Spectral least-squares for conformal mapping and potential theory

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Ten digit algorithms

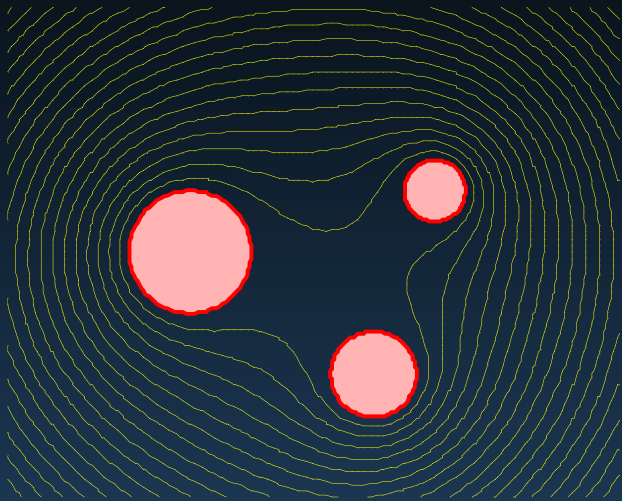
"Ten digits,

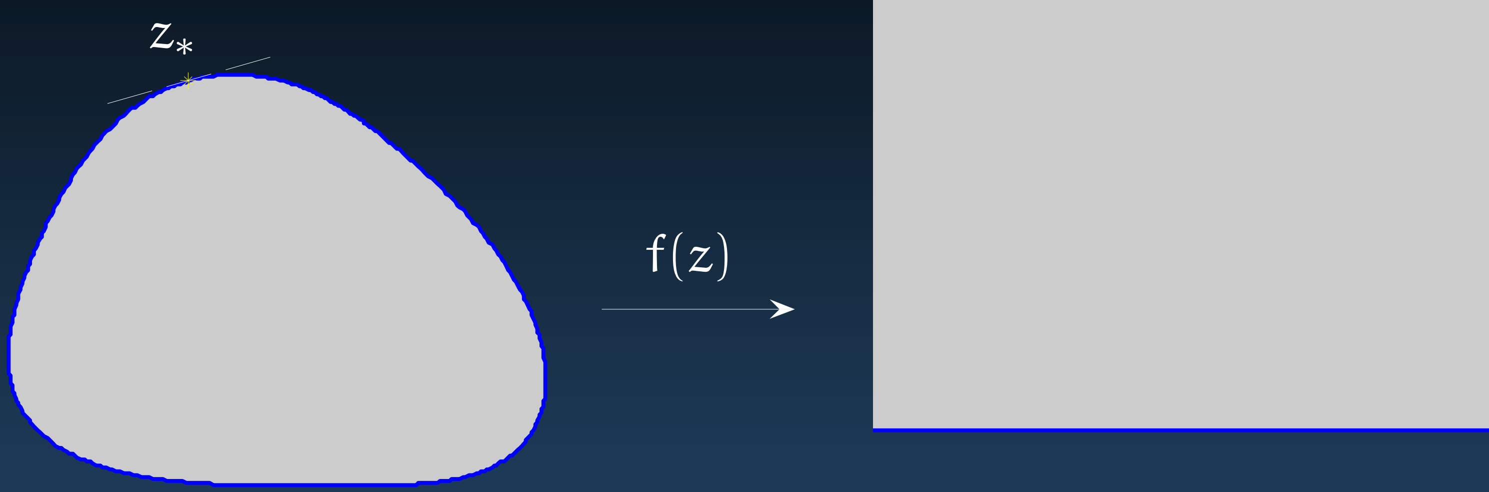
Five seconds,

And just one page."

