MAT 203: HOMEWORK 4

Section 12.4

2. (Drawing)

8. $\mathbf{r}(t) = 6\cos t\mathbf{i} + 2\sin t\mathbf{j}$

Want: $\mathbf{T}(\frac{\pi}{3})$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'(\frac{\pi}{3})}{\|\mathbf{r}'(\frac{\pi}{3})\|}$$

$$\begin{aligned}
\mathbf{r}'(t) &= -6\sin t\mathbf{i} + 2\cos t\mathbf{j} \\
\mathbf{r}'(\frac{\pi}{3}) &= -6\sin\frac{\pi}{3}\mathbf{i} + 2\cos\frac{\pi}{3}\mathbf{j} \\
&= -6\left(\frac{\sqrt{3}}{2}\right)\mathbf{i} + (2)\left(\frac{1}{2}\right)\mathbf{j} \\
&= -3\sqrt{3}\mathbf{i} + \mathbf{j}
\end{aligned}$$

$$\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\| = \sqrt{27+1} = \sqrt{28}$$

$$\therefore \mathbf{T}\left(\frac{\pi}{3}\right) = \left(\frac{-3\sqrt{3}}{28}\right)\mathbf{i} + \left(\frac{1}{\sqrt{28}}\right)\mathbf{j}$$

28. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$

Want: $\mathbf{N}(0)$

$$\mathbf{N}(0) = \frac{\mathbf{T}'(0)}{\|\mathbf{T}'(0)\|}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$\mathbf{r}'(t) = \sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{2 + e^{2t} + e^{-2t}}$$

$$\mathbf{T}(t) = \frac{\sqrt{2}}{\sqrt{2+e^{2t}+e^{-2t}}}\mathbf{i} + \frac{e^t}{\sqrt{2+e^{2t}+e^{-2t}}}\mathbf{j} + \frac{-e^{-t}}{\sqrt{2+e^{2t}+e^{-2t}}}\mathbf{k}$$

$$\mathbf{T}'(t) = \frac{d}{dt}\left[\frac{\sqrt{2}}{\sqrt{2+e^{2t}+e^{-2t}}}\right]\mathbf{i} + \frac{d}{dt}\left[\frac{e^t}{\sqrt{2+e^{2t}+e^{-2t}}}\right]\mathbf{j} + \frac{d}{dt}\left[\frac{-e^{-t}}{\sqrt{2+e^{2t}+e^{-2t}}}\right]\mathbf{k}$$

$$\begin{aligned}\frac{d}{dt}\left[\frac{\sqrt{2}}{\sqrt{2+e^{2t}+e^{-2t}}}\right] &= \sqrt{2}\,\frac{d}{dt}\left[(2+e^{2t}+e^{-2t})^{-1/2}\right] \\ &= \frac{-1}{\sqrt{2}}(2+e^{2t}+e^{-2t})^{-3/2}(2e^{2t}-2e^{-2t})\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left[\frac{e^{t}}{\sqrt{2+e^{2t}+e^{-2t}}}\right] &= \frac{d}{dt}\left[e^{t}(2+e^{2t}+e^{-2t})^{-1/2}\right] \\ &= e^{t}\frac{d}{dt}\left[(2+e^{2t}+e^{-2t})^{-1/2}\right]+e^{t}(2+e^{2t}+e^{-2t})^{-1/2} \\ &= e^{t}(-\frac{1}{2}(2+e^{2t}+e^{-2t})^{-3/2})+e^{t}(2+e^{2t}+e^{-2t})^{-1/2}\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left[\frac{-e^{-t}}{\sqrt{2+e^{2t}+e^{-2t}}}\right] &= -\frac{d}{dt}\left[e^{-t}(2+e^{2t}+e^{-2t})^{-1/2}\right] \\ &= -\left(e^{-t}\frac{d}{dt}[(2+e^{2t}+e^{-2t})^{-1/2}]+(2+e^{2t}+e^{-2t})^{-1/2}\cdot(-1)\right) \\ &= -\left(e^{-t}\cdot-\frac{1}{2}(2+e^{2t}+e^{-2t})^{-3/2}\cdot(2e^{2t}-2e^{-2t})-(2+e^{2t}+e^{-2t})^{-1/2}\right) \\ &= \frac{1}{2}e^{-t}(2e^{2t}-2e^{-2t})(2+e^{2t}+e^{-2t})^{-3/2}+(2+e^{2t}+e^{-2t})^{-1/2}\end{aligned}$$

