MAT341 (Applied Real Analysis) Midterm I
Stony Brook University Fall 2007

Write your name and student ID on the Examination Book. You have 80 minutes to finish the exam. No calculators are allowed.

1.

(a) [**6 pts**] Find the general solution for

$$xy' - y = 1$$

(b) [**5 pts**] What is the solution that is bounded on the interval $(-\infty, \infty)$?

(a)

$$y' - \frac{1}{x}y = \frac{1}{x}$$

Multiplying the integration factor $e^{\int(-\frac{1}{x})dx} = e^{-\ln x} = \frac{1}{x}$, we get $\frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x^2}$

$$\left(\frac{1}{x}y\right)' = \frac{1}{x^2}$$

$$\frac{1}{x}y = -\frac{1}{x} + C$$

$$y = -1 + Cx$$

(b) If $C \neq 0$, $\lim_{x\to\pm\infty} |y| = \infty$. So $C = 0$, and the bounded solution is $y = -1$.

(Alternate method)

$$xy' = 1 + y$$

$$\frac{dx}{dy}\frac{1}{1+y} = \frac{1}{x}$$

$$\int \frac{1}{1+y}dy = \int \frac{1}{x}dx$$

$$\ln(1+y) = \ln(x) + C$$

$$1 + y = Cx$$

$$y = Cx - 1$$

2.

$$x^2y'' + 3xy' + (1 + x^2)y = 0$$

(a) [**7 pts**] Suppose $y = x^r \sum_{n=0}^{\infty} a_n x^n$ is a solution to the above equation, with $a_0 \neq 0$. Find the value of r.

(b) **[5 pts]** Find the recursive relation between a_n and a_{n+2}. You don't have to simplify. (DO NOT solve the recursive relation.)

(a) $y' = \sum_{n=0}^{\infty} a_n(r+n)x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{n+r-2}$. Therefore,

$$\sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{n+r} + \sum_{n=0}^{\infty} 3a_n(r+n)x^{n+r} + (1+x^2)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} a_n\{(r+n)(r+n-1) + 3(r+n)\}x^{n+r} + (1+x^2)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} a_n\{(r+n)(r+n+2)\}x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} a_n\{(r+n)(r+n+2)+1\}x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=-2}^{\infty} a_{n+2}\{(r+n+2)(r+n+4)+1\}x^{n+r+2} + \sum_{n=2}^{\infty} a_n x^{n+r+2} = 0$$

$$a_0\{r(r+2)+1\}x^r + a_1\{(r+1)(r+3)+1\} + \sum_{n=0}^{\infty}[a_{n+2}\{(r+n+2)(r+n+4)+1\} + a_n]x^{n+r+2} = 0$$

Since each coefficients have to vanish, the first term says that

$$a_0\{r^2+2r+1\} = 0$$

Since $a_0 \neq 0$, we divide by a_0 and get $r^2+2r+1=0$. This factors as $(r+1)^2=0$, so $r=-1$.
(Note) Since $r=-1$, the second term says that $a_1=0$. This, together with the recurrence relation between a_{n+2} and a_n, implies that all a_n's with odd n vanish.
(b) Since $r=-1$ the recurrence relation can be read off from the sum as

$$a_{n+2}\{(-1+n+2)(-1+n+4)+1\} + a_n = 0$$

or

$$a_{n+2}\{n^2+4n+4\} + a_n = 0$$

3. **[8 pts]** Let a be some positive real number. Show by calculation that there are no nontrivial solutions to the boundary value problem

$$y'' - \lambda^2 y = 0$$

$$y'(0) = 0, \; y'(a) = 0$$

for any nonzero real number λ.

The general solution is $y = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$. So $y' = C_1 \lambda e^{\lambda x} - C_2 \lambda e^{-\lambda x}$. The boundary conditions say

$$\lambda(C_1 - C_2) = 0, \qquad \lambda(C_1 e^{\lambda a} - C_2 e^{-\lambda a}) = 0$$

Since $\lambda \neq 0$, the first equation implies $C_1 - C_2 = 0$. So substituting $C_2 = C_1$ into the second equation, we get

$$C_1(e^{\lambda a} - e^{-\lambda a}) = 0$$

Now, if $e^{\lambda a} - e^{-\lambda a} = 0$, then $e^{\lambda a} = e^{-\lambda a}$ so $\lambda a = -\lambda a$. This only holds if $a = 0$ or $\lambda = 0$ none of which is true. So $e^{\lambda a} - e^{-\lambda a} \neq 0$ and the above equation says $C_1 = 0$. Then $C_2 = C_1 = 0$. Hence $y = 0 \cdot e^{\lambda x} + 0 \cdot e^{-\lambda x} = 0$, which is a trivial solution.

4.

(a) [**9 pts**] Find the Fourier series of odd periodic extension of

$$f(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ 0 & \frac{1}{2} \leq x < 1. \end{cases}$$

Answer must be simplified for full credit.

