4-Manifolds without Einstein Metrics

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Abstract

It is shown that there are infinitely many compact orientable smooth 4-manifolds which do not admit Einstein metrics, but nevertheless satisfy the strict Hitchin-Thorpe inequality $2\chi > 3|\tau|$. The examples in question arise as non-minimal complex algebraic surfaces of general type, and the method of proof stems from Seiberg-Witten theory.

1 Introduction

A smooth Riemannian metric g is said to be *Einstein* if its Ricci curvature r is a constant multiple of the metric:

$$r = \lambda g.$$

Not every 4-manifold admits such metrics. A necessary condition for the existence of an Einstein metric on a compact oriented 4-manifold is that the Hitchin-Thorpe inequality $2\chi(M) \geq 3|\tau(M)|$ must hold [3]. Moreover, equality can hold only if M manifold is finitely covered by a torus or K3 surface. We will say that M satisfies the *strict Hitchin-Thorpe inequality* if $2\chi(M) > 3|\tau(M)|$.

The purpose of this note is to prove the following result:

Theorem A *There are infinitely many compact simply-connected smooth 4-manifolds which do not admit Einstein metrics, but nevertheless satisfy the strict Hitchin-Thorpe inequality.*

The examples we shall consider arise as non-minimal complex surfaces of general type. The proof hinges on scalar curvature estimates that come from Seiberg-Witten theory.

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2 Scalar Curvature and Topology

In this section, we will develop a certain lower bound for the L^2-norm of the scalar curvature of all Riemannian metrics on a non-minimal complex surfaces of general type. Let us begin by reviewing some definitions and results.

Definition 1 *Let M be a smooth compact oriented 4-manifold. A* polarization *of M is a linear subspace $H^+ \subset H^2(M, \mathbf{R})$ on which the restriction of the intersection form is positive-definite, and which is a maximal subspace with this property.*

The example of interest is the following: let g be a Riemannian metric, and let $H^+(g)$ be the space of harmonic self-dual 2-forms with respect to g. Then $H^+(g)$ is a polarization. If H^+ is a given polarization, and if $H^+(g) = H^+$, we will say that g is adapted to H^+.

If J is an orientation-compatible almost-complex structure on M, J induces a spin^c-structure c on M, and for every metric g one then has a pair of rank-2 complex vector bundles $V_\pm$ which formally satisfy

$$V_\pm = \mathbf{S}_\pm \otimes L^{1/2},$$

where $\mathbf{S}_\pm$ are the left- and right-handed spinor bundles of g, and L is the anti-canonical line bundle of J. For each unitary connection θ on L, we have a Dirac operator $D_\theta : C^\infty(V_+) \to C^\infty(V_-)$, and one can then consider the Seiberg-Witten equations [13]

$$\begin{aligned} D_\theta \Phi &= 0 \\ F_\theta^+ &= i\sigma(\Phi) \end{aligned}$$

for an unknown section Φ of V_+ and an unknown unitary connection θ. Suppose that H^+ is a polarization such that the orthogonal projection c^+ of $c_1(L)$ into H^+ is non-zero. Let g be any H^+-adapted metric, and consider the moduli space of solutions of a generic perturbation of the Seiberg-Witten equations modulo gauge equivalence. This moduli space consists of a finite number of oriented points, and the Seiberg-Witten invariant $n_c(M, H^+)$ is defined to be the number of points in moduli space, counted with signs. This is independent of all choices. Indeed, if $b^+(M) > 1$, it is even independent of H^+.

A Weitzenböck argument yields the following curvature estimate [8]:

Theorem 1 *Let (M, H^+, c) be a smooth compact oriented polarized 4-manifold with spin^c structure such that $n_c(M, H^+) \neq 0$. If $c_1(L) \in H^2(M, \mathbf{R})$ is the anti-canonical class of this structure, let $c_1^+ \neq 0$ be its orthogonal projection to H^+ with respect to the intersection form. Then every H^+-adapted Riemannian metric g satisfies*

$$\int_M s^2 \, d\mu \geq 32\pi^2 (c_1^+)^2,$$

with equality iff g is Kähler with respect to a c-compatible complex structure and has constant negative scalar curvature.

