Let U, V be unitary n×n matrices such that UV = ζVU, where ζ is a primitive n-th root of unity. Prove that any matrix that commutes with both U and V must be a multiple of the identity matrix I.

Proof)

Suppose v is a eigenvector of U with eigen value α. Then

$UVv = \zeta VUv = \zeta V(\alpha v) = \alpha\zeta Vv$.

Hence Vv is again an eigenvector of U.

Repeating this process, we find that $V^k v$ is always an eigenvector of U with eigenvalue

$\alpha\zeta^k$.

Define $v_0=v$, $v_k = V^k v$ for k=1,2,3,... .

Since ζ is a primitive n-th root of unity, all these eigenvalues are distinct and hence the eigenvectors $v_0, v_1, \ldots\ldots,$ v_{n-1} are linearly independent. Hence the following matrix is invertible.

$$T=\begin{pmatrix} | & | & & & | \\ v_0 & v_1 & \cdots & \cdots & v_{n-1} \\ | & | & & & | \end{pmatrix}$$

Verify by easy computation that $UT = T\begin{pmatrix} \alpha & & & & \\ & \alpha\zeta & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \alpha\zeta^{n-1} \end{pmatrix}$.

So we have $T^{-1}UT=\begin{pmatrix} \alpha & & & & \\ & \alpha\zeta & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \alpha\zeta^{n-1} \end{pmatrix}$.

Now, we also have defined our v_k's so that $v_{k+1} = Vv_k$, and also note that v_n must be a multiple of v_0, since it has the same eigenvalue as v_0 and all eigenvalues of U are distinct (i.e. if a n×n matrix has n distinct eigenvalues, then any two eigenvectors having the same eigenvalue must be multiples of one another. Simple exercise.) So $v_n = \beta v_0$ for some β.

This implies that

$$VT=T\begin{pmatrix} 0 & & & & \beta \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}.$$

So if we had any matrix B commuting with both U and V, then

BU=UB, BV=VB. So we have two equalities

1) $T^{-1}BT\, T^{-1}UT = T^{-1}UT\, T^{-1}BT$

2) $T^{-1}VT\, T^{-1}BT = T^{-1}BT\, T^{-1}VT$.

From the first equality, you get that $T^{-1}BT$ must be diagonal and from the second equality, you get that all the diagonal entries of $T^{-1}BT$ must be same.

(I will not demonstrate the detailed calculation since this is almost straight forward. Be convinced yourself by writing down each matrix carefully.)

So we must have $T^{-1}BT = \lambda I$. Hence $B = T\lambda I\, T^{-1} = \lambda I$. ■