(b) [**5 pts**] For the Fourier series obtained in part (a), for what values of x in the interval $[0, 1]$ does the series converge to function $f(x)$? When the series does not converge to $f(x)$, what value(s) does it converge to?

(c) [**8 pts**] Find the Fourier series of even periodic extension of

$$g(x) = \begin{cases} x & 0 \leq x < \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \leq x < 1. \end{cases}$$

You may use the fact that $g(x) = \int_0^x f(t)dt$.

(a) Odd extension produces Fourier Sine Series, i.e. $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$ where $b_n = 2\int_0^1 f(x) \sin n\pi x dx$.

$$b_n = 2\int_0^{\frac{1}{2}} 1 \cdot \sin n\pi x\, dx + 2\int_{\frac{1}{2}}^1 0 \cdot \sin n\pi x\, dx$$

$$= -\frac{2}{n\pi} \cos n\pi x \Big|_0^{\frac{1}{2}} = \frac{2}{n\pi}(1 - \cos \frac{n\pi}{2})$$

Case 1) n is even, i.e. $n = 2k$.
Then $\cos \frac{n\pi}{2} = \cos(k\pi) = (-1)^k$. So $b_n = \frac{2}{n\pi}(1 - (-1)^k) = \frac{2}{n\pi}(1 - (-1)^{n/2})$.
Case 2) n is odd, i.e. $n = 2k - 1$.
Then $\cos \frac{n\pi}{2} = 0$, and so $b_n = \frac{2}{n\pi}$.
So in conclusion, the answer is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

where $b_n = \frac{2}{n\pi}(1-(-1)^{n/2})$, when n is even and $b_n = \frac{2}{n\pi}$ when n is odd. Using k as in case 1 and 2, you may (but you don't have to) write this as

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k\pi}(1-(-1)^k)\sin 2k\pi x + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi}\sin(2k-1)\pi x$$

(b) Since $f(x)$ is piecewise smooth, the Fourier series converges to $f(x)$ whereever $f(x)$ is continuous, i.e. $0 < x < \frac{1}{2}$, $\frac{1}{2} < x < 1$. Now, for $x = 0, \frac{1}{2}$, and 1, $f(x)$ has jump discontinuity, so it converges to $\frac{f(x+0)+f(x-0)}{2}$. For $x = 0$ and $x = 1$, this is $\frac{1+(-1)}{2} = 0$. For $x = \frac{1}{2}$, this is $\frac{1+0}{2} = \frac{1}{2}$. So in conclusion, the Fourier series converges to $f(x)$ for every x in $[0,1]$ except $0, \frac{1}{2}$ or 1, where for $x = 0, 1$ it converges to 0 and for $x = \frac{1}{2}$ it converges to $\frac{1}{2}$.
(c) Since $g'(x) = f(x)$, $g(x) = \sum b_n \left(-\frac{1}{n\pi}\cos n\pi x\right) + C$. To get C, $C = a_0 = 2\int_0^1 g(x)dx = \frac{3}{8}$. Therefore,

$$g(x) = \frac{3}{8} + \sum c_n \cos n\pi x$$

where $c_n = -\frac{2}{n^2\pi^2}(1-(-1)^{n/2})$ when n is even and $c_n = -\frac{2}{n^2\pi^2}$.

5. We have the following PDE.

$$\frac{\partial^2}{\partial x^2}u(x,t) = \frac{\partial}{\partial t}u(x,t) \quad (0 < x < 1,\ 0 < t) \tag{1}$$

$$u(0,t) = 0,\ u(1,t) = 1 \quad (0 < t) \tag{2}$$

$$u(x,0) = x + 1 \quad (0 < x < 1) \tag{3}$$

(a) **[6 pts]** Find the steady state solution of the PDE.

(b) **[5 pts]** Let $v(x)$ be the steady state solution and $w(x,t) = u(x,t) - v(x)$. Write down the PDE for $w(x,t)$.

(c) **[9 pts]** Find all nontrivial solutions of

$$\frac{\partial^2}{\partial x^2}w(x,t) = \frac{\partial}{\partial t}w(x,t) \quad (0 < x < 1,\ 0 < t)$$

$$w(0,t) = 0,\ w(1,t) = 0 \quad (0 < t)$$

that has the following form: $w(x,t) = \phi(x)T(t)$.

(d) **[9 pts]** Find the solution $u(x,t)$ of equation (1)-(3).

(a) $v(x) = \lim_{t\to\infty} u(x,t)$. Then,

$$\frac{\partial^2}{\partial x^2}v(x) = \frac{\partial}{\partial t}v(x).$$

and since $\frac{\partial}{\partial t}v(x) = 0$, $\frac{\partial^2}{\partial x^2}v(x) = 0$. Integrating by x twice, we get $v(x) = Ax + B$ for some A and B. The boundary conditions $u(0,t) = 0, u(1,t) = 1$ $(0 < t)$ becomes $v(0) = 0$ and $v(1) = 1$. So $A \cdot 0 + B = 0$ and $A + B = 1$. Solving for A and B we get $A = 1$ and $B = 0$.

