MAT 203: HOMEWORK 3

Section 11.6

6. Answer: (a) (The equation defines a hyperbolic paraboloid.)

66. *Explain why the curve of intersection of the surfaces*

$$x^2 + 3y^2 - 2z^2 + 2y = 4$$

and

$$2x^2 + 6y^2 - 4z^2 - 3x = 2$$

lies in a plane.

Every point on the curve of intersection satisfies the two equations

$$\begin{cases} x^2 + 3y^2 - 2z^2 + 2y & = & 4 \\ 2x^2 + 6y^2 - 4z^2 - 3x & = & 2 \end{cases}$$

or

$$\begin{cases} 2x^2 + 6y^2 - 4z^2 + 4y & = & 8 \\ 2x^2 + 6y^2 - 4z^2 - 3x & = & 2 \end{cases}$$

or

$$2x^2 + 6y^2 - 4z^2 = 8 - 4y = 2 + 3x$$

Hence every point on the curve lies in the plane defined by

$$8 - 4y = 2 + 3x$$

Section 12.1

2. *Find the domain of the function*

$$\mathbf{r}(t) = (\sqrt{4 - t^2})\mathbf{i} + t^2\mathbf{j} - 6t\mathbf{k}$$

$$\begin{aligned} \text{Domain} &= \{t \in \mathbb{R} : 4 - t^2 \geq 0\} \\ &= \{t \in \mathbb{R} : |t| \leq \sqrt{2}\} \end{aligned}$$

12. *Let* $\mathbf{r}(t) = (\sqrt{t})\mathbf{i} + t^{3/2}\mathbf{j} + e^{-t/4}\mathbf{k}$. *Evaluate:*

(a) $\mathbf{r}(0)$

$$\begin{aligned} \mathbf{r}(0) &= (\sqrt{0})\mathbf{i} + 0^{3/2}\mathbf{j} + e^{-0/4}\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k} \\ &= \mathbf{k} \end{aligned}$$

(b) $\mathbf{r}(4)$

$$\begin{aligned}\mathbf{r}(4) &= (\sqrt{4})\mathbf{i} + 4^{3/2}\mathbf{j} + e^{-4/4}\mathbf{k} \\ &= 2\mathbf{i} + 8\mathbf{j} + e^{-1}\mathbf{k}\end{aligned}$$

(c) $\mathbf{r}(c+2)$

$$\mathbf{r}(c+2) = (\sqrt{c+2})\mathbf{i} + (c+2)^{3/2}\mathbf{j} + e^{-(c+2)/4}\mathbf{k}$$

(d) $\mathbf{r}(9+\Delta t) - \mathbf{r}(9)$

$$\begin{aligned}\mathbf{r}(9+\Delta t) - \mathbf{r}(9) &= (\sqrt{9+\Delta t})\mathbf{i} + (9+\Delta t)^{3/2}\mathbf{j} + e^{-(9+\Delta t)/4} - (\sqrt{9}\mathbf{i} + 9^{3/2}\mathbf{j} + e^{-9/4}\mathbf{k}) \\ &= (\sqrt{9+\Delta t} - \sqrt{9})\mathbf{i} + ((9+\Delta t)^{3/2} - 9^{3/2})\mathbf{j} + (e^{-(9+\Delta t)/4} - e^{-9/4})\mathbf{k} \\ &= (\sqrt{9+\Delta t} - 3)\mathbf{i} + ((9+\Delta t)^{3/2} - 27)\mathbf{j} + (e^{-(9+\Delta t)/4} - e^{-9/4})\mathbf{k}\end{aligned}$$

17. (b)
18. (c)
19. (d)
20. (a)

58. The simplest solution is probably the pair of functions $\mathbf{r}, \mathbf{u}$ defined by

$$\begin{array}{ll}\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\mathbf{j} & \text{for } 0 \le t \le 1 \\ \mathbf{u}(t) = (1-t)\mathbf{i} + (1-t)\mathbf{j} & \text{for } 0 \le t \le 1\end{array}$$

.

How did we come up with this? Let's consider $\mathbf{r}$. We want to find a set of parametric equations for the curve $y = \sqrt{x}$ over the interval $0 \le x \le 1$; that is, a pair of functions $x(t), y(t)$ such that such that for every value of t the point $(x(t), y(t))$ (the terminal point of the vector $x(t)\mathbf{i} + y(t)\mathbf{j}$) lies on the curve, and such that for every point (x_0, y_0) on the curve there exists some t_0 such that $x_0 = x(t_0)$ and $y_0 = y(t_0)$. Of course, since the points $(x(t), y(t))$ are constrained to lie on the curve $y = \sqrt{x}$, we know that whatever $x(t)$ is, we must have $y(t) = \sqrt{x(t)}$. So we really just need to find the one function $x(t)$.

