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THE

RIEMANN MAPPING THEOREM

FOR $\mathbb{R}^3$

University of Maryland
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Nice to be back at SUNY. I was here 10 years ago when I had some very preliminary results to talk about[1]. There was an inspiring conversation with Sullivan who explained how he and Thurston had developed a QC mapping theorem for the domains in $\mathbb{R}^3$ which are "inscribed by balls"(in analogy to their work in $\mathbb{R}^2$, also described in this conference).

This work has been developed over 10 years, preliminary versions were given at the Hayman fest in London (2002) and Purdue. Final version was announced at UM colloquium. First time before international QC experts was the conference in Israel (2006). I will be describing 5 long papers. Forgive me if it sounds all conceptual I emphaisize there was a myraid of technical problems were solved.

On www.quasiconformal.com you'll find links to papers. At the moment it has the original annoucement on the ARKIV but shortly this will be will be updated.

[1]Reifenberg conditions on a surface were equivalent to a bilip reflection, at least for small constants

The problem of finding good co-ordinates is one of the oldest in mathematics. As the earth is not perfect sphere Gauss raised the question of finding orthogonal co-ordinate systems on arbitrary surfaces.

This question was finally completely solved by Ahlfors and Bers (Annals, 1960).

"Measurable Riemann Mapping"

The analytic case is the Uniformization Theorem: Hilbert's XXII problem(1900 ICM). [2]

Proof by Paul Koebe (1907)

(Riemann Mapping Theorem)
Any simply connected domain ($\neq \mathbb{R}^2$) is conformally equivalent to $\mathbb{B}^2$

We tend to forgot about the requirement of "non-empty boundary". However in higher dimensions we find that the boundary properties are crucial.

[2] Hilbert asked about higher dimensions too, but only for several complex variables!

Now by Liouville the only conformal mappings of domains of $\mathbb{R}^3$ are elementary: rotations, inversions etc.

Instead we consider K**-quasiconformal mappings**

$$F : \mathbf{D} \to \mathbf{R}$$

homeomorphisms mapping small balls onto ellipsoids of bounded eccentricity. (e.g. PL maps, diffeomorphisms)

In general we consider Quasisymmetric (QS) mappings:

$$F : \mathbb{R}^m \to \mathbf{S} \subset \mathbb{R}^n$$

here $n \geq m$. Now for any conformal mapping

$$L : \mathbb{R}^m \to \mathbb{R}^m$$

the renormalization is

$$\tilde{F} = \{F \circ L - F \circ L(0)\} /t$$

where normalizing factor $t = ||F \circ L(\mathbf{i}) - F \circ L(0)||$.

F is quasisymmetric (QS) iff the renormalizations are precompact in space of homeomorphisms. [3]

[3] The idea is familar from calculus where one definition of derivative says the "blowups" converge to linear functions.

Some facts about QS:

1. n=m $\Rightarrow$ QS = QC.

2. Any QS map $F : \mathbb{R}^m \to \mathbb{R}^m$ extends to a QC mapping of $\mathbb{R}^{m+1}$ (see versions by Ahlfors, Carleson as well as Väisälä ,Tukia, Douady and Earle)

Väisälä and Gehring (Acta, 1964) produced many necessary and different sufficient conditions for a domain to be a QC image of $\mathbb{B}^3$, i.e. a QC -ball.

Ahlfors (1964) to asked:

Characterize QC-balls.

This question was repeated by Ahlfors in 1978 ICM plenary talk and Gehring in his 1986 ICM plenary talk.

THE FOLLOWING ARE QC-BALLS:

