Review Sheet - MAT 125

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November 11, 2008

1 Precalculus Material.

Notation.

We write $a \cdot b$ or simply ab for a times b.

We write $f^{-1}(x)$ for the inverse function to $f(x)$, which exists as long as f passes the horizontal line test.

We write $[a, b]$ for the numbers between a and b, including a and b. We write (a, b) for the numbers between a and b, *not* including a and b. Can you guess what $[a, b)$ means?

Fractions.

$$\frac{1}{(a/b)} = \frac{b}{a} \tag{1}$$

$$\frac{a}{b} = \frac{ac}{bc} \tag{2}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \tag{3}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \tag{4}$$

Exponents and Logarithms.

Let a be a number greater than 1. Consider the function $f(x) = a^x$. This function is increasing. So it passes the horizontal line test. So it has an inverse, which we call $f^{-1}(x) = \log_a(x)$. (In other words, logs allow you to "cancel out" exponentiation.) Here are properties associated with these functions:

If $a = 10$, we write $\log_a(x) = \log(x)$. If $a = e$, we write $\log_a(x) = \ln(x)$.

$$\log_a (a^x) = a^{\log_a(x)} = x \tag{5}$$

$$a^0 = 1 \qquad\qquad 0 = \log_a(1) \tag{6}$$

$$a^1 = a \qquad\qquad 1 = \log_a(a) \tag{7}$$

$$a^{(b+c)} = a^b \cdot a^c \qquad\qquad \log_a(b) + \log_a(c) = \log_a(b \cdot c) \tag{8}$$

Remark. There is no rule for simplifying $\log_a(b + c)$.

$$a^{-1} = \frac{1}{a} \qquad\qquad -1 = \log_a(1/a) \tag{9}$$

$$\left(a^b\right)^c = a^{bc} \qquad\qquad n \log_a(x) = \log_a(x^n) \tag{10}$$

$$a^{-b} = \frac{1}{a^b} \qquad\qquad -\log_a(x) = \log_a(1/x) \tag{11}$$

$$a^{1/2} = \sqrt{a} \qquad a^{1/n} = \sqrt[n]{a} \qquad\qquad \log_a(\sqrt[n]{x}) = \frac{1}{n}\log_a(x) \tag{12}$$

$$a^c b^c = (a \cdot b)^c \qquad\qquad \log_a(x) = \frac{\log_b(x)}{\log_b(a)} \tag{13}$$

$$\text{DOMAIN}\,(a^x) = \text{RANGE}\,(\log_a(x)) = (-\infty, \infty) \tag{14}$$

$$\text{RANGE}\,(a^x) = \text{DOMAIN}\,(\log_a(x)) = (0, \infty) \tag{15}$$

Trigonometry.

We often write $\theta =$ "theta" for angles. In this course, and in future math courses, we'll use *radians* instead of degrees. The definition is

$$x^\circ = \frac{\pi x}{180}\text{radians} \tag{16}$$

So $90^\circ = \frac{\pi}{2}$ radians. We simply write θ instead of "θ radians". Now we'll review trigonometry. Suppose we have a triangle with a right angle in it. Let its longest side have length 1. Let θ be one of the other angles. Then we denote the length of the side adjacent to θ by $\cos(\theta)$ and that of side opposite to it by $\cos(\theta)$.

Sines and cosines enjoy the following properties:

$$\sin^2(\theta) + \cos^2(\theta) = 1 \tag{17}$$

$$\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a) \tag{18}$$

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b) \tag{19}$$

Therefore,

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \tag{20}$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 \tag{21}$$

Therefore, we have

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2} \tag{22}$$

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \tag{23}$$

2 Calculus.

Limits and Continuity. If a function is continuous, we can just evaluate a limit by "plugging in":

$$\lim_{x \to a} f(x) = f(a) \tag{24}$$

In fact, this is the definition of continuity. Intuitively, continuity means you can draw a function without taking your pencil off the page.

For a limit to exist, we must have

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \tag{25}$$

The lefthand side is called the limit from the left, and the righthand side is the limit from the right. If these exist, their common value is called the limit. So when you look at the limit at a point, follow the graph from the left and from the right. If these limits are equal, so does the limit.

