem is $u \equiv 0$.

$$\triangle u = 0 \quad (\text{ in } D\ )$$

$$u = 0 \quad (\text{ on } \partial D\ )$$

Proof. Before we go on to the proof, I mention that in this problem, we tacitly assume that u is continuous. We start from the first Green's identity

$$\int_D \nabla f \cdot \nabla g dV = \int_{\partial D} f \frac{\partial g}{\partial \mathbf{n}} d\sigma - \int_D f\triangle g dV.$$

Setting $f = g = u$, we have

$$\int_D \nabla u \cdot \nabla u dV = \int_{\partial D} u \frac{\partial u}{\partial n} d\sigma - \int_D u\triangle u dV$$

On the right hand side, the first integral vanishes because $u = 0$ on ∂D and second integral vanishes because $\triangle u = 0$. Also $\nabla u \cdot \nabla u = ||\nabla u||^2$. Hence

$$\int_D ||\nabla u||^2 dV = 0.$$

Now, suppose there was any point p in D such that $\nabla u(p) \neq 0$. Then $||\nabla u(p)||^2 > 0$ and by continuity of u, we actually have $||\nabla u||^2 > 0$ in some small neighborhood of p. Then we will have $\int_D ||\nabla u||^2 dV \quad > 0.$, which contradicts the above formula. Therefore, ∇u must be zero everywhere on the disc. Hence $u = C$ for some constant C, and by continuity to the boundary, $u \equiv 0$. □

b) Prove the uniqueness theorem for the Laplace equation with Dirichlet boundary condition.

Proof. Suppose the Dirichet BVP

$$\triangle v = 0 \quad (\text{ in } D\ )$$

$$v = f \quad (\text{ on } \partial D\ )$$

had two solutions v_1 and v_2. Then $u = v_1 - v_2$ satisfies

$$\triangle u = \triangle(v_1 - v_2) = 0 \quad (\text{ in } D\ )$$

$$u = v_1 - v_2 = f - f = 0 \quad (\text{ on } \partial D\ )$$

which brings us to the equation handled in part a). In part a) we proved that $u \equiv 0$. So $v_1 - v_2 = 0$, i.e. $v_1 = v_2$. □

3. a) Find the divergence of $\mathbf{F} = (x + y, x^2)$.
Solution)

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x + y) + \frac{\partial}{\partial y}(x^2) = 1$$

b) Compute the following integral

$$\int_{\partial D(\mathbf{0},\epsilon)} \mathbf{F} \cdot \mathbf{n} ds$$

where $D(\mathbf{0}, \epsilon)$ is a disc of radius ϵ centered at the origin and $\mathbf{n}$ is the outward unit normal vector.
Solution) Note: I should have added "without using the divergence theorem" for this problem, because this problem itself is about understanding why divergence theorem works. Here is how you do it without using divergence theorem. First, we take polar coordinates because the integral is over a circle. On the boundary of the circle of radius ϵ,

$$x = \epsilon \cos\theta \quad , \quad y = \epsilon \sin\theta.$$

Hence

$$\mathbf{F}(x,y) = \mathbf{F}(\epsilon\cos\theta, \epsilon\sin\theta) = (\epsilon\cos\theta + \epsilon\sin\theta, \epsilon^2\cos^2\theta)$$

Also, draw the picture of the circle and a point (x,y) on it to convince yourself that the direction of the outward normal vector is parallel to (x,y), i.e.

$$\mathbf{n} = (\cos\theta, \sin\theta).$$

The measure for the disc is $rdrd\theta$ and from this we can read off that the measure around the circumference is $rd\theta$ with $r = \epsilon$ in our context. (If you don't like this argument, you can also use $ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$ which you learned from vector calculus.)
So

$$\int_{\partial D(\mathbf{0},\epsilon)} \mathbf{F}\cdot\mathbf{n}ds = \int_0^{2\pi} (\epsilon\cos\theta + \epsilon\sin\theta, \epsilon^2\cos^2\theta)\cdot(\cos\theta,\sin\theta)\ \epsilon d\theta$$

$$= \epsilon^2 \int_0^{2\pi} \cos^2\theta + \sin\theta\cos\theta + \epsilon\cos^2\theta\sin\theta\ \epsilon d\theta = \pi\epsilon^2$$

Note that in computing the above integrals, we need to use $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$. Using the orthogonality relations for the Fourier series simplifies the computation greatly.

c) Find the limit

$$\lim_{\epsilon\to 0} \frac{1}{\text{Area}(D(\mathbf{0},\epsilon))} \int_{\partial D(\mathbf{0},\epsilon)} \mathbf{F}\cdot\mathbf{n}ds.$$