-- Nick Trefethen

```
 J = 3; c = [-2 2+1i 1-2i]; r = [1
.5 .7];
N = 10;
npts = 30;
circ = exp((1:npts)'*2i*pi/npts);
z = [];
for j = 1:J
     z = [z; c(j)+r(j)*circ];
end
A = ones(size(z));
for j = 1:J
    A = [A log(abs(z-c(j)))];
    for k = 1:N
      zck = (z-c(j)).^(-k);
      A = [A real(zck) imag(zck)];
    end
end
X = -A(:,[1 3:end])\A(:,2);
X = [X(1); 1; X(2:end)]/...

(1+sum(X(2*N+2:2*N+1:end)));
```





$$f(z) = \frac{e^{i\theta}}{z - z_*} + \text{analytic} \approx \frac{e^{i\theta}}{z - z_*} + \sum_{m=1}^{M} c_m z^m$$

Discretize boundary and evaluate basis functions:

$$A = \left[\frac{e^{i\theta}}{z - z_*} \quad z \quad z^2 \quad \cdots \quad z^M\right]; \qquad f(z) \approx A \begin{bmatrix} 1 \\ c \end{bmatrix} = A\tilde{c}$$

Impose condition:

$$\text{constant} = \text{imag}\, f(z) \approx \begin{bmatrix} A_I & A_R \end{bmatrix} \begin{bmatrix} \tilde{c}_R \\ \tilde{c}_I \end{bmatrix}$$

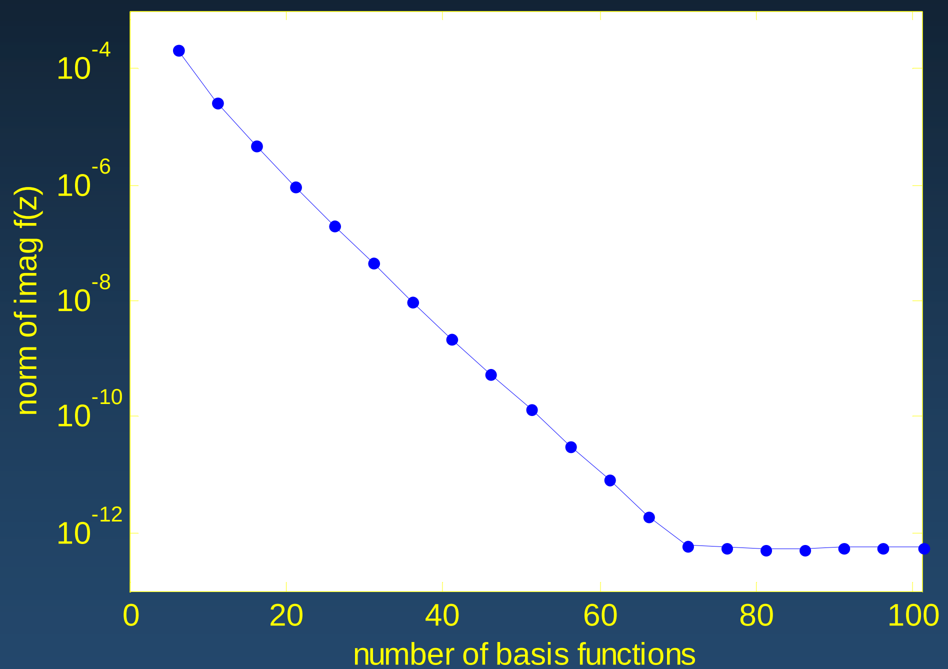
$$0 \approx \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} A_I & A_R \end{bmatrix} \begin{bmatrix} \tilde{c}_R \\ \tilde{c}_I \end{bmatrix} = B\tilde{c}$$