1. Smooth topological balls

2. Half spaces

3. Cylinders

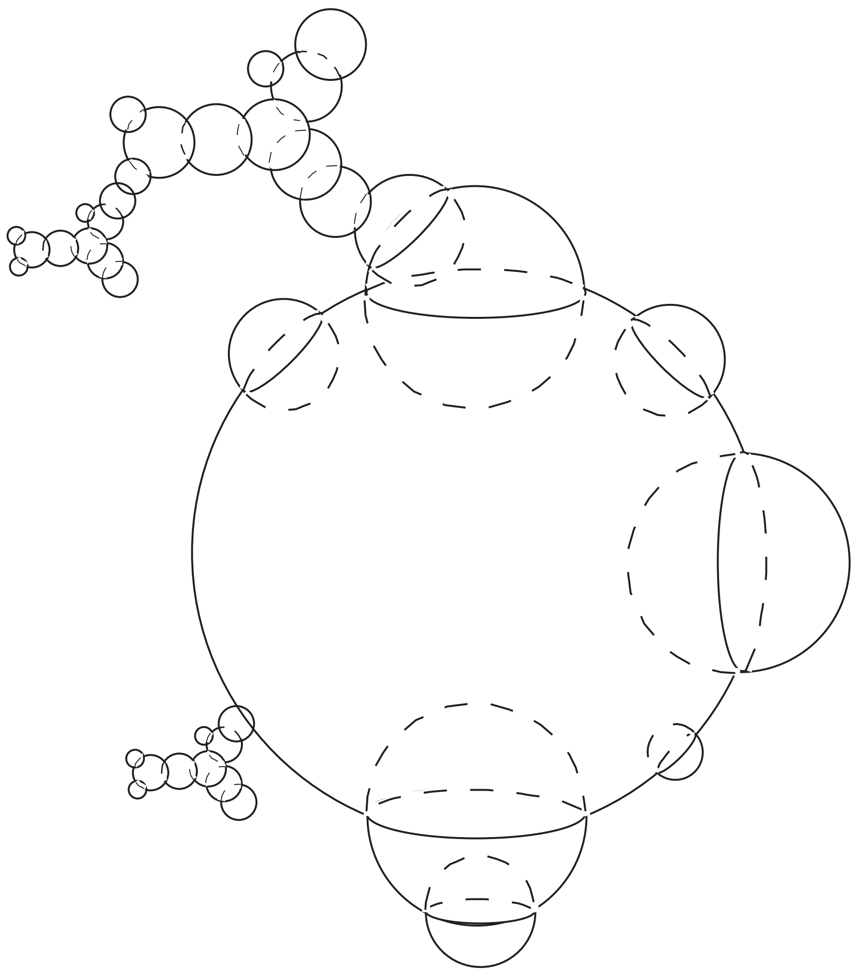
4. Smooth WITH outward cusp.

BUT THESE ARE NOT:

1. Space between parallel planes

2. Smooth WITH inward cusp.

More sophisticated examples of QC-balls come from the Suliivan-Thurston method of inscribed balls, i.e. soap bubble domains. You begin with a ball. Then take a circle packing of its boundary. Each circle is now extended to a full ball. Then you take a circle packing of the exposed boundaries of the new balls... obviously you have to be careful to have new balls always disjoint but otherwise iterating this process always gives you a QC-ball in the limit.



A Sullivan-Thurston Domain :
i.e a tree of inscribed balls

Global Problem

Consider a so called QC mapping, i.e.

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

The unit sphere $\mathbb{S}^{n-1}$ is transformed to a "quasisphere".

For $n = 2$ Ahlfors (Acta, 1964):
$\mathbf{T}$ is a " quasicircle" iff there is a constant k so that

$$\frac{|z_1 - z_2| + |z_2 - z_3|}{|z_1 - z_3|} \leq k$$

for all ordered $z_1, z_2, z_3 \in \mathbf{T}$.

Equivalently, the renormalized $\mathbf{T}$ are precompact in the space of Jordan curves on the sphere. It is easy to see that this is also equivalent to $\mathbf{T}$ being the QS image of $\mathbb{S}^1$, i.e. a quasisymmetric circle.

In his 1978 plenary talk at the ICM Ahlfors asked
"which domains are QC equivalent to the unit ball"
i.e. characterize quasispheres.

This question was discussed in depth by Gehring in his talk at the 1986 ICM.

The obvious generalization, i.e. that a quasisphere is the QS image of $\mathbb{S}^2$ is necessary but not sufficient. This is because of the phenomena of "wild spheres":

There are topological spheres

$$\mathbf{T} \subset \widehat{\mathbb{R}}^3$$

whose complementary domains are NOT simply connected. The standard example is the Alexander Horned Sphere but if you try harder the example can be a QS sphere.

A tame sphere has simply connected complementary domains.