Next we need to choose a set of values for t; to make things simple, we'll let $0 \le t \le 1$. The direction of the arrow drawn in the picture tells us that we should *start* at $(0,0)$ and *end* at $(1,1)$ – in other words that $x(0)$ should be 0 and $x(1)$ should be 1. Now, can you think of a function $x(t)$ such that $x(0) = 0, x(1) = 1$, and $0 \le x(t) \le 1$ for all t satisfying $0 \le t \le 1$? That's easy, right? Just let $x(t) = t$. Which is what we did.

(For $\mathbf{u}$, the process was the same, except that we needed to start at $(1,1)$ and end at $(0,0)$, and lie on the curve $y = x$.)

Section 12.2

30. *Find the interval(s) on which the function*

$$\mathbf{r}(t) = \frac{1}{t-1}\mathbf{i} + 3t\mathbf{j}$$

is smooth.

"Smooth" means that the derivative exists and is continuous; hence to determine on what intervals a function is smooth, we have to find the derivative:

$$\mathbf{r}'(t) = \frac{-1}{(t-1)^2}\mathbf{i} + 3\mathbf{j}$$

The function $\mathbf{r}'$ is defined and continuous for all $t \neq 1$; hence $\mathbf{r}$ is smooth on $(-\infty, 1)$ and $(1, \infty)$.

46. *Use the definition of the derivative to differentiate the function*

$$\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \frac{3}{t}\mathbf{j} - 2t\mathbf{k}$$

Since this course assumes knowledge of single-variable calculus, we won't quite go all the way back to first principles. Instead we'll reduce the problem to a Calculus I problem (actually three Calculus I problems) by using the definition on p. 840 of the book, together with the properties of limits discussed in section 12.1:

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[(\sqrt{t+h}\mathbf{i} + \frac{3}{t+h}\mathbf{j} - 2(t+h)\mathbf{k}) - (\sqrt{t}\mathbf{i} + \frac{3}{t}\mathbf{j} - 2t\mathbf{k})\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[(\sqrt{t+h} - \sqrt{t})\mathbf{i} + \left(\frac{3}{t+h} - \frac{3}{t}\right)\mathbf{j} + (-2(t+h) - (-2t))\mathbf{k}\right] \\
&= \lim_{h\to 0}\left[\left(\frac{\sqrt{t+h} - \sqrt{t}}{h}\right)\mathbf{i} + \left(\frac{\frac{3}{t+h} - \frac{3}{t}}{h}\right)\mathbf{j} + \left(\frac{(-2(t+h) - (-2t)}{h}\right)\mathbf{k}\right] \\
&= \lim_{h\to 0}\left[\left(\frac{\sqrt{t+h} - \sqrt{t}}{h}\right)\mathbf{i}\right] + \lim_{h\to 0}\left[\left(\frac{\frac{3}{t+h} - \frac{3}{t}}{h}\right)\mathbf{j}\right] + \lim_{h\to 0}\left[\left(\frac{(-2(t+h) - (-2t)}{h}\right)\mathbf{k}\right] \\
&= \left(\lim_{h\to 0}\frac{\sqrt{t+h} - \sqrt{t}}{h}\right)\mathbf{i} + \left(\lim_{h\to 0}\frac{\frac{3}{t+h} - \frac{3}{t}}{h}\right)\mathbf{j} + \left(\lim_{h\to 0}\frac{(-2(t+h) - (-2t)}{h}\right)\mathbf{k} \\
&= \left(\frac{d}{dt}[\sqrt{t}]\right)\mathbf{i} + \left(\frac{d}{dt}\left[\frac{3}{t}\right]\right)\mathbf{j} + \left(\frac{d}{dt}[-2t]\right)\mathbf{k} \\
&= \frac{1}{2\sqrt{t}}\mathbf{i} - \frac{3}{t^2}\mathbf{j} - 2\mathbf{k}
\end{aligned}$$

83. True

84. False. (The definite integral of a vector-valued function is a *vector*.)

85. False. (Take $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$.)

86. False. (Take $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$ and $\mathbf{u}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$.)