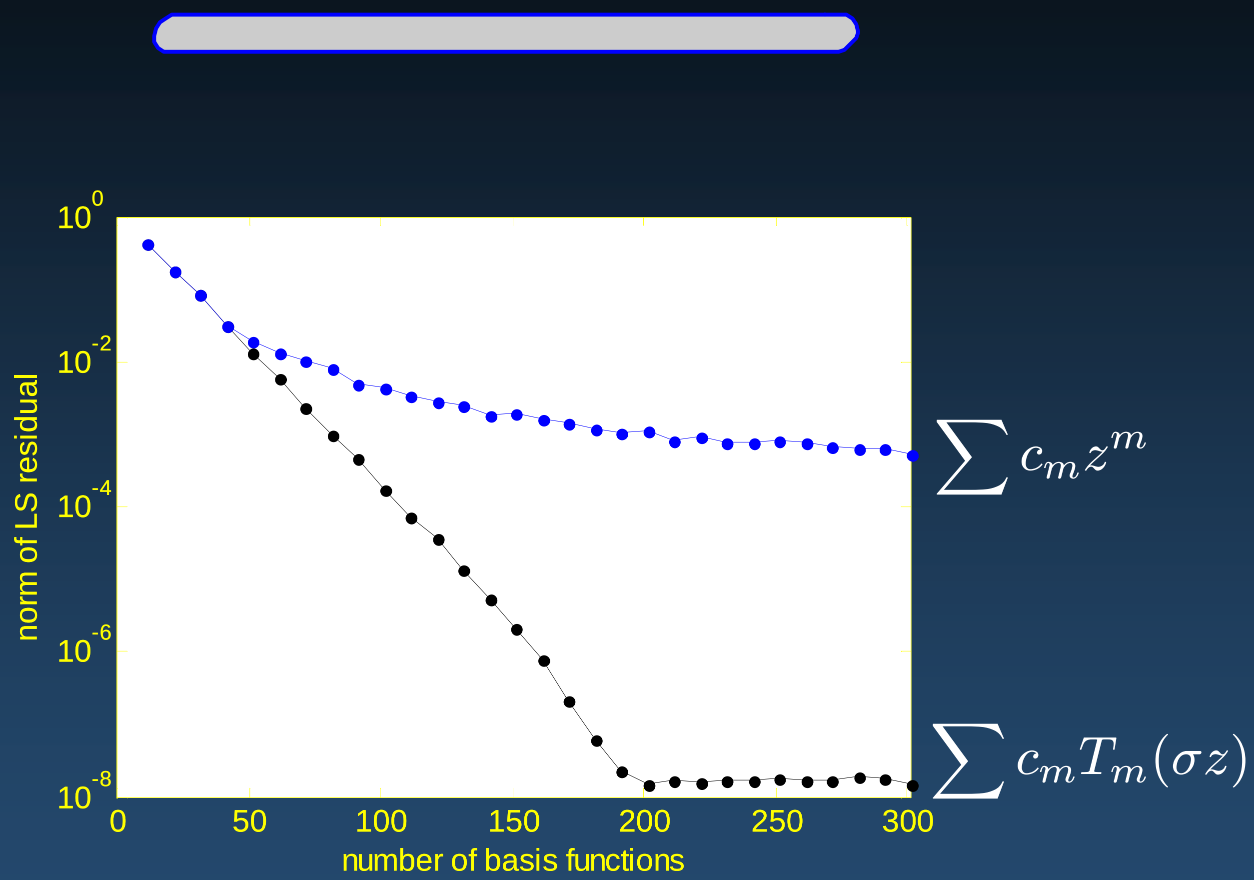
$\tilde{c}_1 = 1$ leaves a standard matrix least-squares problem.

```
M = 10;    % analytic degree
P = 500;   % # of boundary points

% Boundary points
load region_blob
zstar = zb(0);
z = zb( (1:P-1)'/P );

% Basis functions
A(:,1) = tangent./(z-zstar);
for n = 1:M
  A(:,n+1) = z.^n;
end
```

```
% LS matrix
B = diff( [imag(A) real(A)] );

% Impose c(1)=1
b = -B(:,1);
B(:,[1 M+2]) = [];

% Solve & parse
cRI = B\b;
c = cRI(1:M)+1i*cRI(M+1:2*M);

% Check result
w = A*[1;c];
w = w - mean(w);
err = norm(imag(w))/norm(w)
```