Our insight comes from studying reflections

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$

1. sense reversing homeomorphisms

2. idempotent : $F \circ F = \mathbf{I}$

e.g. $F(x_1, x_2, x_3) = (x_1, x_2, -x_3)$ has fixed set

$$\mathbf{T} = \{X : F(X) = X\} = \{(x_1, x_2, 0)\}$$

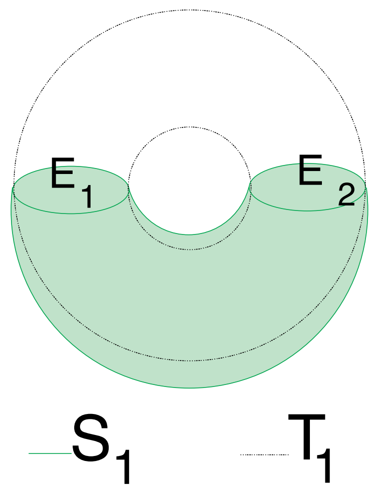
By Smith (1941) any reflection of $\mathbb{R}^3$ has fixed set $\mathbf{T} =$ topological sphere with two complementary domains.

Smith conjectured that $\mathbf{T}$ is tame.

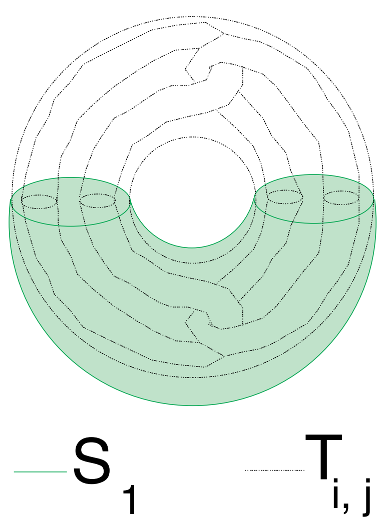
Disproved by Bing (Annals, 1952) by constructing a "wild reflection", he asked for an explicit example.

Then next few pages illustrate our explicit example:

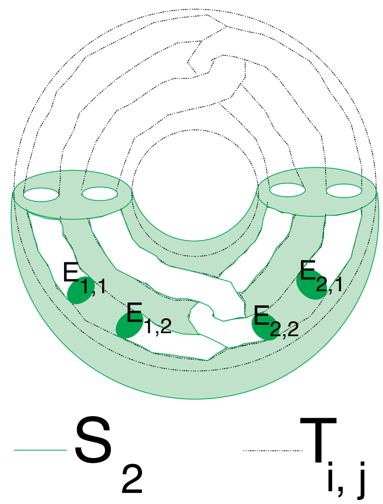
At the first step we cut a bagle in half. There is a reflection of its outside into the inside.



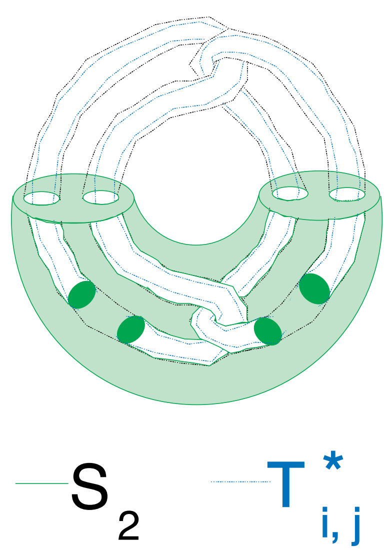
Then we introduce two hooked guide tubes



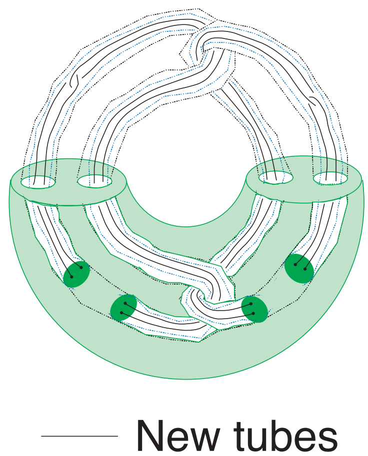
We drill down the tubes to opposing faces



