1. **[10 pts]** Let $f(x,y,z) = x^2y + z$. x,y,z are functions of t and they satisfy $\frac{dx}{dt} = \frac{1}{2}$, $\frac{dy}{dt} = 1$ and $\frac{dz}{dt} = 2$. Find $\frac{df}{dt}$ when $(x,y,z) = (1,2,-1)$.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 2xy \cdot \frac{1}{2} + x^2 \cdot 1 + 1 \cdot 2 \mid_{(x,y,z)=(1,2,-1)} = 2 + 1 + 2 = 5$$

2. **[15 pts]** Find the minimum value of $f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$ under constraint $x + y + z = 3$.

The Lagrange multiplier says $\nabla f = \lambda \nabla g$, where g is the constraint, i.e. $g(x,y,z) = x + y + z$.

$$\nabla f = x\mathbb{i} + y\mathbb{j} + z\mathbb{k}$$

$$\nabla g = \mathbb{i} + \mathbb{j} + \mathbb{k}.$$

So the Lagrange multiplier reads $x\mathbb{i} + y\mathbb{j} + z\mathbb{k} = \lambda(\mathbb{i} + \mathbb{j} + \mathbb{k})$. Comparing componentwise, we have $x = \lambda$,$y = \lambda$ and $z = \lambda$. Now plug in these relations to the constraint $x+y+z = 3$ to get $3\lambda = 3$, i.e. $\lambda = 1$. Hence the minimum, if it exists, must be obtained when $x = y = z = 1$. Hence $f(1,1,1) = \frac{3}{2}$ is the minimum.

3. **[15 pts]** Evaluate the integral $\int_0^1 \int_{x^2}^1 \frac{3}{2} e^{y^{3/2}}\, dydx$. Note that it is necessary to change the order of integration.

$$\int_0^1 \int_{x^2}^1 \frac{3}{2} e^{y^{3/2}}\, dydx = \int_0^1 \int_0^{\sqrt{y}} \frac{3}{2} e^{y^{3/2}}\, dxdy$$

$$= \int_0^1 \frac{3}{2} e^{y^{3/2}} x \mid_{x=0}^{x=\sqrt{y}} dy = \int_0^1 \frac{3}{2} \sqrt{y} e^{y^{3/2}} dy$$

Using change of variables $u = y^{3/2}$, we have $du = \frac{3}{2}\sqrt{y}dy$ and hence the integral is equal to

$$= \int_0^1 e^u du = e^u \mid_0^1 = e^1 - e^0 = e - 1$$

4. **[10 pts]** Let C be a line parametrized by $\mathbf{r}(t) = (t, 2t)$, $0 \leq t \leq 1$. Evaluate the integral $\int_C (x + y) ds$.

$\int_C (x+y)ds = \int_0^1 (x+y)\frac{ds}{dt}dt$. Now $\frac{ds}{dt} = \| \frac{d\mathbf{r}}{dt} \| = \|(1,2)\| = \sqrt{1^2 + 2^2} = \sqrt{5}$. So $\int_C (x + y)ds = \int_0^1 (x + y)\frac{ds}{dt}dt = \int_0^1 (t + 2t)\sqrt{5}dt = \frac{3}{2}t^2\sqrt{5} \mid_0^1 = \frac{3\sqrt{5}}{2}$.

5.a) [**10 pts**]Let $\mathbf{F}(x,y) = (2xy+1)\mathbb{i} + (x^2+2y)\mathbb{j}$. Find a function $f(x,y)$ such that $\nabla f(x,y) = \mathbf{F}(x,y)$.

$$f(x,y) = \int (2xy+1)dx = x^2y + x + g(y).$$
$$\frac{\partial f}{\partial y}(x,y) = x^2 + g'(y)$$

So $g'(y)$ must equal to $2y$. One such choice for $g(y)$ is $g(y) = y^2$. Hence $f(x,y) = x^2y + x + y^2$ satisfies $\nabla f(x,y) = \mathbf{F}(x,y)$. (Note: More generally, for any constant k, $f(x,y) = x^2y + x + y^2 + k$ satisfies the equation.)

b) [**10 pts**] Let C be the curve parametrized as $\mathbf{r}(t) = (4t^4 - 4t^3 + 2t^2 - t, t^4 + t^3 - t^2)$, $0 \le t \le 1$. Evalutate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Hint: Use the fact that the field $\mathbf{F}$ is conservative.)