Now if (M, J) is a complex surface of Kähler type with $b^+ > 1$, and if c is the spin^c structure induced by J, then $n_c(M, H^+) = n_c(M) = 1$. For complex surfaces with $b^+ = 1$, the picture is more complicated, but can be summarized as follows. The set of classes $\alpha \in H^2(M, \mathbf{R})$ with $\alpha^2 := \alpha \cdot \alpha > 0$ consists of two connected components. One component contains the Kähler classes of all Kähler metrics on M; let us call the elements of this component *future pointing*, and the elements of the other *past-pointing*. Then $n_c(M, H^+) = 1$ if c^+ is past-pointing, and that $n_c(M, H^+) = 0$ if c^+ is future pointing [4, 5].

We now come to the technical heart of the article:

Theorem 2 *Let X be a minimal complex algebraic surface of general type, and let $M = X \# k\overline{\mathbf{CP}}_2$ be obtained from X by blowing up $k > 0$ points. Then any Riemannian metric on M satisfies*

$$\int_M s^2 d\mu > 32\pi^2(2\chi + 3\tau + k),$$

where χ and τ are respectively the Euler characteristic and signature of M.

Proof. Let us think of M concretely as obtained from X by blowing up k distinct points $p_1, \ldots, p_k$, so that M comes equipped with an integrable complex structure J. The key observation [4] is that instead of merely considering this complex structure alone, it is natural to consider 2^k distinct complex structures, each of which is the pull-back of J via a diffeomorphism $M \to M$. To this end, choose a biholomorphism between a neighborhood of $p_j \in X$ and the unit ball in $\mathbf{C}^2 = \mathbf{R}^4$. Let $\psi_j : X \to X$ be the identity outside this neighborhood, and act by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \pi u(r) & 0 & -\sin \pi u(r) \\ 0 & 0 & 1 & 0 \\ 0 & \sin \pi u(r) & 0 & \cos \pi u(r) \end{bmatrix}$$

on the ball itself; here r is the distance from the origin in $\mathbf{R}^4$, and the smooth function u satisfies $u(r) \equiv 1$ for $r \leq \frac{1}{3}$ and $u(r) \equiv 0$ for $r \geq \frac{2}{3}$. Since ψ_j is complex anti-linear in a neighborhood of p_j, it induces a diffeomorphism $\phi_j : M \to M$. Assuming that the neighborhoods in question are pairwise disjoint, the ϕ_j's commute with each other, and if $S \subset \{1, \ldots k\}$ is any subset, we may therefore unambiguously define ϕ_S to be the composition of those ϕ_j's for which $j \in S$. Now $J_S = \phi_S^* J$ is an integrable complex structure on M for each $S \subset \{1, \ldots k\}$; for example, $J_\emptyset = J$.

Let $c_1(X)$ denote the pull-back to M of the first Chern class of X via the blowing-down map $M \to X$, and let $E_1, \ldots, E_k$ be the Poincaré duals of the exceptional divisors corresponding to $p_1, \ldots, p_k$. The first Chern class of $T^{1,0}_{J_S}M$ is then

$$c_1(M, J_S) = c_1(X) + \sum \epsilon_j,$$

where

$$\epsilon_j = \begin{cases} E_j & \text{if } j \in S \\ -E_j & \text{if } j \notin S. \end{cases}$$