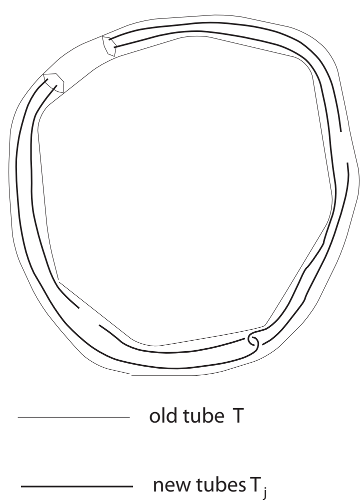
Erect new guide tubes and change the reflection so it fixes the new inner surface, same as old reflection outside previous set of guide tubes



From the faces we extend new hooked tubes

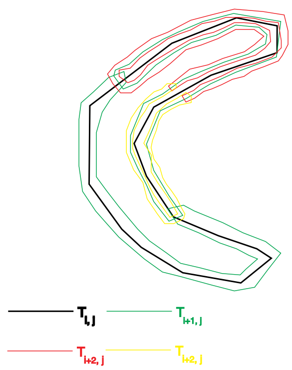


This is continued iteratively with generic step which looks like

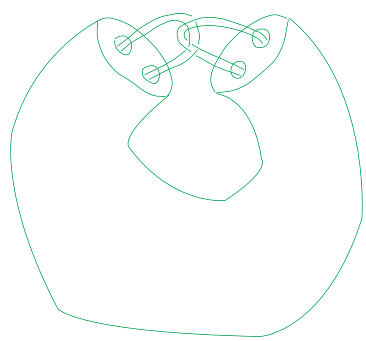


By roughly halving the length each time we find a sequence of reflections converging to a reflection.

This shows how the loops are folded up



The outside looks like sea cucumber



Heinonen, Semmes conjectured that there exist wild K-quasiconformal reflections, indeed everyone I asked (Tukia, Sullivan) were sure they existed. After all can't all homeomorphisms be approximated by QC mappings?

In fact Sullivan made suggestions which morphed into my explicit example. Although the example can be shown to be bihölder, it cannot be a QC reflection.

This is because we see long thin tubes. Renormalizing would yield a bilip reflection fixing an infinite cylinder! As it turns out any wild reflection has long thin tubes.

THEOREM 1 (Hamilton) *QC reflections are tame.*

Step 1:(Väisälä,Tukia) may assume wild reflection is bilip.

Step 2: (Heinonen, Yang) $\mathbf{D}, \mathbf{D}^*$ are "uniform" $\Rightarrow$ fixed set $\mathbf{T}$ approximated by "uniform handle-bodies" made up of Whitney type cubes. The scale factor r is called the resolution.

Step 3: $\mathbf{T}$ is linearly locally connected (LLC)
$\exists k$ so that $\forall X, Y \in \mathbf{T}$ there is continuium $\mathbf{E} \subset \mathbf{T}$:

$$dia(\mathbf{E}) \leq k||X - Y||$$

from this we prove "No long handles":

