1.a) [**5 pts**]In dimension 2, find the vector $\mathbf{v}$ such that the magnitude is $\|\mathbf{v}\| = 2$ and the angle it makes with the positive x-axis is $\frac{\pi}{4}$, lying in the first quadrant.

The unit vector in the direction described is $(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. To achieve size 2, multiply 2 to get $2(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = (\sqrt{2}, \sqrt{2})$.

b) [**5 pts**]Find the midpoint between $(1, 5, 7)$ and $(-3, 1, 5)$.

$\frac{1}{2}\{(1, 5, 7) + (-3, 1, 5)\} = (-1, 3, 6)$.

c) [**5 pts**]Find $(\mathbb{i} - \mathbb{k}) \times (\mathbb{i} - \mathbb{j})$ where $\mathbb{i}$,$\mathbb{j}$, $\mathbb{k}$ are standard unit vectors.

$$\begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} = (0\cdot 0 - (-1)\cdot(-1))\mathbb{i} - (1\cdot 0 - 1\cdot(-1))\mathbb{j} + (1\cdot(-1) - 0\cdot 1)\mathbb{k} = -\mathbb{i} - \mathbb{j} - \mathbb{k}$$

2. a) [**5 pts**]Find both parametric and symmetric equation of the line passing through $(1, 2, 3)$ and is perpendicular to $2x + y - 2z = 4$.

The normal vector to $2x + y - 2z = 4$ is $(2, 1, -2)$. So we have a line to the direction $(2, 1, -2)$ passsing $(1, 2, 3)$, $\mathbf{r}(t) = (1, 2, 3) + t(2, 1, -2) = (1 + 2t, 2 + t, 3 - 2t)$. So the answer in parametric form is : $x = 1 + 2t$, $y = 2 + t$, and $z = 3 - 2t$. Solving these equations by t we get, $t = \frac{x-1}{2}$, $t = y - 2$ and $t = \frac{3-z}{2}$. So $\frac{x-1}{2} = y - 2 = \frac{3-z}{2}$.

b) [**5 pts**]Find an equation given in rectangular coordinates for the equation $r = 3\cos\theta$ given in cylindrical coordinates.

Multiply r to each side and get $r^2 = 3r\cos\theta$. Now, $r^2 = x^2 + y^2$ and $r\cos\theta = x$, so $x^2 + y^2 = 3x$.

c) **[5 pts]**Find the indefinite integral $\int \sin 2t\hat{\imath} + \frac{1}{\sqrt{1-t^2}}\hat{\jmath}\, dt$

Looking at the integration table included in the cover page, we get the answer :

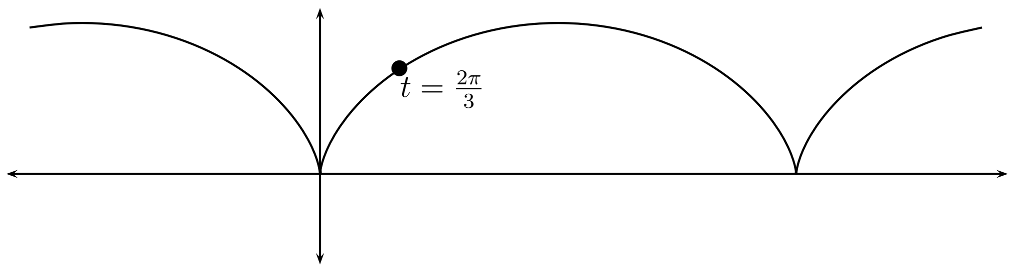
$$\left(-\frac{1}{2}\cos(2t) + C_1\right)\hat{\imath} + (\arcsin t + C_2)\,\hat{\jmath}$$

Check also that differentiating $-\frac{1}{2}\cos(2t)$ gives you $\sin 2t$.

3. **[8 pts]**The graph of $y = x^2$ can be parametrized as $\mathbb{r}(t) = (t, t^2)$. Find the curvature at $(1, 1)$.

$$K = \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}}$$

$y' = 2x$, and $y'' = 2$. So when $x = 1$ and $y = 1$, $K = \frac{2}{(1+2^2)^{\frac{3}{2}}} = \frac{2}{5^{\frac{3}{2}}}$.



4. **[12 pts]** Above is the graph of $\mathbf{r}(t) = (t - \sin t, 1 - \cos t)$. Find its unit tangent vector and its principal unit normal vector at $t = \frac{2\pi}{3}$.
(**Hint** You might want to use the following fact: $\|\mathbf{r}'(t)\| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t} = 2\sin\frac{t}{2}$ when $0 \le t \le \pi$.)