What is it equal to? Does this happen for any vector field $\mathbf{F}$? (Yes or no.)
Solution)

$$\lim_{\epsilon\to 0} \frac{1}{\text{Area}(D(\mathbf{0},\epsilon))} \int_{\partial D(\mathbf{0},\epsilon)} \mathbf{F}\cdot\mathbf{n}ds = \lim_{\epsilon\to 0} \frac{1}{\pi\epsilon^2}\pi\epsilon^2 = 1$$

This is equal to the divergence of $\mathbf{F}$ at (0,0). For any vectorfield $\mathbf{F}$, the divergence is equal to the limit of the flux divided by the area. So the answer is 'yes'.

4. Verify the divergence theorem for the unit ball in $\mathbb{R}^3$ with the vector field

$$\mathbf{F} = (xy, yz, zx)$$

Solution)

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx) = y + z + x$$

In spherical coordinates

$$x = \rho \cos\theta \sin\phi \ \ , y = \rho \sin\theta \sin\phi \ \ , z = \rho \cos\phi$$

and hence

$$\int_B \nabla \cdot \mathbf{F} dV = \int_0^{\pi} \int_0^{2\pi} \int_0^1 (\cos\theta \sin\phi + \rho \sin\theta \sin\phi + \rho \cos\phi) \rho^2 \sin\phi dr d\theta d\phi = 0$$

by using the fact that $\int_0^{2\pi} \cos\theta d\theta = 0$ and other similar facts. Since the question asks us to verify the divergence theorem, we need to compute

$$\int_{\partial B} \mathbf{F} \cdot \mathbf{n} d\sigma$$

Convince yourself that the outward normal vector $\mathbf{n}$ at point (x, y, z) is parallel to (x, y, z). So we need to normalize (x, y, z) to make it a unit vector, but (x, y, z) is on the unit sphere and therefore it is already a unit vector. (If you are not happy with this approach, another way to do this is to think of the unit ball as a level surface of some function and take the gradient of that function.)
So

$$\mathbf{n} = (x, y, z) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

Therefore,

$$\int_{\partial B} \mathbf{F} \cdot \mathbf{n} d\sigma = \int_0^{\pi} \int_0^{2\pi} (xy, yz, zx) \cdot (x, y, z) \sin\phi d\theta d\phi = \int_0^{\pi} \int_0^{2\pi} (x^2 y + y^2 z + z^2 x) \sin\phi d\theta d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} (\cos^2\theta \sin^3\phi \sin\theta + \sin^2\theta \sin^2\phi \cos\phi + \cos^2\phi \cos\theta \sin\phi) \sin\phi d\theta d\phi$$

$$= \int_0^{\pi} 0 + \pi \sin^3\phi \cos\phi + 0 d\phi = \int_0^{\pi} \pi \frac{1 - \cos 2\phi}{2} \sin\phi \cos\phi d\phi$$

$$= \int_0^{\pi} \pi \frac{1 - \cos 2\phi}{2} \times \frac{\sin 2\phi}{2} d\phi = 0$$

Therefore, the divergence theorem

$$\int_{\partial B} \mathbf{F} \cdot \mathbf{n} d\sigma = \int_B \nabla \cdot \mathbf{F} dV$$

holds!

5. Let D be the unit disc. Solve the following boundary value problem

$$\triangle u = 0 \quad (\text{ in } D\ )$$

$$u = f(\theta) \quad (\text{ on } \partial D\ )$$

where

$$f(\theta) = \begin{cases} \cos\theta & 0 \le \theta < \pi. \\ 0 & \text{otherwise} \end{cases}$$

Solution) Changing everything into polar coordinates, we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial^2 \theta} = 0, \quad (0 \le r < 1) \tag{1}$$

$$u(1,\theta) = f(\theta) \tag{2}$$

$$u(r,\theta + 2\pi) = u(r,\theta), \quad (0 < r < 1) \tag{3}$$

$$u(r,\theta) \text{ is bounded as } r \to 0+. \tag{4}$$

Use separation of variables as $u(r,\theta) = R(r)Q(\theta)$ and equation (1) becomes

$$\frac{r(rR')'}{R} + \frac{Q''}{Q} = 0.$$

So both terms must be constant. Now $\frac{Q''}{Q} = -\lambda^2$ because if it were a positive constant we will get an exponential function not a periodic function. So we obtain