Thus,

$$\mathbf{T}'(t)=\left[\frac{-1}{\sqrt{2}}(2+e^{2t}+e^{-2t})^{-3/2}(2e^{2t}-2e^{-2t})\right]\mathbf{i}+\left[e^{t}(-\frac{1}{2}(2+e^{2t}+e^{-2t})^{-3/2})+e^{t}(2+e^{2t}+e^{-2t})^{-1/2}\right]\mathbf{j}$$
$$+\left[\frac{1}{2}e^{-t}(2e^{2t}-2e^{-2t})(2+e^{2t}+e^{-2t})^{-3/2}+(2+e^{2t}+e^{-2t})^{-1/2}\right]\mathbf{k}$$

so

$$\mathbf{T}'(0)=\left[\frac{1}{\sqrt{2}}(2+1+1)^{-3/2}(2-2)\right]\mathbf{i}+\left[-\frac{1}{2}(2+1+1)^{-3/2}(2-2)+(2+1+1)^{-1/2}\right]\mathbf{j}$$
$$+\left[\frac{1}{2}(2-2)(2+1+1)^{-3/2}+(2+1+1)^{-1/2}\right]\mathbf{k}$$

$$=4^{-1/2}\mathbf{j}+4^{-1/2}\mathbf{k}=\frac{1}{2}\mathbf{j}+\frac{1}{2}\mathbf{k}$$

$$\|\mathbf{T}'(0)\|=\sqrt{\frac{1}{4}+\frac{1}{4}}=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}$$

$$\mathbf{N}(0)=\frac{1}{\|\mathbf{T}'(0)\|}\mathbf{T}'(0)=\frac{1}{2\sqrt{2}}\mathbf{j}+\frac{1}{2\sqrt{2}}\mathbf{k}$$

56. $\mathbf{r}(t) = (e^t \sin t)\mathbf{i} + (e^t)\cos t)\mathbf{j} + (e^t)\mathbf{k}$

Want: $\mathbf{T}(0), \mathbf{N}(0), a_{\mathbf{T}}(0), a_{\mathbf{N}}(0)$

Following the hint, we shall find $\mathbf{a}(0), \mathbf{T}(0)$, and $a_{\mathbf{N}}(0)$, and then solve for $\mathbf{N}(0)$ in the equation $\mathbf{a}(0) = a_{\mathbf{T}}(0)\mathbf{T}(0) + a_{\mathbf{N}}(0)\mathbf{N}(0)$.

$$\mathbf{a}(t) = \mathbf{r}''(t)$$
$$\mathbf{r}'(t) = (e^t \cos t + e^t \sin t)\mathbf{i} + (-e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$$
$$\mathbf{r}''(t) = (-e^t \sin t + e^t \cos t + e^t \cos t + e^t \sin t)\mathbf{i} + (-e^t \cos t - e^t \sin t - e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$$
$$\mathbf{a}(0) = \mathbf{r}''(0) = (0 + 1 + 1 + 0)\mathbf{i} + (-1 - 0 - 0 + 1)\mathbf{j} + 1\mathbf{k} = 2\mathbf{i} + \mathbf{k}$$
$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|}$$
$$\mathbf{r}'(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$$
$$\|\mathbf{r}'(0)\| = \sqrt{1+1+1} = \sqrt{3}$$
$$\therefore \mathbf{T}(0) = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