If g is any Riemannian metric on M, the projection of $c_1(M, J_S)$ into the space $H^+(g) \subset H^2(M, \mathbf{R})$ of self-dual harmonic 2-forms is therefore $c_1(M, J_S)^+ = c_1(X)^+ + \sum \epsilon_j^+$. However, $c_1(X)^2 = c_1^2(X) > 0$, so $c_1(X)^+ \neq 0$. Now choose S so that

$$c_1(X)^+ \cdot \epsilon_j^+ \geq 0.$$

If c is the spin^c structure associated with this choice of S, the Seiberg-Witten invariant of (M, H^+, c) is non-zero [4], and Theorem 1 tells us that

$$\begin{aligned} \frac{1}{32\pi^2}\int_M s^2 d\mu &\geq (c_1(M, J_S)^+)^2 \\ &= (c_1(X)^+ + \sum \epsilon_j^+)^2 \\ &= (c_1(X)^+)^2 + 2\sum (c_1(X)^+ \cdot \epsilon_j^+) + (\sum \epsilon_j^+)^2 \\ &\geq (c_1(X)^+)^2 \\ &\geq (c_1(X))^2 = c_1^2(X) \\ &= 2\chi + 3\tau + k \end{aligned}$$

because the intersection form is positive definite on $H^+ = H^+(g)$.

Now suppose we have a metric g for which this inequality is actually an equality. Then each of the inequalities in the above calculation is an equality, and Theorem 1, applied to the first of these, tells us that g is Kähler with respect to a complex structure J_g compatible with c, and hence satisfying $c_1(M, J_g) = c_1(M, J_S)$. By the same reasoning, $(\sum \epsilon_j^+)^2 = 0$, and hence $\sum \epsilon_j^+ = 0$. In particular, $c_1(M, J_S)^+ = c_1(M, J_{\tilde{S}})^+$, where $\tilde{S} = \{1, \ldots, k\} - S$, so, even if $b^+ = 1$, the Seiberg-Witten invariant of $(M, H^+, \tilde{c})$ is also non-zero, where $\tilde{c}$ is the spin^c structure determined by $J_{\tilde{S}}$. The Seiberg-Witten equations for $\tilde{c}$, written with respect to the Kähler metric g, therefore have an irreducible solution; but this says [13, 4] that $-\sum \epsilon_j$ represents an effective divisor on (M, J_g). Thus, if $[\omega]$ is the Kähler class of (M, g, J_g), we have $[\omega] \cdot (-\sum \epsilon_j) > 0$, since this expression represents the area of a non-empty holomorphic curve. On the other hand, the Kähler form ω is self-dual with respect to g, so we have $[\omega] = [\omega]^+$; thus $[\omega] \cdot \sum \epsilon_j = [\omega]^+ \cdot \sum \epsilon_j^+ = 0$, in contradiction to the previous assertion. Our assumption was therefore false; the inequality is always strict. ∎

3 Einstein Metrics

Theorem 3 *Let X be a minimal complex algebraic surface of general type, and let $M = X\#k\overline{\mathbf{CP}}_2$ be obtained from X by blowing up $k > 0$ points. If $k \geq \frac{2}{3}c_1^2(X)$, then M does not admit Einstein metrics.*

Proof. For any Riemannian metric g on M, one has the generalized Gauss-Bonnet formula

$$2\chi + 3\tau = \frac{1}{4\pi^2}\int_M \left(2|W_+|^2 + \frac{s^2}{24} - \frac{|r_0|^2}{2}\right) d\mu$$

where s, r_0, and W_+ are respectively the scalar, trace-free Ricci, and self-dual Weyl curvatures of g; pointwise norms are calculated with respect to the metric, and $d\mu$ is the metric volume form. If g is an Einstein metric, $r_0 = 0$ and Theorem 2 therefore implies that

$$\begin{aligned} c_1^2(X) - k = 2\chi + 3\tau &= \frac{1}{4\pi^2}\int_M \left(2|W_+|^2 + \frac{s^2}{24}\right) d\mu \\ &> \frac{32\pi^2}{4\cdot 24\pi^2}(2\chi + 3\tau + k) \\ &= \frac{1}{3}c_1^2(X), \end{aligned}$$

so that

$$\frac{2}{3}c_1^2(X) > k,$$

contradicting our assumption. Hence M cannot admit an Einstein metric. ∎

Our main result now follows.