So $v(x) = x$.
(b) $u(x,t) = w(x,t) + v(x)$. Now plugging this into the above PDE gives us

$$\frac{\partial^2}{\partial x^2} w(x,t) = \frac{\partial}{\partial t} w(x,t) \quad (0 < x < 1,\ 0 < t)$$
$$w(0,t) = 0, \quad w(1,t) = 0 \ (0 < t)$$
$$w(x,0) = x + 1 - x = 1 \quad (0 < x < 1)$$

(c) The boundary condition gives

$$\phi(0) = 0, \quad \phi(1) = 0$$

We also have

$$\phi''(x)T(t) = \phi(x)T'(t)$$
$$\frac{\phi''(x)}{\phi(x)} = \frac{T'(t)}{T(t)} = -\lambda^2$$

for some real value λ. This can be written as two differential equations

$$\phi''(x) + \lambda^2\phi(x) = 0, \qquad T'(t) + \lambda^2 T(t) = 0.$$

The general solutions are

$$\phi(x) = C_1\cos(\lambda x) + C_2\sin(\lambda x), \quad T(t) = e^{-\lambda^2 t}$$

When boundary condition $\phi(0) = 0$ is used, we get $C_1 = 0$. When we use $\phi(1) = 0$ we get,

$$C_2\sin(\lambda) = 0.$$

If $C_2 = 0$ we get a trivial solution. So $\lambda = n\pi$ for some natural number n. Hence

$$w(x,t) = \phi(x)T(t) = C\sin(n\pi x)e^{-(n\pi)^2 t}$$

(d) Part (c) says that the solution to the differential equation in part (b) is in the form of

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)e^{-(n\pi)^2 t}.$$

The initial condition $w(x,0) = 1$ gives us that

$$1 = \sum_{k=1}^{\infty} c_n \sin(n\pi x)$$

So c_n can be gotten by

$$c_n = 2\int_0^1 \sin(\frac{n\pi}{a}x)dx = \frac{2}{\pi}(1 - (-1)^n)$$

If $w(x,t)$ satisfies the PDE of part (b), $u(x,t) = w(x,t) + v(x)$ satisfies PDE (1)-(3). Therefore,

$$u(x,t) = x + \sum_{n=1}^{\infty} \frac{2}{\pi}(1 - (-1)^n)\sin(n\pi x)e^{-(n\pi)^2 t}$$

6. **[18 pts]** Solve the following PDE.

$$\frac{\partial^2}{\partial x^2}u(x,t) = \frac{\partial}{\partial t}u(x,t) \quad (0 < x < \pi,\ 0 < t)$$

$$u(0,t) = 0,\ \frac{\partial}{\partial x}u(\pi,t) = 0 \quad (0 < t)$$

$$u(x,0) = 1 \quad (0 < x < \pi)$$

You may need the following fact: If $f(x)$ is piecewise smooth on the interval $[0,\pi]$,

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{2n-1}{2}x\right)$$

where

$$c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin\left(\frac{2n-1}{2}x\right)dx.$$

We first look for solutions of the form $u(x,t) = \phi(x)T(t)$ satisfying the heat equation and the boundary conditions. The heat equation gives $\phi''(x)T(t) = \phi(x)T'(t)$ and we get

$$\frac{\phi''(x)}{\phi(x)} = \frac{T'(t)}{T(t)} = -\lambda^2$$

for some real value λ. This can be written as two differential equations

$$\phi''(x) + \lambda^2\phi(x) = 0, \qquad T'(t) + \lambda^2 T(t) = 0.$$

The general solutions are

$$\phi(x) = C_1\cos(\lambda x) + C_2\sin(\lambda x), \quad T(t) = e^{-\lambda^2 t}$$

with boudary condition implying

$$\phi(0) = 0, \quad \phi'(\pi) = 0.$$

Hence we get $C_1 = 0$ and $C_2\cos(\lambda a) = 0$. Since $\phi(x)$ has to be a nontrivial solution, $C_2 \neq 0$ and we get

$$\cos(\lambda\pi) = 0$$

This only holds when $\lambda\pi = \frac{2n-1}{2}\pi$. So we get the solution $u(x,t) = \sin\left(\frac{2n-1}{2}x\right)e^{-\left(\frac{2n-1}{2}\right)^2 t}$. Using the principle of superposition, the following function satisfies the heat equation and the boundary conditions:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\tfrac{2n-1}{2}x\right)e^{-\left(\frac{2n-1}{2}\right)^2 t}.$$

Now, we want $u(x,t)$ to satisfy the initial condition $u(x,0) = 1$, so plugging in $t = 0$ to above, we have

$$1 = \sum_{n=1}^{\infty} c_n \sin\left(\tfrac{2n-1}{2}x\right)$$

So c_n can be calculated by

$$c_n = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin\left(\tfrac{2n-1}{2} x\right) dx = \frac{4}{(2n-1)\pi}$$

and finally

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin\left(\tfrac{2n-1}{2} x\right) e^{-\left(\frac{2n-1}{2}\right)^2 t}.$$