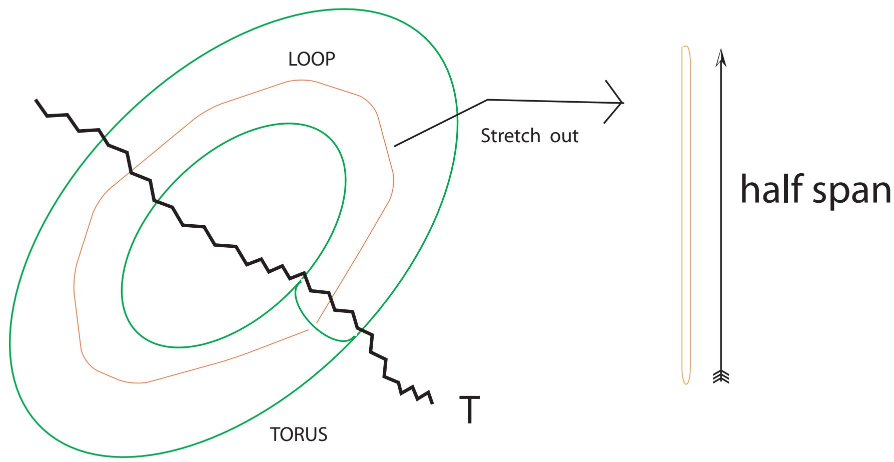
$$Length(Handle) \leq Cr$$

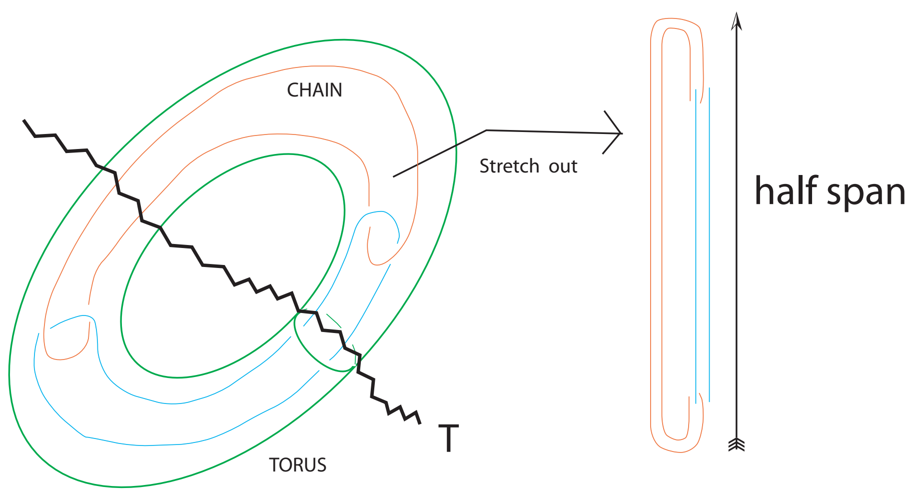
where r is the resolution, i.e. size of cubes.

Step 4: Now we come to the main idea of the proof: measure the handle-bodies approximating the fixed set. Remember we have a reflection so there are approximating handle-bodies on each side of $\mathbf{T}$. Furthermore handles occur in pairs. One could run a loop so it transverses a pair of handles.

It turns out that length is not the best way to measure the inside of a torus. Now length is measured by the min length of loops transversing the inside of the torus. More generally use chains of loops. Consider all isometric deformations of these chains. The extremal case is when the deformed chain has max diameter. The span of the chain is the twice this extremal diameter. The span of the torus is then the infinium of the span of these transversing chains. For technical details see the appendix.

The idea of span:





DEFINITION 1 *We say that a flat sphere* $\mathbf{T}$ *is uniform if its family of "renormalizations"* $\widetilde{\mathbf{T}}$ *is precompact in the space of flat spheres wrt the hausdorff metric between compact sets*

Equivalently, at all scales $\mathbf{T}$ can be squeezed between "uniform" polyhedra from the complementary domains.

From Theorem 1 we see that the fixed set of a QC reflection is a uniform sphere.

The converse is the characterization:

THEOREM 2 *(Hamilton)*
$\mathbf{T}$ *is the fixed set of a QC reflection iff it is a uniform sphere.*

This also characterizes the fixed set of bilip reflections, Poincaré (1898), Jones (Acta 1981) who asked about this.

Example: The product $\mathbb{R}^1 \times \Gamma$ for any two dimensional quasicircle $\mathbf{\Gamma}$ through ∞ gives the fixed set of a bilip reflection of $\mathbb{R}^3$.

Sets vs Functions:

A uniform sphere $\mathbf{T}$ has blowups giving only topological spheres in the limit. Consider parametrizations

$$H : \mathbb{S}^2 \to \mathbf{T}$$

However there are uniform spheres so any parametrization has NON precompact blowups.

On the other hand QS $\Rightarrow$ precompact renormalizations but not geometric uniformity.

Quasispheres

These methods solve the Ahlfors-Gehring problem of charactering $F(\mathbb{S}^2)$ for QC maps of $\widehat{\mathbb{R}}^3$:

THEOREM 3 *(Hamilton)* $\mathbf{T}$ *is quasisphere iff:* $\mathbf{T}$ *is a uniform AND QS sphere.*

This generalizes the Ahlfors problem of extending a quasisymmetric mapping $F : \mathbb{S}^2 \to \mathbb{S}^2$. i.e. if we have a QS mapping from the sphere onto a uniform sphere then it extends to a QC mapping.