Limits Preserve Inequalities. Say we are interested in the point $x = a$. Suppose we look at a small neighborhood $(a - c, a + c)$ (where c can be any positive number), and we observe that

$$f(x) \leq g(x) \tag{26}$$

there, with the possible exception of $x = a$ itself. Then we can conclude:

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \tag{27}$$

Squeeze Theorem. Therefore, suppose $f(x) \leq g(x) \leq h(x)$ on a small neighborhood $(a - c, a + c)$ of $x = a$, with the possible exception of $x = a$. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \tag{28}$$

then

$$L \leq \lim_{x \to a} g(x) \leq L \tag{29}$$

So we also must have

$$\lim_{x \to a} g(x) = L \tag{30}$$

For example,

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \tag{31}$$

except at $x = 0$.

Since $x^2 \geq 0$, we have

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2 \tag{32}$$

Taking the limit as $x \to 0$, we conclude

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 \tag{33}$$

Some Very Special Limits.

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \tag{34}$$

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0 \tag{35}$$

$$\lim_{x \to \infty} \frac{1}{x} = 0 \tag{36}$$

$$\lim_{x \to -\infty} e^x = 0 \tag{37}$$

$$\lim_{x \to \infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \ldots + b_0} = \frac{a_n}{b_n} \tag{38}$$

and that includes 0 if $a_n = 0$ but $b_n \neq 0$, and ∞ if $b_n = 0$ and $a_n > 0$, and $-\infty$ if $b_n = 0$ and $a_n < 0$.

We say $f(x)$ has $x = a$ as a *horizontal asymptote* if

$$\lim_{x\to\infty} f(x) = a \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = a \tag{39}$$

We say $f(x)$ has a *vertical asymptote* at $x = a$ if

$$\lim_{x\to a} f(x) = \infty \qquad \text{or} \qquad \lim_{x\to a} f(x) = -\infty \tag{40}$$

Limit Laws.

$$\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \tag{41}$$

$$\lim_{x\to a} c \cdot x = c \cdot \lim_{x\to a} x \tag{42}$$

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x) \tag{43}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \tag{44}$$

if $\lim_{x\to a} g(x) \neq 0$.

Derivatives.

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{45}$$

is the *derivative* of $f(x)$ with respect to x. Geometrically, it is the slope of the line tangent to $f(x)$ at $x = a$.

The equation of the tangent line to $f(x)$ at the point $x = a$ is given by

$$y = f(a) + f'(a)(x - a) \tag{46}$$

Sum and Difference Rules.

$$(f(x) + g(x))' = f'(x) + g'(x) \tag{47}$$

$$(f(x) - g(x))' = f'(x) - g'(x) \tag{48}$$

Product and Quotient Rules.

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \tag{49}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \tag{50}$$

Power Rule.

$$(x^a)' = ax^{a-1} \tag{51}$$

Notice, this rule can **NOT** be used on a^x. Rather, for that we have

$$(a^x)' = a^x \ln(a) \tag{52}$$

Thus $\ln(x)$ comes up naturally when taking derivatives, which justifies the name "natural logarithm". We therefore have

$$(e^x)' = e^x \tag{53}$$

Also,

$$(\ln(x))' = \frac{1}{x} \tag{54}$$

Since $\log_a(x) = \ln(x)/\ln(a)$ we have

$$(\log_a(x))' = \frac{1}{\ln(a)x} \tag{55}$$

Trigonometric Derivatives.

Since

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}=\lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \tag{56}$$

$$=\lim_{h\to 0}\left(\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h}\right)=\cos(x) \tag{57}$$

we have

$$(\sin(x))'=\cos(x) \tag{58}$$

By similar reasoning,

$$(\cos(x))'=-\sin(x) \tag{59}$$

Notice the that the derivative of sine is *minus* cosine. Now that we know these two derivatives, we can use the Quotient Rule to find the other trig derivatives:

$$(\tan(x))'=\left(\frac{\sin(x)}{\cos(x)}\right)'=\frac{(\sin(x))'\cos(x)-\sin(x)\,(\cos(x))'}{\cos^2(x)}=\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} \tag{60}$$