$$Q'' + \lambda^2 Q = 0$$

$$Q(\theta + 2\pi) = Q(\theta)$$

which has the general solution $Q(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta)$ and the periodic condition forces

$$\lambda = \lambda_n = n\ , \quad n = 1, 2, 3, \ldots$$

For R we have

$$r(rR')' - n^2 R = 0$$

$$R(r) \text{ is bounded as } r \to 0+.$$

This equation is called Cauchy-Euler equation which it is known that it has the solution of the form r^α. Plugging in $R = r^\alpha$ we get

$$(\alpha(\alpha - 1) + \alpha - n^2)r^\alpha = 0$$

and we obtain $\alpha = n,\ -n$. So the solution is $R(r) = C_1 r^n + C_2 r^{-n}$, but the bounded condtion as $r \to 0+$ forces $C_2 = 0$. Hence $u(r,\theta) = R(r)Q(\theta)$ is of the form

$$1, \quad r^n \cos(n\theta), \quad r^n \sin(n\theta)$$

therefore the solution u in general are the superpositions of these solutions, i.e.

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n r^n \sin(n\theta).$$

Now, it remains to find what a_n and b_n are. We use the boundary condition at $r = 1$ to get

$$f(\theta) = u(1,\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta).$$

This is nothing else than the Fourier series, and hence we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta = \frac{1}{2\pi}\int_0^{\pi} \cos\theta d\theta = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)d\theta = \frac{1}{\pi}\int_0^{\pi} \cos(\theta)\cos(n\theta)d\theta = \begin{cases} 1/2 & n = 1. \\ 0 & \text{otherwise} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta = \frac{1}{\pi}\int_0^{\pi} \cos(\theta)\sin(n\theta)d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}(\sin(n\theta+\theta) + \sin(n\theta-\theta))d\theta = \begin{cases} \frac{2n}{(n^2-1)\pi} & n = even \\ 0 & n = odd. \end{cases}$$

5-II. Let D be the rectangle given by

$$D = \{(x,y)\Big| 0 \le x \le 2,\ 0 \le y \le 3\}.$$

Consider the following wave equation

$$\triangle u = \frac{\partial^2}{\partial^2 t}u \qquad (\text{ in } D\ )$$

$$u = 0 \qquad (\text{ on } \partial D\ ).$$

Use separation of variables and find the five lowest harmonics of this system.
Solution) Putting $u(x,y,t) = \phi(x,y)T(t)$ we get

$$\frac{\triangle\phi}{\phi} = \frac{T''}{T}.$$

So both sides must be constant. Since we are looking for harmonics, which are solutions that are periodic with respect to time, we let $\frac{T''}{T} = -\lambda^2$, so that

$$T'' + \lambda^2 T = 0$$

which yields $T(t) = A\cos\lambda t + B\sin\lambda t$.
Now let $\phi(x,y) = X(x)Y(y)$. Then $\triangle\phi = -\lambda^2\phi$ becomes

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2$$

which implies that each term $\frac{X''}{X}$ and $\frac{Y''}{Y}$ are constants. Now the boundary conditions imply that

$$X(0) = X(2) = 0 \quad , \quad Y(0) = Y(3) = 0.$$

So in order to have nontrivial solutions, the constants need to be negative, i.e.

$$\frac{X''}{X} = -\mu^2 \quad , \quad \frac{Y''}{Y} = -\nu^2$$

These differential equations along with the boundary conditions yield

$$X = C_1 \sin\mu x, \quad Y = C_2 \sin\nu y$$

where $\mu = \mu_m = \frac{m\pi}{2}$ and $\nu = \nu_n = \frac{n\pi}{3}$.
Hence the periodic solutions $u(x,y,t) = X(x)Y(y)T(t)$ have the form

$$u(x,y,t) = \sin\mu_m x \, \sin\nu_n y \, (A\cos\lambda_{mn}t + B\sin\lambda_{mn}t)$$

where the frequency $\lambda_{mn}^2 = \mu_m^2 + \nu_n^2 = \left(\frac{m\pi}{2}\right)^2 + \left(\frac{n\pi}{3}\right)^2 = (9m^2 + 4n^2)\frac{\pi^2}{36}$.
Therefore the lowest five harmonics corresponds to the above solution $u(x,y,t)$ with $(m,n) = (1,1),(1,2),(2,1),(1,3)$ and $(2,2)$ (in increasing order respect to λ_{mn}).