The "if-you-had-all-the-time-in-the-world" way)

$$\frac{d\mathbf{r}}{dt} = (16t^3 - 12t^2 + 4t - 1, 4t^3 + 3t^2 - 2t)$$

$$\mathbf{F}(\mathbf{r}(t)) = (1+2t^3-6t^4+10t^5-12t^6+8t^8, -t^2-2t^3+14t^4-24t^5+32t^6-32t^7+16t^8)$$

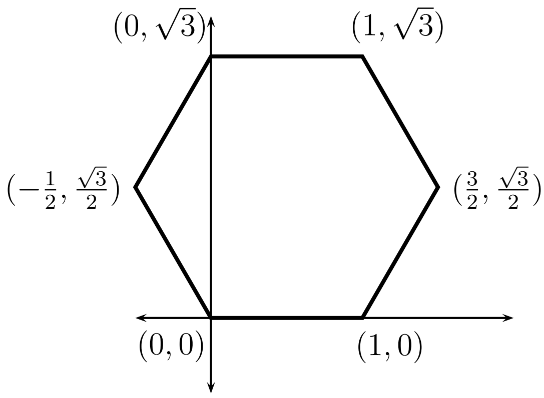
So $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (1 + 2t^3 - 6t^4 + 10t^5 - 12t^6 + 8t^8, -t^2 - 2t^3 + 14t^4 - 24t^5 + 32t^6 - 32t^7 + 16t^8) \cdot (16t^3 - 12t^2 + 4t - 1, 4t^3 + 3t^2 - 2t)dt = \int_0^1 -1 + 4t - 12t^2 + 16t^3 + 15t^4 - 96t^5 + 238t^6 - 344t^7 + 360t^8 - 160t^9 - 176t^10 + 192t^11\, dt = 3$.

The "use-the-property-of-conservative-field" way)

Note that $\mathbf{r}(0) = (0,0)$ and $\mathbf{r}(1) = (1,1)$. So the curve joins $(0,0)$ and $(1,1)$. Since the value of the integral stays the same even if different path is chosen, we use the straight line from $(0,0)$ to $(1,1)$ instead. This is parametrized as $\mathbf{r}(t) = (t,t)$ with $0 \le t \le 1$. $\frac{d\mathbf{r}}{dt} = (1,1)$ and $\mathbf{F}$ evaluated on this path is $\mathbf{F}(\mathbf{r}(t)) = (2t^2+1, t^2+2t)$. Therefore, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^2+1, t^2+2t) \cdot (1,1)dt = \int_0^1 3t^2 + 2t + 1dt = 3$.

The easiest solution)

We use part a). $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1) - f(0,0) = 1^21 + 1 + 1^2 - 0 = 3$.



6. a) **[15 pts]** Let C be the path starting at point $(0,0)$, going around the hexagon shown left in counterclockwise direction and ending when it returns to point $(0,0)$. Evaluate the integral

$$\int_C (x+y)dx + (3x-2y)dy$$

(Hint: Use the Green's theorem and the fact that the area of the hexagon is $\frac{3\sqrt{3}}{2}$.)

By Green's theorem,

$$\int_C (x+y)dx + (3x-2y)dy = \int_{\hexagon}\int \frac{\partial}{\partial x}(3x-2y) - \frac{\partial}{\partial y}(x+y)dxdy$$

$$= \int_{\hexagon}\int 2dxdy = 2\int_{\hexagon}\int dxdy = 2\text{Area}(\hexagon) = 3\sqrt{3}.$$

b) **[8 pts]** Let C_1 be the line on the x-axis, from $(0,0)$ to $(1,0)$. Evaluate the integral $\int_{C_1}(x+y)dx + (3x-2y)dy$

We parametrize the line as $\mathbf{r}(t) = (x(t), y(t)) = (t, 0)$, $0 \le t \le 1$.
$\int_{C_1}(x+y)dx + (3x-2y)dy = \int_0^1 \left((x+y)\frac{dx}{dt} + (3x-2y)\frac{dy}{dt}\right) dt$
$= \int_0^1 (t+0)\cdot 1 + (3t - 2\cdot 0)\cdot 0dt = \int_0^1 tdt = \frac{1}{2}t^2 \mid_0^1 = \frac{1}{2}.$

c)* [**10 pts**] Let C_2 be the path on the hexagon shown in part a), that begins at point $(1,0)$, follows the hexagon counterclockwise and ends at point $(0,0)$. (So this path does not pass through the line joining $(0,0)$ and $(1,0)$). Evaluate the integral $\int_{C_2}(x+y)dx+(3x-2y)dy$