Convergence as a function of the polynomial degree:

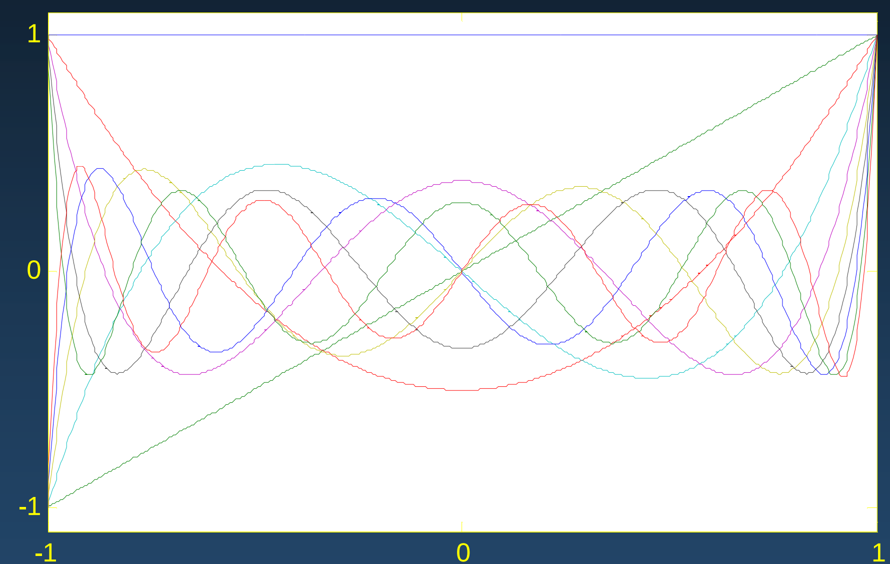


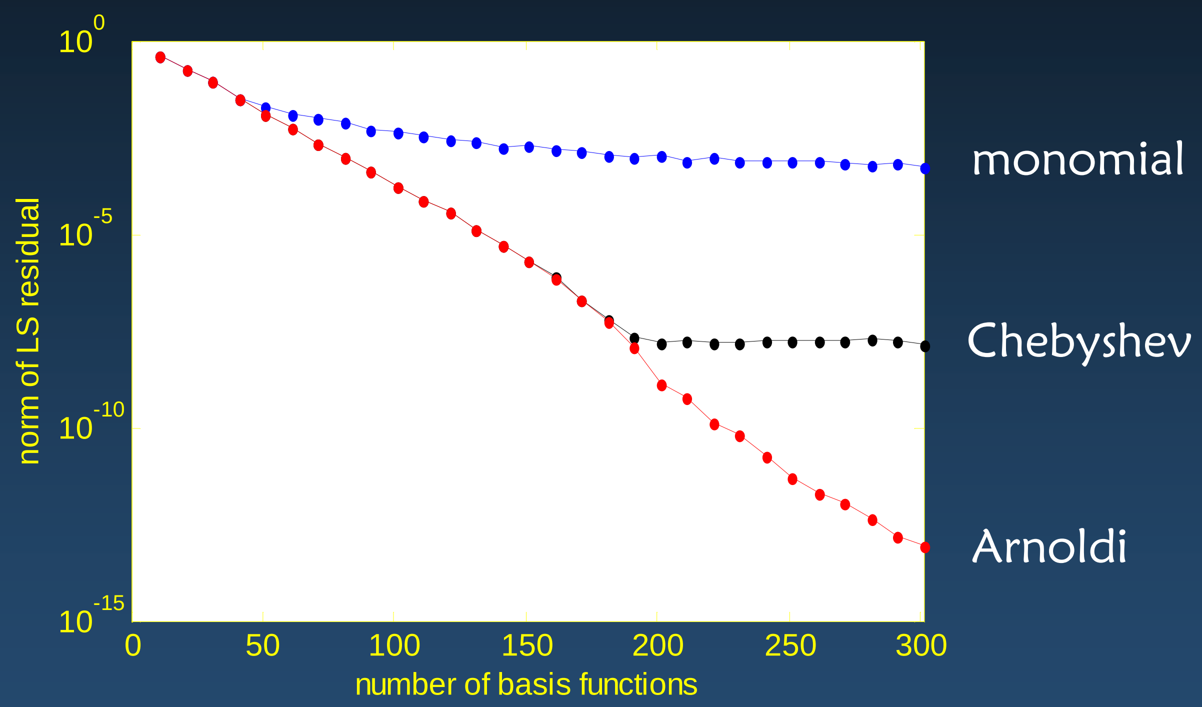


10^0
10^-2
10^-4
10^-6
10^-8
norm of LS residual
0
50
100
150
200
250
300
number of basis functions
$\sum c_m z^m$
$\sum c_m T_m(\sigma z)$

The Arnoldi iteration creates a numerically orthogonal representation of the span of $1, z, z^2, \ldots$

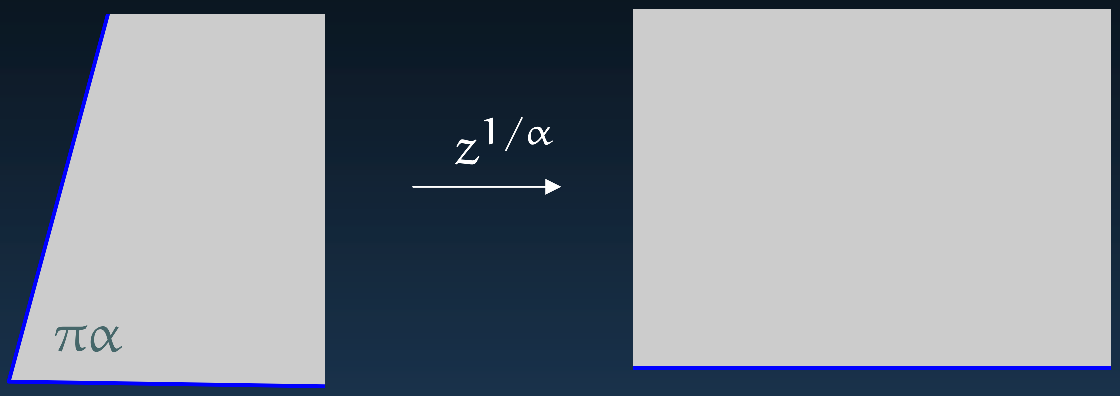
```
x = linspace(-1,1,500)';
q = x.^0;
Q(:,1) = q/norm(q);
for n = 1:9
  q = x .* Q(:,n);
  r = Q(:,1:n)'*q;
  q = q - Q(:,1:n)*r;
  Q(:,n+1) = q/norm(q);
end

