Review Sheet - MAP 103

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1 Introduction.

This is a review sheet for **MAP 103: Proficiency Algebra** which I will continue to develop throughout the semester. You can skip the first section, especially if you're studying for a midterm at the last minute, but it is informative, so do read it if you get a chance. If you get stuck going from one step to the next in an argument, don't give up, but rather try to figure out what has changed from the old step to the new. Usually there will only be one possibility.

2 Properties of the Real Numbers.

The real numbers are a *field*, which means that they are endowed with operations $+$ and $\cdot$ such that for any real numbers a, b, c,

1) *Associativity of Addition and Multiplication.* $(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$

2) *Commutativity of Addition.* $a+b = b+a$ and $ab = ba$

3) *Existence of Zero and One.* ...there are numbers 0 and 1 such that $a+0 = 0+a = a$ and $a1 = 1a = a$

4) *Existence of Additive and Multiplicative Inverses.* For every number a there is a number $-a$ such that $a+(-a) = (-a)+a = 0$. For every number $a \neq 0$ there is a number a^{-1} such that $aa^{-1} = a^{-1}a = 1$.

5) *Distributivity of Multiplication Over Addition.* a(b+c)=ab+ac

The real numbers are ordered, which means that the exists a relation $\leq$ such that, for any real numbers a, b, c,

6) $a \leq b$ or $b \leq a$

7) $a \leq b$ and $b \leq a$ together imply $a = b$

8) $a \leq b$ implies that $a + c \leq b + c$

9) $0 \leq a$ and $0 \leq b$ together imply that $0 \leq ab$

These are all properties that are important to know. There is a last property which, together with the others, characterizes the real numbers, but you do not need to know it unless maybe you major in math. That property is,

Completeness. Let S be a collection of real numbers contained in some interval $(-\infty, b]$. Then we can choose a smallest number b' for which S is contained in $(-\infty, b']$.

Why are these properties important? Well, we can use it to define the *absolute value* function:

$|x| = x$ if $0 \leq x$. $|x| = -x$ if $x \leq 0$.

This function is defined for any real number x by property (6), since we have $x \leq 0$ or $0 \leq x$. It just amounts to taking the positive part (or keeping it zero). So $|1| = 1$ and $|-2| = 2$ and $|0| = 0$.

We can also use these properties to prove that the square of a real number can't be negative. Suppose $x^2 \leq 0$. By property (6) we have $0 \leq x$ or $x \leq 0$. If $0 \leq x$, then by proprety (9), $0 \leq x^2$, so by property (7) $x^2 = 0$.

On the other hand, if $x \leq 0$, then by property (8) $x - x \leq 0 - x$, so that $0 \leq -x$. Then by property (7) $0 \leq (-x)^2$. So $0 \leq x^2$. And again by property (7) $x^2 = 0$.

So the square of a real number can't be negative.

3 Polynomials.

A *polynomial* is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \tag{1}$$

where the a_i are real numbers, and n is a nonnegative whole number. So the following are polynomials:

$$p(x) = 2x + 3 \tag{2}$$

$$p(x) = x^6 + x^5 + 2x^3 + \pi x^2 + 9.6 \tag{3}$$

but **NOT** x^π or $2x^{3.5} + 9x$ or x^x or

$$x^{1/a} \tag{4}$$

which equals $\sqrt[a]{x}$ or

$$x^{-1} \tag{5}$$

which equals $1/x$. Sometimes things that are polynomials may not look like polynomials. For example,

$$\frac{x^{-3}y^{-1/2}}{x^{-6}y^{-3/2}} = x^3 y \tag{6}$$

is a polynomial–in both x and y.

The highest exponent n is called the *degree* or *order* of the polynomial. We will mainly concern ourselves with first and second degree polynomials. To add polynomials, simply combine like terms as the following example illustrates:

$$\left(x^4 + 3x^2 + \frac{1}{2}x + 9\right) + \left(67x^3 - 5x^2 - \frac{1}{3}x + 4.5\right) \tag{7}$$

$$= x^4 + 67x^3 - 2x^2 + \left(\frac{1}{2} - \frac{1}{3}\right)x + 13.5 = x^4 + 67x^3 - 2x^2 + \frac{1}{6}x + 13.5 \tag{8}$$