The QC Riemann Mapping Theorem is a one sided version of this. We characterize quasiconformal images of the unit ball, i.e. QC-balls.

For $\mathbb{R}^2$ only need simply connectedness (with non empty boundary). One difference between $\mathbb{R}^2$ & $\mathbb{R}^3$ is that any topological 2-ball in $\widehat{\mathbb{R}}^2$ is "uniformly simply connected " (USC): its renormalizations are precompact in the space of disks.

However in $\mathbb{R}^3$ renormalizations of a topological ball could converge to a torus. It is standard to show that a QC-ball is USC.

However we expect conditions on the boundary. Gehring (1986) showed that if $\mathbf{D}$ is a QC-ball

$$\widehat{\mathbb{R}}^3 - \mathbf{D}$$

is "linearly locally connected" (LLC).

We find that a USC + LLC domain has boundary parametrized by a homeomorphism $H : \mathbb{S}^2 \to \partial D$ [4]

[4] i.e. the prime-end boundary $\partial\widehat{\mathbf{D}}$ is a topological sphere. (Zorich)

One case known: Väisälä (Acta, 1988) a cylinder

$$\mathbf{D} = \mathbf{A} \times \mathbb{R}^1$$

is QC-ball iff boundary is parametrized by:

$$H : \mathbb{S}^2 \to \partial \mathbf{D}$$

QS wrt to the inner distance metric. (Least diameter of inside arcs joining points X, Y)

A general concept of quasisymmetry is required.

On a USC + LLC domain there is a Gromov metric on $\mathbf{D}$ which extends to the prime-end boundary. [5] This is obtained by showing that $\mathbf{D}$ is Gromov uniform tree, i.e. we approximate $\mathbf{D}$ from the inside by disjoint uniform polyhedra connected as a tree. This happens uniformly at all scales. So under higher resolutions we have layers of these trees.

The Gromov metric is obtained by fixing ∞ and its containing polyhedra, and weighting every other polyhedra by c^{-n} where n is the number of steps or layers in the tree to get to ∞, and $c < 1$ is a constant which actually depends on the USC+ LLC bounds.

[5] Similar to Gromov's theory of Hyperbolic Trees (discrete groups)

The distance between X, Y will then be approximately c^{-n} where this is the weight of the last uniform polyhedra to cut off X, Y from ∞.

The following is one of several versions of the mapping theorem:

THEOREM 4 *(Hamilton)* $\mathbf{D}$ *is the QC-ball iff* $\mathbf{D}$ *is a USC + LLC domain with boundary a (Gromov) QS sphere.*

Of course one hoped for a natural metric instead of something depending on the USC+LLC bounds. However being simply connected is not a very strong notion in $\widehat{\mathbb{R}}^3$. A simply connected domain domain can “look” like it is non-simply connected for arbitrary long stretches. The point of USC+ LLC is to bound this.

In the Sullivan Thurston construction of domains inscribed by balls we also get a tree structure. But these balls are the “0-constant ” case. This means they give a $K-QC$ map with K independent of the domain. In this case you do not even need QS mapping of the boundary.

e.g. 1: Our theorem generalizes Väisälä cylinder result:

COROLLARY 1 *A USC domain is a QC ball if its boundary is a QS sphere wrt inner distance.*

Notice that a cylinder domain is automatically USC [6].

e.g. 2: "Manhattan" domain

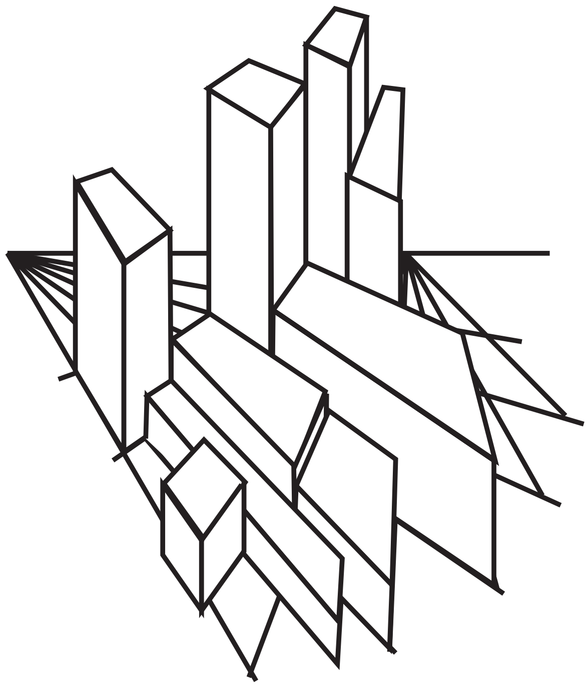
QC-balls which cannot be inscribed by round balls: consider a packing of $\mathbb{R}^2$ by rectangles $\mathbf{A_j}$ of bounded eccentricity. We form $\mathbf{D}$ by adjoining the skyscrappers

