MAT 203: HOMEWORK 2

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Section 11.4

8. *Let* $\mathbf{u} = 3\mathbf{i} + 5\mathbf{k}, \mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ *Find:*

(a) $\mathbf{u} \times \mathbf{v}$

By definition,

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

Here, $u_1 = 3, u_2 = 0, u_3 = 5$ and $v_1 = 2, v_2 = 3, v_3 = -2$. Thus,

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= ((0)(-2) - (5)(3))\mathbf{i} - ((3)(-2) - (5)(2))\mathbf{j} + ((3)(3) - (0)(2))\mathbf{k} \\ &= (-15)\mathbf{i} - (-6 - 10)\mathbf{j} + (9)\mathbf{k} \\ &= -15\mathbf{i} + 16\mathbf{j} + 9\mathbf{k}\end{aligned}$$

(b) $\mathbf{v} \times \mathbf{u}$

We use the identity $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$ (Thorem 11.7, p.791) to conclude that

$$\mathbf{v} \times \mathbf{u} = -(-15\mathbf{i} + 16\mathbf{j} + 9\mathbf{k}) = 15\mathbf{i} - 16\mathbf{j} - 9\mathbf{k}$$

(c) $\mathbf{v} \times \mathbf{v}$

The cross product of a vector with itself is 0. Always. (Theorem 11.7)

14. *Given* $\mathbf{u} = \langle -10, 0, 6\rangle$ *and* $\mathbf{v} = \langle 7, 0, 0\rangle$, *find* $\mathbf{u} \times \mathbf{v}$ *and show that it is orthogonal to* $\mathbf{u}$ *and* $\mathbf{v}$.

We have

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \langle (u_2v_3 - u_3v_2), -(u_1v_3 - u_3v_1), (u_1v_2 - u_2v_1)\rangle \\ &= \langle (0)(0) - (6)(0), -((-10)(0) - (6)(7)), (-10)(0) - (0)(7)\rangle \\ &= \langle 0, -(-42), 0\rangle = \langle 0, 42, 0\rangle\end{aligned}$$

And we see that

$$\mathbf{u} \cdot \langle 0, 42, 0\rangle = \langle -10, -, 6\rangle \cdot \langle 0, 42, 0\rangle = (-10)(0) + (0)(42) + (6)(0) = 0 + 0 + 0 = 0$$

and

$$\mathbf{v} \cdot \langle 0, 42, 0\rangle = \langle 7, 0, 0\rangle \cdot \langle 0, 42, 2\rangle = (7)(0) + (42)(0) + (0)(0) = 0 + 0 + 0 = 0$$

so that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, as desired.

32. *Verify that the four points* $(2, -3, 1), (6, 5, -1), (3, -6, 4), (7, 2, 2)$ *are the vertices of a parallelogram, and find the area of the parallelogram.*

Let

$$\begin{aligned} P &= (2, -3-1) \\ Q &= (6, 5, -1) \\ R &= (3, -6, 4) \\ S &= (7, 2, 2) \end{aligned}$$

Then

$$\begin{aligned} \overrightarrow{PQ} &= \langle 6-2, 5--3, -1-1 \rangle = \langle 4, 8, -2 \rangle \\ \overrightarrow{RS} &= \langle 7-3, 2--6, 2-4 \rangle = \langle 4, 8, -2 \rangle \end{aligned}$$

Hence $\overrightarrow{PQ} = \overrightarrow{RS}$, so the points are vertices of a parallelogram.

The area is given by $\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$, so we compute:

$$\overrightarrow{PR} = \langle 3-2, -6--3, 4-1 \rangle = \langle -1, -3, 3 \rangle$$

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \langle 4, 8, -2 \rangle \times \langle 1, -3, 3 \rangle \\ &= \langle (8)(3) - (-2)(-3), -((4)(3) - (-2)(1)), (4)(-3) - (8)(1) \rangle \\ &= \langle 24 - 6, -(12 - (-2)), -12 - 8 \rangle \\ &= \langle 18, -14, -20 \rangle \end{aligned}$$

Thus

$$\begin{aligned} \|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \|\langle 18, -14, -20 \rangle\| &= \sqrt{18^2 + 14^2 + 20^2} \\ &= \sqrt{(2 \cdot 9)^2 + (2\dot{7})^2 + (2 \cdot 10)^2} \\ &= \sqrt{(2^2)(9^2) + (2^2)(7^2) + (2^2)(10^2)} \\ &= \sqrt{(2^2)(9^2 + 7^2 + 10^2)} \\ &= 2\sqrt{81 + 49 + 100} \\ &= 2\sqrt{100 + 80 + 40 + 1 + 9} \\ &= 2\sqrt{100 + 120 + 10} \\ &= 2\sqrt{100 + 130} \\ &= 2\sqrt{230} \end{aligned}$$

46. *Given* $\mathbf{u} = \langle 1, 3, 1 \rangle, \mathbf{v} = \langle 0, 6, 6 \rangle, \mathbf{w} = \langle -4, 0, -4 \rangle$*, find the volume of the parallelepiped having adjacent edges* $\mathbf{u}$*,* $\mathbf{v}$*, and* $\mathbf{w}$*.*

Let V be the volume. By Theorem 11.10 (p.795),

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

Here,

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \langle v_2 w_3 - v_3 w_2, -(v_1 w_3 - v_3 w_1), v_1 w_2 - v_2 w_1 \rangle \\ &= \langle (6)(-4) - (6)(0), -((10)(-4) - (6)(-4)), (0)(0) - (6)(-4) \rangle \\ &= \langle -24, -24, 24 \rangle \end{aligned}$$

so that

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\mathbf{u}\cdot\langle -24,-24,24\rangle &= \langle 1,3,1\rangle\cdot\langle -24,-24,24\rangle\\ &= (1)(-24)+(3)(-24)+(1)(24)\\ &= 4(-24)+24\\ &= 3(-24)=-72\end{aligned}$$

whence $V=|-72|=72$.