plot( x, Q*diag(1./Q(end,:)) )
```




10^0
10^-5
10^-10
10^-15
norm of LS residual
0
50
100
150
200
250
300
number of basis functions
monomial
Chebyshev
Arnoldi

Ill-conditioned spectrally accurate bases:

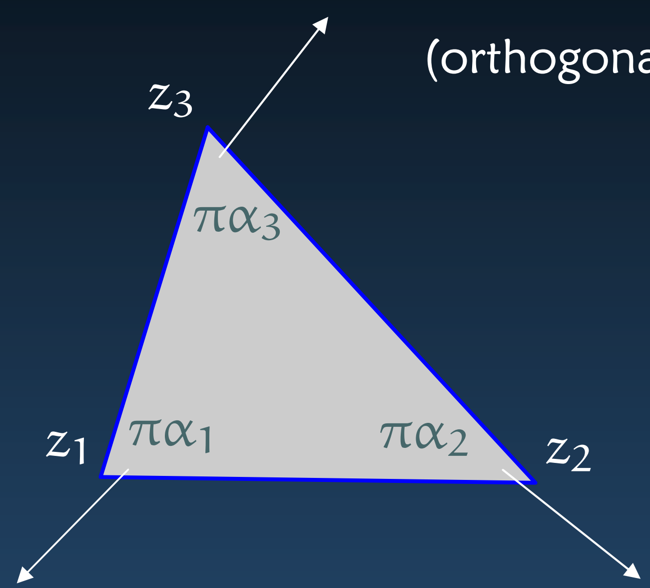
- monomial basis for polynomial interpolation
- radial basis functions for surface interpolation
  - (Fornberg, Schaback, Buhmann)
- fundamental solutions/charge simulation for BVPs
  - (Kitagawa, Amano)
- method of particular solutions for eigenvalues
  - (Barnett, Betcke)
- plane-wave bases for Maxwell's equations
  - (Cessenat, Despres, Monk, Perrey-Debain)



