MAT 514 - Analysis for Teachers II - (Complex Analysis)

Paul Cernea

July 20, 2009

1 Introduction.

This is a course on complex analysis. A needlessly foreboding term, I prefer to call it *Calculus With Imaginary Numbers*. Since you are most likely training to become K-12 teachers, you won't often be using the theory from this course directly. Nevertheless, there are several reasons to learn it. It is highly elegant in its own right; it cultivates a deeper understanding of the usual single-real-variable calculus that you'll probably teach; and, perhaps most importantly, it teaches you the techniques behind rigorous mathematical proof, as well as the need for it. If there's one thing I want you to take away from this course, it's how to create a rigorous proof, and why it's important. Furthermore, I hope you will pass this on to your own students when the time comes.

2 Course Logistics.

Time/Location: Tuesdays and Thursdays 1:30 PM - 4:55 PM / Physics Building P-125

Textbook: *Basic Complex Analysis* 3rd Edition by Hoffman and Marsden, available in the bookstore.

Instructor: Paul Cernea

Office: Math Building 5-125 A ; Tuesdays and Thursdays 5:10 PM

Email: pcernea@gmail.com

Homework will be assigned Thursdays and collected the following Thursday. There will be one take-home midterm, and a non-take-home final. The grade will be distributed as follows:

Homework (60 %) / Take-Home Midterm 1 (20 %) / Final (20 %)

3 Course Schedule.

First we will define the complex numbers. These form a coordinate plane whose x-axis is the real numberline. We will find that all the nice elementary functions we know from first-semester calculus have natural extensions to the complex plane. We will figure out how to differentiate complex-valued functions, and deduce that the derivative formulas for elementary functions are the same as the familiar ones from calculus. The functions that are differentiable on open neighborhoods are said to be analytic on those neighborhoods. If a function is analytic on the whole complex plane, it is called entire.

Day 1. Begin Chapter 1, Analytic Functions. Section 1.1, Introduction to Complex Numbers. Section 1.2, Properties of Complex Numbers. Section 1.3, Some Elementary Functions.

Day 2. Section 1.4, Continuous Functions.

Day 3. Section 1.5, Basic Properties of Analytic Functions. Section 1.6, Differentiation of Elementary Functions. End Chapter 1.

Having learned how to differentiate analytic functions, we will study how to integrate them. We cannot just proceed in a naive fashion, instead we have to recall our knowledge of vector calculus. The all-important Cauchy's Theorem is a consequence of Green's Theorem. You will have to learn the proof of this for the exam. Cauchy's Theorem allows us, among other things, to take antiderivatives.

Day 4. Begin Chapter 2, Cauchy's Theorem. Section 2.1, Contour Integrals. Section 2.2, Cauchy's Theorem–A First Look.

From Cauchy's Theorem we can deduce Cauchy's Integral Formula, which allows us to write an analytic function in the form of a certain integral. We can differentiate this integral as many times as we like giving us convenient formulas for the k*th derivative of an analytic function. This is already a striking difference between real and complex analysis. Unlike in the real case, if a function of complex variables is once-differentiable, it is infinitely differentiable. From Cauchy's Integral Formulas for derivatives, we can deduce Liouville's Theorem which says that any bounded entire function is constant. A remarkable consequence of this is the Fundamental Theorem of Algebra!*

Day 5. Section 2.3, A Closer Look At Cauchy's Theorem. Section 2.4, Cauchy's Integral Formula. Section 2.5, Maximum Modulus Theorem, Harmonic Functions. End Chapter 2.

Day 6. Review. I can/will hold out-of-class review sessions by popular demand. *Assign Take-Home Midterm!!!*

Day 7. *Collect Take-Home Midterm!!!* Begin Chapter 3, Series Representation of Analytic Functions. Section 3.1, Convergent Series of Analytic Functions. Section 3.2, Power Series and Taylor's Theorem.

For the second part of the course we'll talk about infinite series. Any function analytic about a point z will agree with its Taylor series around that point, up to its radius of convergence. This is another vast difference between real and complex variables. A real-valued function can be infinitely differentiable and disagree with its Taylor series. This makes analytic functions in some sense very rare! We'll discuss Laurent series, which are like Taylor series, but with negative powers too. The coefficient of the (-1)st term is called the residue. The Residue Theorem is an immensely powerful tool that allows us to evaluate tricky definite integrals!

Day 8. Section 3.3, Laurent Series and Classification of Singularities. End Chapter 3. Begin Chapter 4, Calculus of Residues. Section 4.1, Calculation of Residues.

Day 9. Section 4.2, Residue Theorem. Section 4.3, Evaluation of Definite Integrals. Section 4.4, Evaluation of Infinite Series and Partial Fraction Expansions. End Chapter 4.

We'll briefly discuss some more advanced topics. The Riemann Mapping Theorem is an incredible theorem that perhaps only the far-reaching genius of Riemann could've foreseen! It says that if A is any open, simply-connected subset of the complex plane (besides the whole plane or the empty set), and B is another such region, then there is a one-to-one analytic function whose domain is A and whose range is B (and vice-versa). In particular, this mapping preserves angles. This allows us to translate problems from complicated regions Σ to simple regions like the unit disc, solve them there, and translate the solution back.

We will also discuss analytic continuation. Remember I said that analytic functions are very rare, in stark contrast to smooth real-valued functions? For example, there are infinitely-differentiable functions which are bumpy between 1 and -1, but constant for x with $|x| > 1$. Not so for analytic functions. If $f(z)$ is analytic on an open connected region A, and you patch an open connected region B onto A, there is at most one way to extend $f(z)$ to B. I am only going to briefly touch upon these advanced topics, so I don't expect you to have any deep knowledge of them, but I'll expect you to know the statements of the main results, and perhaps have a vague idea of what they mean.

Day 10. Introduction to Chapter 5, Conformal Mapping. Section 5.1, Basic Theory of Conformal Mappings. Introduction to Chapter 6, Further Development of the Theory. 6.1, Analytic Continuation and Elementary Riemann Surfaces.

Day 11. If you ladies and gents still have the energy, we can shoot the breeze about Optional Topics like the Gamma Function, Riemann-Zeta Function. Those won't be on the test. Also, we'll have Review. I can/will hold out-of-class review sessions by popular demand. Also, we can watch a movie or do something cool.

Final Time/Location: TBA.

$$\maltese \quad \maltese \quad \maltese \quad \maltese \quad \maltese \quad \maltese \quad \maltese \quad \maltese \quad \maltese \quad \maltese \quad \maltese \quad \maltese \quad \maltese \quad \maltese \quad \maltese \quad \maltese \tag{1}$$