To find $a_{\mathbf{N}}(0)$, we find $a_{\mathbf{T}}(0)$, then we use

$$a_{\mathbf{N}}(0) = \sqrt{\|\mathbf{a}(0)\|^2 - (a_{\mathbf{T}}(0))^2}$$

$$\begin{aligned} a_{\mathbf{T}}(0) = \mathbf{a}(0) \cdot \mathbf{T}(0) &= (2\mathbf{i} + \mathbf{k}) \cdot (\frac{1}{\sqrt{3}}\mathbf{i} + \sqrt{3}\mathbf{j} + \sqrt{3}\mathbf{k}) \\ &= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3} \end{aligned}$$
$$\|\mathbf{a}(0)\|^2 = 2^2 + 1^2 = 5$$

Hence

$$a_{\mathbf{N}}(0) = \sqrt{5-3} = \sqrt{2}$$

We now solve:

$$\mathbf{a}(0) = a_{\mathbf{T}}(0)\mathbf{T}(0) + a_{\mathbf{N}}(0)\mathbf{N}(0)$$

i.e.

$$\begin{aligned} 2\mathbf{i} + \mathbf{k} &= \sqrt{3} \cdot \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \sqrt{2}(n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}) \\ &= (1 + \sqrt{2}n_1)\mathbf{i} + (1 + \sqrt{2}n_2)\mathbf{j} + (1 + \sqrt{2}n_3)\mathbf{k} \end{aligned}$$

$$\rightsquigarrow \begin{cases} 1 + \sqrt{2}n_1 &= 2 \\ 1 + \sqrt{2}n_2 &= 0 \\ 1 + \sqrt{2}n_3 &= 1 \end{cases}$$

so that

$$n_1 = \frac{1}{\sqrt{2}},\ n_2 = \frac{-1}{\sqrt{2}},\ n_3 = 0$$

Thus,

$$\mathbf{N}(0) = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$$

Section 12. 5

18. $\mathbf{r}(t) = 6\cos(\pi t/4)\mathbf{i} + 2\sin(\pi t/4)\mathbf{j} + t\mathbf{k}$

(a) Want: $\|\mathbf{r}(2) - \mathbf{r}(0)\|$

$$\begin{aligned}
\mathbf{r}(0) &= 6\cos 0\mathbf{i} + 2\sin 0\mathbf{j} + 0\mathbf{k} = 6\mathbf{i} \\
\mathbf{r}(2) &= 6\left(\cos\frac{\pi}{2}\right)\mathbf{i} + 2\left(\sin\frac{\pi}{2}\right)\mathbf{j} + 2\mathbf{k} = 2\mathbf{j} + 2\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\therefore \|\mathbf{r}(2) - \mathbf{r}(0)\| &= \|-6\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\| \\
&= \sqrt{36+4+4} = \sqrt{44} = 2\sqrt{11}
\end{aligned}$$

(b) Want:

$$\|\mathbf{r}(0.5) - \mathbf{r}(0)\| + \|\mathbf{r}(1) - \mathbf{r}(0.5)\| + \|\mathbf{r}(1.5) - \mathbf{r}(1)\| + \|\mathbf{r}(2) - \mathbf{r}(1.5)\|$$

(Lengthy calculation requiring intensive use of trigonometry omitted. It is unlikely that any exam problem will involve such complications.)

(c) Divide $[0, 2]$ into smaller and smaller segments, each time applying the process of part (b).

24. *Calculate the curvature of*

$$\mathbf{r}(t) = \left\langle 4(\sin t - t\cos t), 4(\cos t + t\sin t), \frac{3}{2}t^2 \right\rangle.$$

By Theorem 12.8,

$$K(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

$$\begin{aligned}
\mathbf{r}'(t) &= \langle 4(\cos t - (-t\sin t + \cos t)), 4(-\sin t + (t\cos t + \sin t)), 3t\rangle \\
&= \langle 4(-t\sin t), 4t\cos t, 3t\rangle
\end{aligned}$$

$$\begin{aligned}
\|\mathbf{r}'(t)\| &= \sqrt{16t^2\sin^2 t + 16t^2\cos^2 t + 9t^2} \\
&= \sqrt{16t^2 + 9t^2} \\
&= \sqrt{25t^2} \\
&= 5|t|
\end{aligned}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{5|t|}\langle -4t\sin t, 4t\cos t, 3t\rangle$$

which we rewrite as

$$\begin{cases} \left\langle -\frac{4}{5}\sin t, \frac{4}{5}\cos t, \frac{3}{5}\right\rangle, & \text{if } t \geq 0 \\ \left\langle \frac{4}{5}\sin t, -\frac{4}{5}\cos t, -\frac{3}{5}\right\rangle, & \text{if } t < 0 \end{cases}$$