Section 11.5

6. $(-3,0,2),\mathbf{v}=6\mathbf{j}+3\mathbf{k}$

Want: equations (parametric and symmetric) of the line passing through $(-3,0,2)$ and parallel to $\mathbf{v}$.

Let $P=(x,y,z)$ be a point on the line. The vector with initial point $(-3,0,2)$ and terminal point P must be *parallel* to $\mathbf{v}$ – that is, equal to a scalar multiple $t\mathbf{v}$ of $\mathbf{v}$. We must therefore have

$$(x-(-3))\mathbf{i}+(y-0)\mathbf{j}+(z-2)\mathbf{k}=t(6\mathbf{j}+3\mathbf{k})=6t\mathbf{j}+3t\mathbf{k}$$

so that

$$\begin{aligned}x+3 &= 0\\ y &= 6t\\ z-2 &= 3t\end{aligned}$$

or, in other words,

$$\begin{aligned}x= &\ -3\\ y &= 6t\\ z &= 3t+2\end{aligned}$$

Since x does not depend on t (and hence is also independent of y and z), we can only write down one "symmetric equation":

$$\frac{y}{6}=\frac{z-2}{3}$$

This specifies a plane in which the line is located; to specify the line itself we need to add the equation $x=-3$, which describes another plane. Our desired line is precisely the intersection of these two planes.

18. Want: parametric equations of the line passing through $(-1,4,-3)$ and parallel to $\mathbf{v}=5\mathbf{i}-\mathbf{j}$.

Just as in the previous problem, we must have

$$\begin{aligned} x-(-1) &= 5t \\ y-4 &= -t \\ z-3 &= 0 \end{aligned}$$

or

$$\begin{aligned} x &= 5t-1 \\ y &= -t+4 \\ z &= 3 \end{aligned}$$

40. Want: equation of plane passing through $(3,2,2)$ and perpendicular (orthogonal) to the line given by

$$\frac{x-1}{4} = y+2 = \frac{z+3}{-3}$$

We first need to express the line as the set of scalar multiples of some vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ with initial point $P = (p_1, p_2, p_3)$. To do this, write the equations of the line in parametric form, by letting $t = y+2$. Then we have

$$\begin{aligned} x-1 &= 4t \\ y+2 &= t \\ z+3 &= -3t \end{aligned}$$

If (x,y,z) is a point on the line, then the vector from P to (x,y,z) will be a scalar multiple $t\mathbf{v}$ of $\mathbf{v}$. That is,

$$\langle x-p_1, y-p_2, z-p_3 \rangle = t\langle v_1, v_2, v_3 \rangle = \langle tv_1, tv_2, tv_3 \rangle$$

But from the parametric equations above, we see that we can choose $P = (1,-2,-3)$ and $\mathbf{v} = \langle 4, 1, -3 \rangle$. Hence our line is the set of scalar multiples of the vector $\langle 4,1,-3 \rangle$ with initial point $(1,-2,-3)$.

Now we are ready to find the equation of the plane. Let (x,y,z) be a point on the plane. Then the vector from $(3,2,2)$ to (x,y,z) must be orthogonal to $\mathbf{v} = \langle 4,1,-3 \rangle$:

$$\langle x-3, y-2, z-2 \rangle \cdot \langle 4,1,-3 \rangle = 0$$

or

$$4(x-3) + (y-2) - 3(z-2) = 0$$

which is the desired equation.

52. Want: equation of plane passing through $(4,2,1)$ and $(-3,5,7)$ and parallel to the z-axis.

Let (x,y,z) be an arbitrary point on the plane. Then the vector $\mathbf{u}$ with initial point $(4,2,1)$ and terminal point (x,y,z) lies in the plane, as does the vector $\mathbf{v}$ with initial point $(4,2,1)$ and terminal point $(-3,5,7)$. If the plane is parallel to the z-axis, then any vector orthogonal to the plane will be orthogonal to the z-axis (that is, orthogonal to the vector

$\mathbf{k} = \langle 0, 0, 1\rangle$). In particular, the *cross product* of any two vectors in the plane – such as $\mathbf{u}$ and $\mathbf{v}$ – will be orthogonal to $\mathbf{k}$:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{k} = 0$$

$$(\langle x - 4, y - 2, z - 1\rangle \times \langle -7, 3, 6\rangle) \cdot \langle 0, 0, 1\rangle = 0$$

Since taking the dot product with $\langle 0, 0, 1\rangle$ amounts to saying "Give me the $\mathbf{k}$-component", our equation becomes:

$$(-7)(y - 2) - (3)(x - 4) = 0$$

which is just the sort of equation we were looking for.

(Note that the equation $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{k} = 0$ says that the parallelepiped with adjacent edges $\mathbf{u}, \mathbf{v}$, and $\mathbf{k}$ has zero volume – in other words, if these three vectors share a common initial point, then they must lie in a single plane. This is exactly what we wanted.)