Diagonalizing 2×2 Matrices

The purpose of this document is to prepare you for the early exam.

Problem 1 Diagonalize the matrix $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

Step 1. (Find eigenvalues)

To find the eigenvalue, one solves the eigenvalue equation

$$\boxed{Det[A - \lambda I] = 0}$$

This is the equation to solve when one wants to find all the eigenvalues of any n×n matrix. 'Det' stands for determinant and there are some complicated formulas for general n×n matrices. In our case we only need to know that $Det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

Let's use this formula to this problem.

$$Det[A - \lambda I] = Det\left[\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = Det\begin{pmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) - 2\cdot 1$$

So the eigenvalue equation becomes $\lambda^2 - 5\lambda + 4 = 0$.

Using the quadratic formula $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2}$, we get $\lambda = 1$ or $\lambda = 4$.

So we get the eigenvalues 1 and 4.

Step 2. (Find eigenvectors)

For each eigenvalue λ there is at least one non-zero column vector which satisfies $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$. This equation always turns out to be under-determined, which means you have infinitely many solutions. So you may set $x = 1$ and then solve for y. (If it does not work then set $y = 1$ instead.)

Let's look at how it's done.

① for the eigenvalue $\lambda = 1$,

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

So we have $\begin{pmatrix} 2x+2y \\ 1 \cdot x+3y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Equating the first row we get $2x+2y=x$

Equating the second row we get $x+3y=y$

They both simplify as $x+2y=0$, so we actually have only one equation and two unknowns (see what I mean by under-determined?).

Since we are looking for just one solution among infinitely many possible solutions, we set $x=1$, and then from $x+2y=0$, we get $y=-1/2$.

Writing the answer down as a column matrix, we have the eigenvector for the eigenvalue 1

$$\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

② for the eigenvalue $\lambda = 4$,

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4 \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Setting $x=1$, and equating the top row, we get

$2 \cdot 1 + 2y = 4 \cdot 1$

Hence $y=1$ and the eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Step 3. Diagonalizing

So far we have found that following equations hold.

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(Try doing the actual calculation to see that these are correct. If not, please email me so I can correct it!)

Now, the above matrix equations can be put into one matrix equation as

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

(You may want to pause here - to check that this equation is same as the prior equations.)

Now, all we have to do is to multiply the inverse of $\begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & 1 \end{pmatrix}$ each side. Then we get a nice looking identity :

$$\boxed{\begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & 1 \end{pmatrix}^{-1}\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}}$$

This the diagonalization of the matrix $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

Since the above solution contains a lot of explanations, let me do a second example to show you how you should solve a problem like this.

Problem 2. Diagonalize $\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$

Step 1. (Find eigenvalues)

$$Det\begin{pmatrix} 1-\lambda & 2 \\ 4 & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda) - 2\cdot 4 = \lambda^2 - 9$$

$\lambda^2 - 9 = 0$

$\lambda = 3$ or $\lambda = -3$

Step 2. (find eigenvectors)

① $\lambda = 3$

$$\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3 \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

top row : $x + 2y = 3x$

let $x = 1$, then $1 + 2y = 3$, so $y = 1$

$$\therefore \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

② $\lambda = -3$

$$\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3 \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

top row : $x + 2y = -3x$

let $x = 1$, then $1 + 2y = -3$, so $y = -2$

$$\therefore \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = -3 \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Step 3. Diagonalize

From step 2, combining results of ① and ②, we get

$$\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$$

Multiplying $\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}^{-1}$ to the left of each side, we get

$$\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} \qquad \Leftarrow \text{Answer!!!}$$