Differentiating yields

$$\mathbf{T}'(t) = \begin{cases} \left\langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\right\rangle, & \text{if } t \geq 0 \\ \left\langle \frac{4}{5}\cos t, \frac{4}{5}\sin t, 0\right\rangle, & \text{if } t < 0 \end{cases}$$

so that

$$\begin{aligned} \|\mathbf{T}'(t)\| &= \sqrt{\left(\frac{4}{5}\right)^2 \cos^2 t + \left(\frac{4}{5}\right)^2 \sin^2 t + 0} \\ &= \sqrt{\left(\frac{4}{5}\right)^2} = \frac{4}{5} \end{aligned}$$

Thus,

$$K(t) = \frac{\frac{4}{5}}{5|t|} = \frac{4}{25|t|}$$

38. *Calculate the curvature of*

$$\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}.$$

$$K(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

$$\mathbf{r}'(t) = 4t\mathbf{i} + \mathbf{j} + t\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{16t^2 + 1 + t^2} = \sqrt{17t^2 + 1}$$

$$\begin{aligned} \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} &= \frac{1}{\sqrt{17t^2+1}}(4t\mathbf{i} + \mathbf{j} + t\mathbf{k}) \\ &= \frac{4t}{\sqrt{17t^2+1}}\mathbf{i} + \frac{1}{\sqrt{17t^2+1}}\mathbf{j} + \frac{t}{\sqrt{17t^2+1}}\mathbf{k} \end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}\left[\frac{4t}{\sqrt{17t^2+1}}\right] &= \frac{4\sqrt{17t^2+1}-4t\frac{d}{dt}\left[\sqrt{17t^2+1}\right]}{17t^2+1} \\
&= \frac{4\sqrt{17t^2+1}-4t\left(\frac{34t}{2\sqrt{17t^2+1}}\right)}{17t^2+1} \\
&= \frac{4\sqrt{17t^2+1}-\frac{68t^2}{\sqrt{17t^2+1}}}{17t^2+1} \\
&= \frac{4}{\sqrt{17t^2+1}}-\frac{68t^2}{(17t^2+1)^{3/2}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\left[\frac{1}{\sqrt{17t^2+1}}\right] &= 34t\left(-\frac{1}{2}(17t^2+1)^{-3/2}\right) \\
&= -17t(17t^2+1)^{-3/2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\left[\frac{t}{\sqrt{17t^2+1}}\right] &= \frac{\sqrt{17t^2+1}-t\cdot\frac{d}{dt}\left[\sqrt{17t^2+1}\right]}{17t^2+1} \\
&= \frac{\sqrt{17t^2+1}-t\cdot\frac{34t}{2\sqrt{17t^2+1}}}{17t^2+1} \\
&= \frac{1}{\sqrt{17t^2+1}}-\frac{17t^2}{(17t^2+1)^{3/2}}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathbf{T}'(t) &= \left(\frac{4}{\sqrt{17t^2+1}}-\frac{68t^2}{(17t^2+1)^{3/2}}\right)\mathbf{i}+\left(\frac{-17t}{(17t^2+1)^{3/2}}\right)\mathbf{j}+\left(\frac{1}{\sqrt{17t^2+1}}-\frac{17t^2}{(17t^2+1)^{3/2}}\right)\mathbf{k} \\
&= \left(\frac{1}{(17t^2+1)^{3/2}}\right)\mathbf{i}+\left(\frac{-17t}{(17t^2+1)^{3/2}}\right)\mathbf{j}+\left(\frac{1}{(17t^2+1)^{3/2}}\right)\mathbf{k}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathbf{T}'(t)\| &= \sqrt{(17t^2+1)^{-3}+289t^2(17t^2+1)^{-3}+(17t^2+1)^{-3}} \\
&= \sqrt{(2+289t^2)(17t^2+1)^{-3}}
\end{aligned}
$$

so that

$$
\begin{aligned}
K(t) &= \frac{\sqrt{(2+289t^2)(17t^2+1)^{-3}}}{\sqrt{17t^2+1}} \\
&= (2+289t^2)^{1/2}(17t^2+1)^{-3/2}(17t^2+1)^{-1/2} \\
&= (2+289t^2)^{1/2}(17t^2+1)^{-2}
\end{aligned}
$$