Theorem A *There are infinitely many compact simply-connected orientable smooth 4-manifolds which do not admit Einstein metrics, but nevertheless satisfy the strict Hitchin-Thorpe inequality.*

Proof. If X is any minimal complex surface of general type with $c_1^2 \geq 3$, there is then at least one integer k satisfying $c_1^2 > k \geq \frac{2}{3}c_1^2$. The complex surface $M = X\#k\overline{\mathbf{CP}}_2$ then satisfies the strict Hitchin-Thorpe inequality $2\chi > 3|\tau|$, but does not admit Einstein metrics by Theorem 3.

Now Seiberg-Witten theory implies [4] that $c_1^2(X)$ is a diffeomorphism invariant of $M = X\#k\overline{\mathbf{CP}}_2$, so it suffices to produce a sequence of simply-connected minimal surfaces X_j of general type such that the sequence of integers $c_1^2(X_j)$ is increasing. One such sequence is given by the Fermat surfaces $w^m + x^m + y^m + z^m = 0$ of degree $m = j + 4$, with $c_1^2 = j^3 + 4j^2$. ∎

4 The Symplectic Case

In order to keep our discussion as concrete and elementary as possible, we have thus far assumed that our 4-manifolds arose as compact complex surfaces. The proof of Theorem 2, however, only depends on the non-vanishing of certain Seiberg-Witten invariants of $M = X\#k\overline{\mathbf{CP}}_2$. Now if X admits a symplectic structure, the symplectic blow-up construction of McDuff [9] supplies a family of such structures on M, and a result of Taubes [10] then provides us with the non-vanishing invariants we need to prove the following:

Theorem 4 *Let (X, ω) be a symplectic manifold, and let $M = X\#k\overline{\mathbf{CP}}_2$. If $b^+(X) = 1$, assume that $c_1(X) \cdot [\omega] < 0$. Then any Riemannian metric on M satisfies*

$$\int_M s^2 d\mu > 32\pi^2 c_1^2(X).$$

Here, of course, $c_1(X)$ is the first Chern class of an almost-complex structure adapted to the symplectic structure. The assumption that $c_1(X) \cdot [\omega] < 0$ if $b^+ = 1$ is needed to compensate for the fact that Taubes' proof involves large perturbations of the Seiberg-Witten equations, whereas the relevant scalar curvature estimates stem from the unperturbed equations.

This immediately yields a generalization of Theorem 3:

Theorem 5 *Let (X, ω) be a symplectic manifold, and let $M = X\#k\overline{\mathbf{CP}}_2$. If $b^+(X) = 1$, assume that $c_1(X) \cdot [\omega] < 0$. If $k \geq \frac{2}{3}c_1^2(X)$, then M does not admit Einstein metrics.*

Of course, this is a trivial consequence of the Hitchin-Thorpe inequality unless $c_1^2(X) > 0$. On the other hand, it is unnecessarily weak if X is itself the blow-up of another symplectic manifold. In analogy with the Enriques-Kodaira classification, it is therefore natural to introduce a definition which characterizes the natural setting for applications of these results:

Definition 2 *A minimal symplectic 4-manifold (X, ω) is of* general type *if*

(a) $c_1^2(X) > 0$; and

(b) $c_1(X) \cdot [\omega] < 0$.

A symplectic 4-manifold of general type is an iterated symplectic blow-up of a minimal symplectic manifold of general type.

If $b^+ > 1$, Taubes [11, 12] has shown that condition (b) is automatic and that (a) fails only for minimal symplectic manifolds with $c_1^2 = 0$. In analogy to the Kodaira classification, the latter class of minimal symplectic manifolds might be conjectured to all arise as elliptic fibrations unless c_1 is a torsion class.