Since path C is same as following path C_1 and then C_2, $\int_C = \int_{C_1} + \int_{C_2}$. So

$$\int_{C_2}(x+y)dx+(3x-2y)dy = \int_C (x+y)dx+(3x-2y)dy - \int_{C_1}(x+y)dx+(3x-2y)dy$$

$$= 3\sqrt{3} - \frac{1}{2}.$$

7. [**14 pts**] The integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} dzdydx$$

is given. Change the order of integration to $dxdydz$. **DO NOT** evaluate the integral. (Hint: Draw the xy-trace,yz-trace,xz-trace of $z = 1 - x^2 - y^2$.)

Answer :

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{\sqrt{1-y^2-z}} dxdydz$$

8. Let S be the lamina bounded by the half circle $y = \sqrt{1-x^2}$ and the x-axis. The density of the lamina is $\rho = 3\sqrt{x^2+y^2} + 3x$.

a) **[15 pts]** Find the mass of the lamina S. (You might find it easier to compute in polar coordinates.)

$Mass = \int\int \rho dxdy$. So mass is

$$\int_0^{\pi}\int_0^1 (3r + 3r\cos\theta) r dr d\theta = \int_0^{\pi} 1 + \cos\theta d\theta = \pi$$

b) **[7 pts]** Setup, but **do not evaluate**, the integral that gives you the x-coordinate of the center of mass. (Zero points will be given if you write the formula for y-coordinate instead. So be careful.)

$$\overline{x} = \frac{1}{\pi}\int_0^{\pi}\int_0^1 r\cos\theta(3r + 3r\cos\theta) r dr d\theta$$

9. **[15 pts]** Let D be the surface in three dimensional space parametrized by $\mathbf{r}(u,v) = (\cos u, v + \sin v, \sin u)$ with $0 \le u \le \pi$ and $0 \le v \le 2\pi$. Find

$$\int\int_D z\,dS$$

We use the fact that $dS = \sqrt{\det A^t A} dudv$. Where A is the matrix

$$A = D\mathbf{r} = \begin{pmatrix} \frac{\partial}{\partial u}\cos u & \frac{\partial}{\partial v}\cos u \\ \frac{\partial}{\partial u}(v+\sin v) & \frac{\partial}{\partial v}(v+\sin v) \\ \frac{\partial}{\partial u}\sin u & \frac{\partial}{\partial v}\sin u \end{pmatrix} = \begin{pmatrix} -\sin u & 0 \\ 0 & 1+\cos v \\ \cos u & 0 \end{pmatrix}$$

Therefore

$$A^t A = \begin{pmatrix} -\sin u & 0 & \cos u \\ 0 & 1+\cos v & 0 \end{pmatrix} \begin{pmatrix} -\sin u & 0 \\ 0 & 1+\cos v \\ \cos u & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2 u + \cos^2 u & 0 \\ 0 & (1+\cos v)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (1+\cos v)^2 \end{pmatrix}$$

$$\det A^t A = \begin{vmatrix} 1 & 0 \\ 0 & (1+\cos v)^2 \end{vmatrix} = (1+\cos v)^2$$

Hence

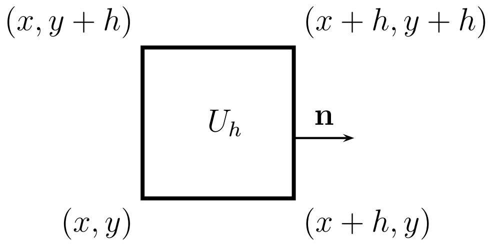
$$\int\int_D z dS = \int_0^{2\pi}\int_0^{\pi} \sin u \sqrt{(1+\cos v)^2} dudv$$

$$= \int_0^{2\pi} -\cos u(1+\cos v)|_0^{\pi} dv = 2\int_0^{2\pi} 1+\cos v dv = 4\pi$$

Note: There is another way of solving this problem, following the text book as $dS = \|\mathbf{r}_u \times \mathbf{r}_v\| dudv$. However, in this case one has to compute a determinant of a 3 by 3 matrix rather than 2 by 2 as above. So the amount of computation is more and it is more complicated. Still, if carried out carefully, one obtains the same answer as above, i.e. $dS = (1+\cos v)dudv$.