$$\mathbf{A_j} \times \{0 \leq x_3 < h_j\}$$

to the the half-space $x_3 < 0$. This is a K-QC ball. Now $\mathbf{D}$ (in general) cannot be inscribed by balls but neither can it be QC transformed into a Sullivan -Thurston domain by a QC mapping of $\mathbb{R}^3$ (for then a packing by squares would be equivalent to packing by circles).

[6] A QS sphere need not bound a simply connected domain let alone one which is USC

A Manhattan Domain: note that all the vertical sides of the skyscrapers are part of the boundary of the domain. Just their rectangular bases are open.



If you can do it to New York, you can do it anywhere

Appendix: Technical details of Theorem 1:

Although the idea of span is simple it is easier to work with a less natural definition. Given a handle-body $\mathbf{\Sigma}$ linked with an unknotted loop α in its exterior we consider the chains of loops inside $\mathbf{\Sigma}$ which link with α. In our situation there is symmetry so there α and its reflection α^*.

To measure the linking we consider chains of loops inside $\mathbf{\Sigma}$. A loop consists of an arc beginning and ending on $\mathbf{T}$ together with the reflection. A chain γ is the union of a finite number of loops and is admissible if it links through α, e.g. for a torus the admissible chains would be a loop going all the way around, or chains built up from loops.

Now we define the span. On the chain we define a metric $\rho \geq 0$ which is constant on each loop and has the property that for any topological disk $\mathbf{S}$ with $\partial \mathbf{S} = \alpha$

$$\sum \rho \geq 1$$

where the sum is taken over all points Z where γ intersects $\mathbf{S}$. The idea is that the total flow through α is at least 1.

The ρ-length of γ is

$$\int \rho |dZ| \ ,$$

i.e the sum of the lengths of loops weighted by its the values of ρ. The span $\lambda(\mathbf{\Sigma}, \alpha)$ is the infinium of such ρ-lengths, taken over all admissible chains γ and metrics. For uniform handle-body one sees that λ is a positive number that scales like ordinary length. The infinium is achieved by some chain, i.e. the loops don't get smaller and smaller. However this minimizing chain may intersect $\partial\mathbf{\Sigma}$.

For a regular torus the span is in fact exactly equal to the length. But even for a twisted torus the span can be strictly less than length.

Next one finds that the minimizing chains γ may be taken to have minimal topological configuration, i.e. no subchain of γ links through α. From this we prove the bifurcation property: that a minimal configuration is built up of subchains which are themselves minimal configurations.

We want our handles breaking to give higher resolution linked handles. This this does not happen if for example there is a is a knot (which can actually occur for bilip images of a sphere). It is surprising but this cannot happen for QC reflections.

This leads to the fundamental property of span:

LEMMA 1 *Suppose that we have a minimal configuration containing two admissible chains γ_L and γ_R linked through the left and right of an unknotted closed curve α. Then γ_L and γ_R are linked through some unknotted closed curves α_j and*

$$max\{\lambda(\gamma_L,\alpha_1),\lambda(\gamma_R,\alpha_2)\} \geq \lambda(\gamma_L+\gamma_R,\alpha)$$

Example: Consider a chain made up of two linked loops. The span of one loop is always at least the span of the chain.

So we now complete the proof of the Theorem 1. Assume that the components are not simply connected. Then there is a nested sequence of linked handlebodies $\mathbf{\Sigma}$ obtained by using higher and higher resolutions. The fundamental lemma is now applied inductively. So we find a handle with at least the span of the original handlebody. But these have small diameter, contradicting "no long handles".