$$=\frac{1}{\cos^2(x)}=\left(\frac{1}{\cos(x)}\right)^2=\sec^2(x) \tag{61}$$

Therefore

$$(\tan(x))'=\sec^2(x) \tag{62}$$

Similarly we can obtain

$$(\cot(x))'=-\csc^2(x) \tag{63}$$

$$(\sec(x))'=\sec(x)\tan(x) \tag{64}$$

$$(\csc(x))' = -\csc(x)\cot(x) \tag{65}$$

An simple trick to remember three rules instead of six. If you know the formula for the derivative of a trig function, you can find the derivative of its "co"-function by replacing the functions in the formula by co-functions and putting a minus sign at the front. For example, say you know the derivative of secant is secant times tangent. Then the derivative of *co*secant is minus *co*secant times *co*tangent.

Chain Rule and Implicit Differentiation.

Recall that if f and g are functions, then you can form a new function, f composed with g, denoted by

$$(f \circ g)(x) = f(g(x)) \tag{66}$$

There is a rule for differentiating the composition of functions. It is called the *Chain Rule*. The rule is

$$(f \circ g)'(x) = f'(g(x))g'(x) \tag{67}$$

So you plug g into f' and then multiply by g'. For example, if we want to find

$$(-\ln(\cos(x))' \tag{68}$$

then we can write $f(x) = -\ln(x)$ and $g(x) = \cos(x)$. Then $f'(x) = -1/x$ and $g'(x) = -\sin(x)$. So it follows that

$$(-\ln(\cos(x))' = -\left(-\frac{1}{\cos(x)}\sin(x)\right) = \tan(x) \tag{69}$$

An unexpected result, impossible to obtain without the Chain Rule!

Now that we have the chain rule, we can differentiate equations like

$$x^2 + y^2 = 1 \tag{70}$$

(with respect to x) even if y is not by itself on the lefthand side. In particular, y does not need to be a function. This is called *implicit differentiation*. By the Chain Rule we have

$$(f(y))' = f'(y)y' \tag{71}$$

So if $f(u) = u^2$, then $f'(u) = 2u$. So we have

$$\left(y^2\right)' = 2yy' \tag{72}$$

Therefore, implicit differentiation works like this:

$$x^2 + y^2 = 1 \tag{73}$$

$$\left(x^2 + y^2\right)' = (1)' \tag{74}$$

$$2x + 2yy' = 0 \tag{75}$$

Solving for y' by itself, we obtain

$$y' = -\frac{x}{y} \tag{76}$$

Let's take another example and implicitly differentiate

$$x^2 + y^2 = xe^y \tag{77}$$

Take the derivative of both sides:

$$\left(x^2 + y^2\right)' = (xe^y)' \tag{78}$$

We simply differentiate like normal, except multiply by y' when we differentiate a function of y:

$$2x + 2yy' = 1 \cdot e^y + xe^y y' \tag{79}$$

We can even solve for y':

$$y' = \frac{2x - e^y}{xe^y - 2y} \tag{80}$$

Sometimes we write $\frac{dy}{dx}$ instead of y' to emphasize that differentiation is with respect to x. Then for example $\frac{dx}{dx}$ is just 1 since the dx's cancel out.

Logarithmic Differentiation. This is just a trick using the Chain Rule that helps us differentiate exponential expressions. Suppose we want to find

$$(x^x)' \tag{81}$$

Neither the Power Rule, nor the rule for a^x is applicable in this case. Please remember that for the exam. Instead, we need to use a trick called logarithmic differentiation. Make the substitution

$$u = (x^x)' \tag{82}$$

By the Chain Rule, we have

$$(\ln(x^x))' = \frac{1}{x^x}u \tag{83}$$

But also

$$\ln(x^x) = x\ln(x) \tag{84}$$

So

$$(\ln(x^x))' = \ln(x) + \frac{x}{x} = 1 + \ln(x) \tag{85}$$

by the Product Rule. Therefore,

$$\frac{1}{x^x}u = \ln(x) + \frac{x}{x} = 1 + \ln(x) \tag{86}$$

In other words,

$$(x^x)' = u = (1 + \ln(x))\,x^x \tag{87}$$

and we have solved the problem.

Also the derivative formulas

$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}} \tag{88}$$

$$(\arctan(x))' = \frac{1}{1+x^2} \tag{89}$$

can be found using implicit differentiation, and drawing right triangles.