10. a) **[15 pts]** Let

$$\mathbf{F}(x,y) = (u(x,y), v(x,y)).$$



Let U_h be the square region with the four corners being (x,y),$(x+h,y)$,$(x+h,y+h)$ and $(x,y+h)$. Prove that

$$\lim_{h\to 0} \frac{\text{Flux}_F(U_h)}{\text{Area}(U_h)} = \text{div}\mathbf{F}$$

where $\text{Flux}_F(U_h) = \int_{\partial U_h} \mathbf{F}\cdot\mathbf{n}\, ds$ and $\mathbf{n}$ is the outward unit normal vector.

See the separate pdf file for this.

b) **[6 pts]** Write down the first and second alternative form of Green's theorem. (The first alternative form is often called the curl form of Green's theorem and the second alternative form is often called the divergence form of Green's theorem).

$$\int_C \mathbf{F} d\mathbf{r} = \int \int_S \text{Curl}\mathbf{F} \cdot \mathbb{k} dxdy$$
$$\int_C \mathbf{F} \cdot \mathbf{N} ds = \int \int_S \text{Div}\mathbf{F} dxdy$$

c)* **Extra Credit [10 pts]** Explain why the divergence form of Green's theorem holds. Your explanation should have part a) involved.

For any given simply connected domain S, we can subdivide it by infinitely small squares. On each infinitesimal square, part a) says that

$$\text{Flux}_{\mathbf{F}} = \text{div}\mathbf{F} dA.$$

Now, if we add this up for all the small squares, the left side turns into a line integral of the boundary because the interiors cancel out, and the right side turns into a double integral. This yields

$$\int_C \mathbf{F} \cdot \mathbf{N} ds = \int \int_S \text{Div}\mathbf{F} dA$$

11. Let $\mathbf{F}(x,y,z) = (-y+z)\mathbb{i} + (x-z)\mathbb{j} + (x-y)\mathbb{k}$. Let surface S be the portion of the surface $z = 4 - x^2 - y^2$ satisfying $z \geq 0$. Let C be the boundary of this surface, oriented counterclockwise. Verify the Stoke's theorem as follows:

a) **[10 pts]** C can be parametrized as $\mathbf{r}(t) = (2\cos t, 2\sin t, 0)$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly, without using the Stoke's theorem.

The curve C goes around a full circle, so that $0 \leq t2\pi$.

$$\frac{d\mathbf{r}}{dt} = (-2\sin t, 2\cos t, 0)$$

$\mathbf{F}$ evaluated on the curve $\mathbf{r}(t)$ is

$$\mathbf{F}(\mathbf{r}(t)) = (-2\sin t + 0)\mathbb{i} + (2\cos t - 0)\mathbb{j} + (2\cos t - 2\sin t)\mathbb{k}$$

hence we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt = \int_0^{2\pi} (-2\sin t\mathbb{i} + 2\cos t\mathbb{j} + (2\cos t - 2\sin t)\mathbb{k})\cdot(-2\sin t, 2\cos t, 0)dt$$

$$= \int_0^{2\pi} 4\sin^2 t + 4\cos^2 t = \int_0^{2\pi} 4dt = 8\pi$$

b) [**15 pts**] Now compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ using a different method, as a surface integral, using the Stoke's theorem.

Stoke's theorem says that $\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S \text{curl}\mathbf{F} \cdot \mathbf{N} dS$. So let us verify the double integral.

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y+z & x-z & x-y \end{vmatrix} = 2\mathbb{k}$$

The upward normal vector can be gotten by taking the surface as a level surface of $g(x,y,z) = z + x^2 + y^2 - 4$ and taking its gradient $\nabla g = (2x, 2y, 1)$. The unit upward normal vector is then $\mathbf{N} = \frac{(2x,2y,1)}{\sqrt{4x^2+4y^2+1}}$. We also need to know what dS is. When the surface S is given by $z = f(x,y)$, $dS = \sqrt{1 + f_x^2 + f_y^2} dxdy$. So $dS = \sqrt{1 + 4x^2 + 4y^2} dxdy$. Now we know everything to compute this integral.

$$\int\int_S \text{curl}\mathbf{F} \cdot \mathbf{N} dS = \int\int_{\bigcirc} (2\mathbb{k}) \cdot \frac{(2x, 2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{1 + 4x^2 + 4y^2} dxdy$$

$$= \int\int_{\bigcirc} 2dxdy = 2\text{Area}(\bigcirc) = 8\pi$$