Problem. Let M be the set of all n×n matrices over $\mathbb{C}$. Prove that every algebraic automorphisms of M is an inner-automorphism (i.e. if $\phi : M \to M$ is an automorphism then there is a $g \in M$ such that $\phi(A) = gAg^{-1}$ for $\forall A \in M$.)

Preliminary)

Before proving, we need to make some definitions. First, I will denote E_{ij} as a matrix which every elements are zero except one element at i-th row, j-th column, which is 1.

$$\begin{array}{c} \\ i-th \rightarrow \end{array} \begin{pmatrix} 0 & & \overset{\displaystyle j-th \atop \downarrow}{} & \\ & \ddots & & \\ & & 1 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

Also I will denote D as a diagonal matrix which has diagonal entries as 1 through n.

$$\begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & \ddots & \\ & & & & n \end{pmatrix}$$

Some equalities to check by direct calculation:

1) $E_{ij}E_{kl} = \delta_{jk}E_{il}$
2) $DE_{ij} = iE_{ij}$
3) $E_{ij}D = jE_{ij}$

I also need a wide known, easy theorem from elementary linear algebra.

Theorem. If an n×n matrix has n distinct eigenvalues, then there exists n linearly independent eigenvectors, hence it can be diagonalized. The eigenvectors form a basis.

And following is an easy exercise.

Lemma. If an n×n matrix has n distinct eigenvalues, any two eigenvectors having the same eigenvalues are collinear.

Proof) Since $DE_{ij} = iE_{ij}$, we have $\phi(D)\phi(E_{ij}) = \phi(DE_{ij}) = i\phi(E_{ij})$

This means that the column vectors of $\phi(E_{ij})$ are the right eigenvectors of $\phi(D)$ with eigenvalue i. Doing this process for all $i = 1, 2, 3 \cdots, n$ we find that $\phi(D)$ has n distinct eigenvalues, hence it has n linearly independent eigenvectors $v_{1,} v_{2,} \cdots, v_n$. Reinvestigating $\phi(E_{ij})$, since all column vectors of $\phi(E_{ij})$ are eigenvectors of the same eigenvalues, by lemma, we see that all the column vectors of $\underline{\phi(E_{ij})}$ are multiples of $\underline{v_i}$.

Again using $E_{ij}D = jE_{ij}$, same argument above will lead us to find left eigenvectors $w_1^T, w_2^T, \cdots, w_n^T$ (because left eigenvectors are row vectors, these are transposed so that they will become row vectors.) We then conclude also that all the row vectors of $\phi(E_{ij})$ are multiples of w_i^T.

So we conclude that $\phi(E_{ij}) = c_{ij} v_i w_j^T$ for some non-zero constant c_{ij}. (note that $v_i w_j^T$ is an n×n matrix while $w_j^T v_i$ is a scalar.)

From $E_{ij}E_{kl} = \delta_{jk}E_{il}$, we have $c_{ij}c_{kl}v_i w_j^T v_k w_l^T = \delta_{jk}c_{kl}v_i w_l^T$. Define $d_{jk} = w_j^T v_k$. Subtract right from left to get $(c_{ij}c_{kl}d_{jk} - \delta_{jk}c_{il})v_i w_l^T = 0$, and since $v_i w_j^T$ is not a zero matrix, $c_{ij}c_{kl}d_{jk} - \delta_{jk}c_{il} = 0$.

Let us inspect some special cases of this equation.

1) If j≠k then $c_{ij}c_{kl}d_{jk} = 0$. Hence we have

Rule 1. $d_{jk} = 0$ if j≠k.

2) If i=j=k=l then $c_{ii}c_{ii}d_{ii} - c_{ii} = 0$. Hence

Rule 2. $d_{ii}c_{ii} = 1$

3) If i=l, j=k then $c_{ij}c_{ji}d_{jj} - c_{ii} = 0$. So.

Rule 3. $c_{ij}c_{ji}d_{jj} = c_{ii}$

Now define, $\mu_i = c_{i1}v_i$ and $\eta_j = c_{1j}d_{11}w_j$. Then we have the following:

#1. $$\begin{aligned}\mu_i\eta_j^T &= c_{i1}c_{1j}d_{11}v_i w_j^T = c_{i1}c_{1j}(w_1^T v_1)v_i w_j^T = c_{i1}c_{1j}v_i(w_1^T v_1)w_j^T = c_{i1}c_{1j}(v_i w_1^T)(v_1 w_j^T)\\ &= \phi(E_{i1})\phi(E_{1j}) = \phi(E_{i1}E_{1j}) = \phi(E_{ij})\end{aligned}$$

#2. $\mu_i^T\eta_i = c_{i1}c_{1i}d_{11}v_i^T w_i = c_{i1}c_{1i}d_{11}d_{ii} = c_{ii}d_{ii} = 1$. (Here, we used rule 3 : $c_{i1}c_{1i}d_{11} = c_{ii}$ and rule 2.)

#3. if i≠j, $\mu_i^T\eta_j = c_{i1}c_{1j}d_{11}(v_i^T w_j) = c_{i1}c_{1i}d_{11}\cdot 0 = 0$

So let us define matrix $g = \begin{pmatrix} | & | & & | \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ | & | & & | \end{pmatrix}$ and $h^T = \begin{pmatrix} | & | & & | \\ \eta_1 & \eta_2 & \cdots & \eta_n \\ | & | & & | \end{pmatrix}$.

#2 and #1 gives us $gh = I$. So $h = g^{-1}$.

Also, check directly that $gE_{ij}h = \mu_i\eta_j^T$.

So from #1 we know that $\phi(E_{ij}) = \mu_i\eta_j^T = gE_{ij}h = gE_{ij}g^{-1}$ for all i,j.

Since E_{ij}'s generate M, we have $\phi(A) = gAg^{-1}$. ■