60. $y = e^x$

$$K(x) = \frac{|e^x|}{(1+e^{2x})^{3/2}} = e^x(1+e^{2x})^{-3/2}$$

(a)

$$\begin{aligned} K'(x) &= e^x \cdot \frac{d}{dx}\left[(1+e^{2x})^{3/2}\right] + (1+e^{2x})^{-3/2}e^x \\ &= e^x\left(-\frac{3}{2}(1+e^{2x})^{-5/2}\cdot 2e^{2x}\right) + (1+e^{2x})^{-3/2}e^x \\ &= -3e^{3x}(1+e^{2x})^{-5/2} + e^x(1+e^{2x})^{-3/2} \end{aligned}$$

$$\begin{gathered} \rightsquigarrow -3e^{3x}(1+e^{2x})^{-5/2} + e^x(1+e^{2x})^{-3/2} = 0 \\ e^x(1+e^{2x})^{-3/2} = 3e^{3x}(1+e^{2x})^{-5/2} \\ e^x(1+e^{2x})^{-1} = 3e^{3x} \\ e^x = 3e^{3x}(1+e^{2x}) \\ e^x = 3e^{3x} + e^{5x} \\ e^{5x} + 3e^{3x} - e^x = 0 \\ e^x(e^{4x} + 3e^{2x} - 1) = 0 \\ \rightsquigarrow e^{4x} + 3e^{2x} - 1 = 0 \end{gathered}$$

$$\begin{aligned} e^{2x} &= \frac{-3 \pm \sqrt{9 - (4)(-1)}}{2} \\ &= \frac{-3 \pm \sqrt{13}}{2} \end{aligned}$$

But since $e^{2x} > 0$, we must have

$$e^{2x} = \frac{-3+\sqrt{13}}{2}$$

so that

$$x = \frac{1}{2}\ln\left(\frac{-3+\sqrt{13}}{2}\right).$$

(b)

$$\begin{aligned} \lim_{x\to\infty} K(x) = \lim_{x\to\infty} \frac{e^x}{(1+e^{2x})^{3/2}} &\leq \lim_{x\to\infty} \frac{e^x}{(1+e^{2x})} \\ &\leq \lim_{x\to\infty} \frac{e^x}{e^{2x}} \\ &= \lim_{x\to\infty} e^{-x} = 0 \end{aligned}$$

But $0 < (e^x)/((1+e^{2x})^{3/2})$ for all x, so it follows from the Squeeze Theorem that

$$\lim_{x\to\infty} \frac{e^x}{(1+e^{2x})^{3/2}} = 0.$$

Section 13.1

8. $g(x, y) = \ln|x + y|$

(a) $g(2, 3) = \ln|2 + 3| = \ln|5| = \ln 5$

(b) $g(5, 6) = \ln|5 + 6| = \ln|11| = \ln 11$

(c) $g(e, 0) = \ln|e + 0| = \ln|e| = \ln e = 1$

(d) $g(0, 1) = \ln|0 + 1| = \ln|1| = \ln 1 = 0$

(e) $g(2, -3) = \ln|2 - 3| = \ln|-1| = \ln 1 = 0$

(f) $g(e, e) = \ln|e + e| = \ln|2e| = \ln(2e) = \ln 2 + \ln e = \ln 2 + 1$

19. $f(x, y) = \arcsin(x + y)$

$$\begin{aligned} \text{domain: } & \{(x, y) \in \mathbb{R}^2 : -1 \leq x + y \leq 1\} \\ \text{range: } & \{z \in \mathbb{R} : -\frac{\pi}{2} \leq z \leq \frac{\pi}{2}\} \end{aligned}$$