

Let $F(x, y) = (u(x, y), v(x, y))$ be a vector field on \mathbb{R}^2 . Let U_h be a square region with corners on $(x, y), (x+h, y), (x+h, y+h)$ and $(x, y+h)$.

Prove that $\lim_{h \rightarrow 0} \frac{\text{Flux}_F(U_h)}{\text{Area}(U_h)} = \text{Div } F$.

Proof) By definition,

$$\text{Flux}_F(U_h) = \oint_{\partial U_h} (F \cdot \vec{n}) ds.$$

∂U_h , the boundary of U_h , consists of four lines

C_1, C_2, C_3 and C_4 as shown right.

The outward unit normal vector \vec{n} for each line is

$(0, -1), (1, 0), (0, 1)$ and $(-1, 0)$ respectively.

Therefore,

$$\begin{aligned} \text{Flux}_F(U_h) &= \int_{C_1} (F \cdot (0, -1)) ds + \int_{C_2} (F \cdot (1, 0)) ds + \int_{C_3} (F \cdot (0, 1)) ds + \int_{C_4} (F \cdot (-1, 0)) ds \\ &= \int_x^{x+h} (u(t, y), v(t, y)) \cdot (0, -1) dt + \int_y^{y+h} (u(x+h, t), v(x+h, t)) \cdot (1, 0) dt \\ &\quad + \int_x^{x+h} (u(t, y+h), v(t, y+h)) \cdot (0, 1) dt + \int_y^{y+h} (u(x, t), v(x, t)) \cdot (-1, 0) dt \\ &= -\int_x^{x+h} v(t, y) dt + \int_y^{y+h} u(x+h, t) dt + \int_x^{x+h} v(t, y+h) dt - \int_y^{y+h} u(x, t) dt \end{aligned}$$

Applying the first fundamental theorem of calculus (i.e. $\lim_{h \rightarrow 0} \frac{1}{h} \int_a^{a+h} f(t) dt = f(a)$) we get:

$$\lim_{h \rightarrow 0} \frac{1}{h} \text{Flux}(U_h) = -v(x, y) + u(x+h, y) + v(x, y+h) - u(x, y).$$

$$\begin{aligned} \text{Hence } \lim_{h \rightarrow 0} \frac{\text{Flux}_F(U_h)}{\text{Area}(U_h)} &= \lim_{h \rightarrow 0} \frac{\text{Flux}_F(U_h)}{h^2} = \lim_{h \rightarrow 0} \frac{\frac{1}{h} \text{Flux}_F(U_h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-v(x, y) + u(x+h, y) + v(x, y+h) - u(x, y)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(x+h, y) - u(x, y)}{h} + \lim_{h \rightarrow 0} \frac{v(x, y+h) - v(x, y)}{h} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \text{Div } F \end{aligned}$$

(QED)