6. Let D be the unit disc. For the heat equation

$$\triangle u = \frac{\partial}{\partial t}u \quad (\text{ in } D)$$

$$u = 0 \quad (\text{ on } \partial D),$$

we want to find a solution of the form $u(r,\theta,t) = \phi(r,\theta)e^{-\lambda^2 t}$.
a) Write down the differential equation and all the boundary condition that

ϕ has to satisfy.
Solution)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial^2\theta}=-\lambda^2\phi, \quad (0\le r<1)$$

$$\phi(1,\theta)=0$$

$$\phi(r,\theta+2\pi)=\phi(r,\theta), \quad (0<r<1)$$

$$\phi(r,\theta) \text{ is bounded as } r\to 0+$$

b) Setting $\phi(r,\theta)=R(r)Q(\theta)$, write down the differential equation and all the boundary conditions that R and Q needs to satisfy.
Solution) Pluggin' in $\phi(r,\theta)=R(r)Q(\theta)$ we get

$$\frac{(rR')'}{rR}+\frac{Q''}{r^2Q}=-\lambda^2,$$

$$R(1)=0$$

$$Q(\theta+2\pi)=Q(\theta)$$

$$R(r) \text{ is bounded as } r\to 0+$$

The differential equation above can be written as

$$\frac{r(rR')'}{R}+\lambda^2r^2=-\frac{Q''}{Q}$$

Since the left hand side only depends on r and right hand side only depends on θ, they must both be constant. So we let $\frac{Q''}{Q}=-\mu^2$, where minus sign was chosen because we have that Q must be periodic. Hence the differential equation for Q is

$$Q''+\mu^2Q=0$$

$$Q(\theta+2\pi)=Q(\theta)$$

and the differential equation for R is

$$r(rR')'+\lambda^2r^2R=\mu^2R$$

$$R(1)=0$$

$$R(r) \text{ is bounded as } r\to 0+$$

c) Solve the equations for R and Q. (You may use the fact that the general solution of the differential equation

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right)+(\lambda^2x^2-\mu^2)y=0$$

is $y = C_1 J_\mu(\lambda x) + C_2 Y_\mu(\lambda x)$ where $J_\mu(x)$ and $Y_\mu(x)$ are Bessel functions of first and second kind respectively.
Solution) For Q,

$$Q(\theta) = A\cos(\mu\theta) + B\sin(\mu\theta)$$

where the periodic condition forces $\mu = \mu_m = m$.
For R,

$$R = C_1 J_\mu(\lambda r) + C_2 Y_\mu(\lambda r) = C_1 J_m(\lambda r) + C_2 Y_m(\lambda r).$$

But the condition that $R(r)$ is bounded as $r \to 0+$ forces $C_2 = 0$. Also, the condition $R(1) = 0$ forces $J_m(\lambda) = 0$. So λ must be zeros of $J_m(x)$. So if we name α_{mn} as the n-th zero of $J_m(x)$, then $\lambda = \lambda_{mn} = \alpha_{mn}$. Hence

$$R = C_1 J_\mu(\alpha_{mn} r)$$

7. Suppose u is a function that depends only on the radius r, and it is unbounded at 0. In addition

$$\triangle u = 0 \qquad (\text{ in } \mathbb{R}^2 - \{0\} \text{ }).$$

Find u.
Solution) In polar coordinates, the Laplace equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial^2 \theta} = 0$$

since u does not depend on θ, $\frac{\partial^2 u}{\partial^2 \theta} = 0$ and so we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0$$

Multiplying by r and integrating by r we have

$$r\frac{\partial u}{\partial r} = C$$

Now divide by r and integrate to get

$$u(r) = C\ln r + C'$$

Since we want unbounded solutions, this means $C \neq 0$.

8. Let D be the unit disc. Let u be the solution of the Dirichlet boundary value problem

$$\triangle u = 0 \quad (\text{ in } D)$$

$$u(x,y) = x^2 \quad (\text{ on } \partial D).$$

Find $u(0,0)$. (Hint: use the mean value theorem from the last homework (equation (13) in page 278)

Solution) The mean value theorem says that

$$u(0,0) = \frac{1}{\text{circumference}} \int_{-\pi}^{\pi} u(1,\theta)(\text{radius})d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(1,\theta)d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2\theta d\theta = \frac{1}{2}$$

Note: Final Exam Location: Harriman Hall 112 (8AM-10:30AM)

You are REQUIRED to know the first and second Green's identities and also the formula for mean value theorem. The following will be given on the exam sheet and therefore you do not have to memorize them

1. Polar and Spherical formula for the Laplacian $\triangle$.
2. $\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$.
3. Product to sum formula for trigonometric functions. (e.g. $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$.)
4. Equation (2)-(5) on page 62.