MAT 203: HOMEWORK 1

Section 11.1

1. (a) *Find the component form of a vector* $\mathbf{v}$ *whose initial point is* $(1,1)$ *and whose terminal point is* $(5,3)$.

In general, if (p_1, p_2) and (q_1, q_2) are the initial and terminal points respectively of a vector $\mathbf{v}$, then the component form is $\langle q_1 - p_1, q_2 - p_2 \rangle$. So, in this case, we have $\mathbf{v} = \langle 5-1, 3-1 \rangle = \langle 4, 2 \rangle$.

(b) *Sketch the vector with its initial point at the origin.*

This is the part where you draw a picture of a directed line segment in the coordinate plane, with initial point $(0,0)$ and terminal point $(4,2)$.

7. *Find the vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *whose initial and terminal points are given. Show that* $\mathbf{u}$ *and* $\mathbf{v}$ *are equivalent.*
$\mathbf{u} : (0,3), (6,-2)$
$\mathbf{v} : (3,10), (9,5)$

We "find" a vector by writing down its component form; two vectors are "equivalent" if their component forms are the same. Here, $\mathbf{u} = \langle 6-0, -2-3 \rangle = \langle 6, -5 \rangle$ and $\mathbf{v} = \langle 9-3, 5-10 \rangle = \langle 6, -5 \rangle$; since $\langle 6, -5 \rangle = \langle 6, -5 \rangle$, the two vectors are equivalent.

39. *Find the unit vector in the direction of* $\mathbf{u}$ *and verify that it has length 1.*

$\mathbf{u} = \langle \frac{3}{2}, \frac{5}{2} \rangle$

By definition, the "unit vector in the direction of $\mathbf{u}$" is the vector $\frac{1}{\|\mathbf{u}\|}\mathbf{u}$; a vector of this form always has length 1, as we will see. We need to calculate the number $\|\mathbf{u}\|$, and then multiply the vector $\mathbf{u}$ by the number $\frac{1}{\|\mathbf{u}\|}$. From the definition, we have

$$\|\langle \frac{3}{2}, \frac{5}{2} \rangle\| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{34}{4}} = \frac{\sqrt{34}}{2}$$

so that

$$\frac{1}{\|\langle \frac{3}{2}, \frac{5}{2} \rangle\|} = \frac{2}{\sqrt{34}}$$

.

We now multiply:

$$\frac{2}{\sqrt{34}}\langle \frac{3}{2}, \frac{5}{2} \rangle = \langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \rangle$$

a.

Finally, we compute the length of this vector:

$$\|\langle\frac{3}{\sqrt{34}},\frac{5}{\sqrt{34}}\rangle\| = \sqrt{\left(\frac{3}{\sqrt{34}}\right)^2+\left(\frac{5}{\sqrt{34}}\right)^2} = \sqrt{\frac{9}{34}+\frac{25}{34}} = \sqrt{\frac{34}{34}} = \sqrt{1} = 1$$

which is what we wanted.

Suggestion: Instead of $\langle\frac{3}{2},\frac{5}{2}\rangle$, try doing this problem in the general case, for any (nonzero) vector $\mathbf{u} = \langle a, b\rangle$.

Section 11.2

4. (I'll skip this one, which just asks you to plot a pair of points.)

8. *Find the coordinates of the point located seven units in front of the yz-plane, two to the left of the xz-plane, and one unit below the xy-plane.*

The yz-plane is the plane where $x = 0$; seven units "in front" yields $x = 7$. The xz-plane is the plane where $y = 0$; two units "to the left" gives you $y = -2$. The xy-plane is the plane where $z = 0$; one unit "below" gives $z = -1$. Hence the coordinates of the point are $(7, -2, 0)$.

37. *Find the standard equation of the sphere with center* $(0, 2, 5)$ *and radius* 2.

Plugging the numbers into the formula, we get

$$(x-0)^2 + (y-2)^2 + (z-5)^2 = 2^2 = 4$$

.

48. *Describe the solid satisfying the condition*

$$x^2 + y^2 + z^2 > -4x + 6y - 8z - 13$$

.

The point here is to rewrite the inequality so that it the set of points satisfying it is easy to describe. To do this, first put everything on one side:

$$x^2 + 4x + y^2 - 6y + z^2 + 8z + 13 > 0$$

then complete the squares:

$$x^2+4x+y^2-6y+z^2+8z+13 = (x^2+4x+4)-4+(y^2-6y+9)-9+(z^2+8z+16)-16+13 > 0$$

Or, in other words,

$$(x+2)^2 - 4 + (y-3)^2 - 9 + (z+4)^2 - 16 + 13 = (x+4)^2 + (y-3)^2 + (z+4)^2 - 16 > 0$$

That is:

$$(x+4)^2 + (y-3)^2 + (z+4)^2 > 16 = 4^2$$

.

From the distance formula, we see that this inequality says: ”the distance from (x, y, z) to $(-2, 3, -4)$ is greater than 4”. Hence, the set of points satisfying the inequality is the exterior of the ball of radius 4 centered at $(-2, 3, 4)$.

Section 11.3

7. *Given that* $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ *and* $\mathbf{v} = \mathbf{i} - \mathbf{k}$ *find the following:*
(a) $\mathbf{u} \cdot \mathbf{v}$

In general, if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$. So, in our case,

$$\mathbf{u} \cdot \mathbf{v} = (2)(1) + (-1)(0) + (1)(-1) = 2 + 0 - 1 = 1$$

(b) $\mathbf{u} \cdot \mathbf{u}$

Same idea:

$$\mathbf{u} \cdot \mathbf{u} = (2)(2) + (-1)(-1) + (1)(1) = 4 + 1 + 1 = 6$$

(c) $\|\mathbf{u}\|^2$

Since $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}$, we just did this in (b).

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$

From (a), $\mathbf{u} \cdot \mathbf{v} = 1$, so $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 1\mathbf{v} = \mathbf{v}$

19. *Determine whether* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal, parallel, or neither.*
$\mathbf{u} = \langle 4, 0 \rangle$ $\mathbf{v} = \langle 1, 1 \rangle$

We calculate (the cosine of) the angle between $\mathbf{u}$ and $\mathbf{v}$, i.e. the quantity

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

So:

$$\frac{\langle 4, 0 \rangle \cdot \langle 1, 1 \rangle}{\|\langle 4, 0 \rangle\|\|\langle 1, 1 \rangle\|} = \frac{(4)(1) + (0)(1)}{\sqrt{4^2 + 0^2}\sqrt{1^2 + 1^2}} = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Since $\frac{1}{\sqrt{2}}$ is neither 1 nor -1 nor 0, the vectors are neither parallel nor orthogonal.

Find the component of $\mathbf{u}$ *that is orthogonal to* $\mathbf{v}$*, given* $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$.
43. $\mathbf{u} = \langle 6, 7 \rangle, \mathbf{v} = \langle 1, 4 \rangle, \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 2, 8 \rangle$.

This could not be simpler. We wish to find $\mathbf{w}$, where

$$\mathbf{u} = \text{proj}_{\mathbf{v}}\mathbf{u} + \mathbf{w}$$

Thus:

$$\mathbf{w} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 6, 7 \rangle - \langle 2, 8 \rangle = \langle 4, -1 \rangle$$

44 - 46: Same thing, different numbers.