The difference of two polynomials is also a polynomial. Remember that when you subtract any kind of sum, you have to distribute the minus sign, since you are really multiplying by minus one:

$$-\left(x^3 - 4x^5 + 696x\right) = -x^3 + 4x^5 - 696x \tag{9}$$

The product of polynomials is also a polynomial. To multiply polynomials, you have to use the distributive property. Here is an example:

$$\left(6x^3 + 4x^2 + x\right)\left(2x^3 + 9x + 7\right) \tag{10}$$

I will rewrite this with brackets. The brackets should help you follow where everything gets multiplied out to:

$$\left([6x^3] + [4x^2] + [x]\right)\left(2x^3 + 9x + 7\right) \tag{11}$$

$$= \left[12x^6 + 54x^4 + 42x^3\right] + \left[8x^5 + 36x^3 + 28x^2\right] + \left[2x^4 + 9x^2 + 7x\right] \tag{12}$$

(...now combine like terms...)

$$= 12x^6 + 8x^5 + 56x^4 + 78x^3 + 37x^2 + 7x \tag{13}$$

4 Where Does a Polynomial Equal Zero?

Let $p(x)$ be a polynomial. If we plug in a number r and $p(r) = 0$, then we say r is a *root* of that polynomial. For example, $\sqrt{10}$ is a root of the polynomial

$$p(x) = x^2 - 10 \tag{14}$$

This is (probably) where the term *square root* comes from. A second-degree polynomial such as the one above is called a *quadratic polynomial.* The most general form of a quadratic polynomial is

$$p(x) = ax^2 + bx + c \tag{15}$$

The *quadratic formula* gives us a formula for the roots of this polynomial, which exist if and only if $b^2 - 4ac \geq 0$. The formula is

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{16}$$

Memorize this. Here is a proof that the formula works which you don't need to know for the exam, but is good FYI. If you learn this proof, you will learn something that only a small percentage of people on this earth know.

Let $ax^2 + bx + c = 0$. First, we apply a trick called *completing the square*. Since

$$a\left(x + \frac{b}{2a}\right)^2 = a\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) \tag{17}$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) = ax^2 + bx + \frac{b^2}{4a} \tag{18}$$

we have

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0 \tag{19}$$

Now let us solve for x:

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c \tag{20}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \tag{21}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{22}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \tag{23}$$

Finally,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{24}$$

5 Dividing One Polynomial By Another.

If you make a fraction with a polynomial on the top (numerator) and one on the bottom (denominator), then the result will not always be a polynomial. We call a fraction of two polynomials a *rational function*. Unlike a polynomial, a rational function might not be defined everywhere. A rational function is undefined precisely where the denominator equals zero. For example,

$$\frac{x^99 + x^{19,832,649} + 89357934x^3 - 4908}{x^2 + 3x + 2} \tag{25}$$

is undefined precisely when $x^2 + 3x + 2 = 0$, in other words at $x = -1$ and $x = -2$.

If you have an equation involving rational functions, you can cross-multiply to get an equation involving polynomials. For example, let's try to solve

$$\frac{x-1}{x-2} = \frac{2x-3}{x-5} \tag{26}$$

Multiplying both sides by $(x-2)(x-5)$ gives us

$$(x-1)(x-5) = (2x-3)(x-2) \tag{27}$$

$$x^2 - 6x + 5 = 2x^2 - 7x + 6 \tag{28}$$

$$0 = x^2 - x + 1 \tag{29}$$

This doesn't have any solutions, because $(-1)^2 - 4 < 0$.

Remember that to add two rational functions together, you just have to make a common denominator. So

$$\frac{x^2 + x^3}{4x^5 + 9} + \frac{x^3 + x^6}{x^3 + 9x} = \frac{(x^3 + 9x)(x^2 + x^3) + (4x^5 + 9)(x^3 + x^6)}{(4x^5 + 9)(x^3 + 9x)} \tag{30}$$

Finally, suppose that $a(x)/b(x)$ is a rational function. Then we can write

$$a(x) = q(x)b(x) + r(x) \tag{31}$$

where $q(x)$ and $r(x)$ are polynomials. We call $q(x)$ the quotient, and $r(x)$ the remainder of the *long division*. (We call $a(x)$ the dividend and $b(x)$ the divisor.) Effectively, what we are doing is long division of polynomials. I will go over the long division in class before Midterm 1, so we can get a proper review on it. It's too unwieldy to type up. Maybe I'll scan a handwritten example later.