$\mathbf{r}'(t) = (1 - \cos t, \sin t)$. Hence $T(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{(1-\cos t, \sin t)}{2\sin\frac{t}{2}}$. Evaluating $\mathbf{T}(t)$ at $t = \frac{2\pi}{3}$ gives $\mathbf{T}(\frac{2\pi}{3}) = \frac{(1-\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3})}{2\sin\frac{pi}{3}} = \frac{(1+1/2, \sqrt{3}/2)}{\sqrt{3}} = (\sqrt{3}/2, 1/2)$. Now the principal normal vector is a unit vector that is orthogonal to this vector, so it is either $(1/2, -\sqrt{3}/2)$ or $-(1/2, -\sqrt{3}/2)$. Then inspecting the graph, we see that the one that points to the concave side is $(1/2, -\sqrt{3}/2)$. Answer: $\mathbf{T}(\frac{2\pi}{3}) = (\sqrt{3}/2, 1/2)$ and $\mathbf{N}(\frac{2\pi}{3}) = (1/2, -\sqrt{3}/2)$.

5. **[12 pts]**$f(x,y,z) = xy + yz$, $x(s,t) = s+t$, $y(s,t) = s-t$ and it is known that z does not depend on t. Suppose also that $\frac{\partial z}{\partial s} = s^2$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial z}{\partial z}\frac{\partial z}{\partial s} = y \cdot 1 + (x+z) \cdot 1 + y \cdot s^2 = y + x + z + ys^2$$

For $\frac{\partial f}{\partial t}$, first notice that $\frac{\partial z}{\partial t} = 0$ since z does not depend on t.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial z}\frac{\partial z}{\partial t} = y \cdot 1 + (x+z) \cdot (-1) + y \cdot 0 = y - x - z$$

6. **[8 pts]** $xy + z\sin y - x^2 = 0$. Find $\frac{\partial z}{\partial y}$ implicitly.

Differentiate both sides by y treating z as a function of x and y and treating x as a constant.

$$x + \frac{\partial z}{\partial y}\sin y + z\cos y = 0$$

. Note that when differentiating the $z\sin y$ term, we have used the product rule.
Now solve for $\frac{\partial z}{\partial y}$ and get

$$\frac{\partial z}{\partial y} = -\frac{x + z\cos y}{\sin y}$$

7. [**10 pts**]Find the equation of the tangent plane of $z = x^2 - 2y^2$ at $(1, 1, -1)$.

Let $F(x, y, z) = x^2 - 2y^2 - z$. Then the above surface is a level surface of $F(x, y, z)$. So we can use the property that gradients are perpendicular to level surfaces. $\nabla F = (2x, -4y, -1)$. So at $(1, 1, -1)$, $\nabla F(1, 1, -1) = (2, -4, -1)$. Hence we found the normal vector to the plane that we are considering. Now, use the standard equation of plane $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$. Answer : $2(x - 1) - 4(y - 1) - (z + 1) = 0$.

8. [**8 pts**] $f(x, y) = x^2 - 2y^2$ and $\mathbf{u} = \cos\frac{\pi}{3}\hat{\imath} + \sin\frac{\pi}{3}\hat{\jmath}$. Find $D_{\boldsymbol{u}}f(1, 1)$.

$\nabla f = (2x, -4y)$. $\mathbf{u} = (1/2, \sqrt{3}/2)$. So $D_{\boldsymbol{u}}f(1, 1) = \nabla f(1, 1) \cdot \mathbf{u} = (2, -4) \cdot (1/2, \sqrt{3}/2) = 1 - 2\sqrt{3}$.

9. **[12 pts]** Find all critical points of the function $f(x, y) = 6xy - x^3 - y^2$. For each critical point, determine whether it is a saddle point, a relative minimum or a relative maximum by using the second partial test.

$\nabla f = (6y - 3x^2, 6x - 2y)$. To find critical points, we solve $\nabla f = 0$. So we get system of equations

$$6y - 3x^2 = 0$$

$$6x - 2y = 0$$

The second equation can be solved for y as $y = 3x$. Plugging this into the first equation, we get $18x - 3x^2 = 0$. This can be factored as $3x(6 - x) = 0$. So we have two solutions $x = 0$ and $x = 6$. Since $y = 3x$, we have $(0, 0)$ and $(6, 18)$ as the critical points.
Now let us compute $d = f_{xx}f_{yy} - (f_{xy})^2$. $d = (-6x)^2 \cdot (-2) - 6^2 = 12x - 36$. So at the point $(0, 0)$, $d = -36 < 0$ and we see that $(0, 0)$ is a saddle point. For $(6, 18)$, we get $d = 36 > 0$ and $f_{xx} = -6 \cdot 6 = -36 < 0$. So $(6, 18)$ is a relative minima.