5 Concluding Remarks

Theorem 1 tells us that any Riemannian metric on a non-minimal complex surface $M = X\# k\overline{\mathbf{CP}}_2$ of general type satisfies the scalar-curvature estimate

$$\int_M s^2 d\mu > 32\pi^2 c_1^2(X),$$

where X is the minimal model for M. In fact, this estimate is sharp, at least if X does not contain any (-2)-curves. Indeed, this last assumption implies [2] that $c_1(X) < 0$; thus X admits [1, 14] a Kähler-Einstein metric $\check{g}$, and one then has

$$\int_X s_{\check{g}}^2 d\mu_{\check{g}} = 32\pi^2 c_1^2(X).$$

Let $p_1, \ldots, p_k \in X$ be distinct points which will be blown up to obtain a smooth model for M, and choose disjoint complex coordinate charts centered on these points so that

$$\check{g} = \delta + O(\varrho^2)$$

where δ and ϱ are respectively the Euclidean metric and radius associated with the chart. Define $h_1 = \delta - \check{g}$. Let h_2 denote the pull-back of the Fubini-Study metric on $\mathbf{CP}_1$ to $\mathbf{C}^2 - 0$ via the tautological projection, and use these same charts to transplant h_2 to a punctured neighborhood of each of the $p_1, \ldots, p_k$. Let $\phi : \mathbf{R} \to \mathbf{R}$ be a non-negative smooth function which is identically 1 on $(-\infty, \frac{1}{2})$ and identically 0 on $(1, \infty)$, and, for each sufficiently small $t > 0$, let g_t be the smooth Riemannian metric on the blow-up M whose restriction to the open dense set $X - \{p_1, \ldots, p_k\}$ is given by

$$g_t = \check{g} + \phi(t^{-1}\varrho)(h_1 + t^4 h_2).$$

For $\varrho < t/2$, this metric coincides up to scale with the Burns metric [6] on the blow-up of $\mathbf{C}^2$ at the origin, and so has scalar curvature $s \equiv 0$; and for $\varrho > t$, it coincides with $\check{g}$. In the transition region $\varrho \in (t/2, t)$, one has

$$\begin{aligned} \|\phi(t^{-1}\varrho)(t^4h_2 - h_1)\| &\leq C_0 t^2 \\ \|\mathbf{D}\left[\phi(t^{-1}\varrho)(t^4h_2 - h_1)\right]\| &\leq C_1 t \\ \|\mathbf{D}^{(2)}\left[\phi(t^{-1}\varrho)(t^4h_2 - h_1)\right]\| &\leq C_2 \end{aligned}$$

where $\mathbf{D}$ is the Euclidean derivative operator associated with the given coordinate system, and the constants C_j are independent of t. Thus $s^2(g_t)$ is uniformly bounded as $t \to 0$, and since the volume of the annular transition region is of order t^4, we conclude that

$$\lim_{t\to 0^+} \int_M s_{g_t}^2 d\mu_{g_t} = \int_X s_{\check{g}}^2 d\mu_{\check{g}} = 32\pi^2 c_1^2(X).$$

The bound is therefore sharp, as claimed.

Even if X contains (-2)-curves, the above conclusion should still hold. Indeed, if $\check{X}$ is the complex orbifold obtained by collapsing all the (-2)-curves in X, then the Aubin-Yau proof, without essential alterations, seems to show that $\check{X}$ admits an orbifold Kähler-Einstein metric $\check{g}$ with singularities modeled on $\mathbf{C}^2/(\pm 1)$. The previous formula for g_t then merely needs to be augmented by using Eguchi-Hanson metrics to smooth out the orbifold singularities of $\check{g}$.

One strategy for finding Einstein metrics on a compact 4-manifold is to try to minimize $\int s^2 d\mu$, since a critical point of this functional is either Einstein or scalar-flat. We have just seen, however, that this strategy will fail on a non-minimal surface of general type because a minimizing sequence can collapse (bubble off) to a metric on a topologically simpler manifold. In light of this and Theorem 3, it seems plausible to conjecture that non-minimal surfaces of general type never admit Einstein metrics. Indeed, making a considerable leap of faith beyond [7], one might conjecture that Einstein metrics on irrational surfaces are always Kähler. Further progress in this direction, however, would seem to require estimates on the norm of the Weyl curvature which for the present remain elusive.

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