Map is locally analytic in the variable $z^{1/\alpha}$.

$$(z - z_3)^{m/\alpha_3}, \; m = 1, 2, \ldots$$

(orthogonalized)



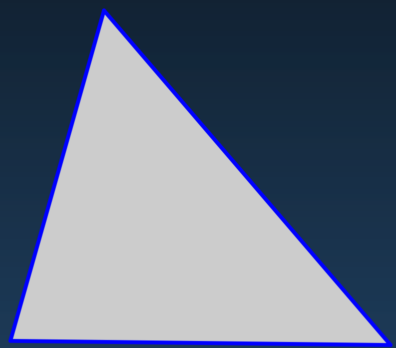
$$(z - z_1)^{m/\alpha_1}, \; m = 1, 2, \ldots$$

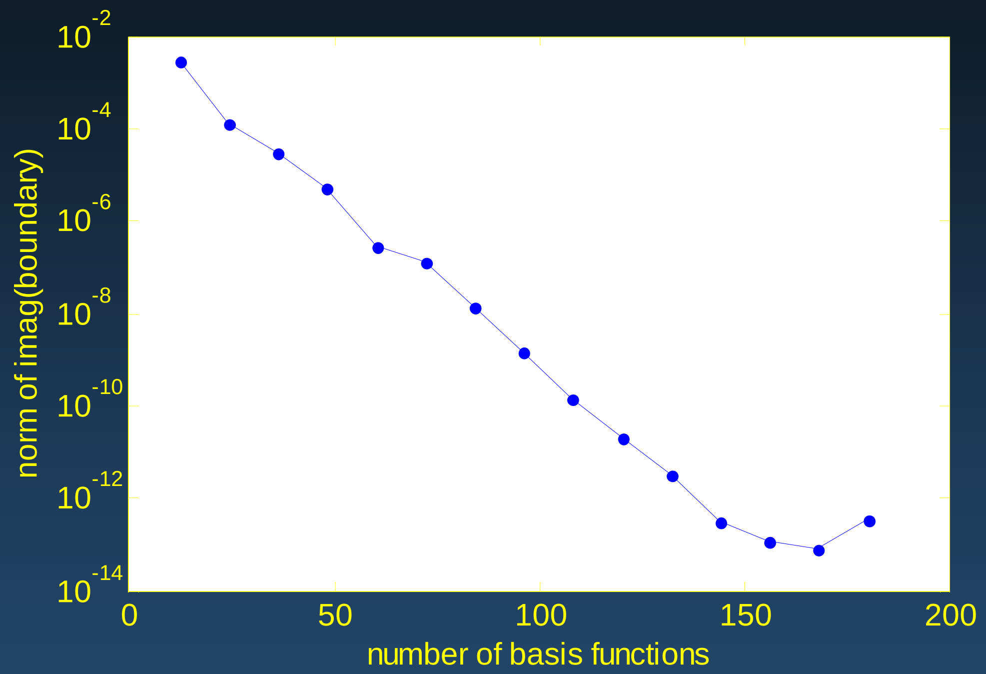
(orthogonalized)

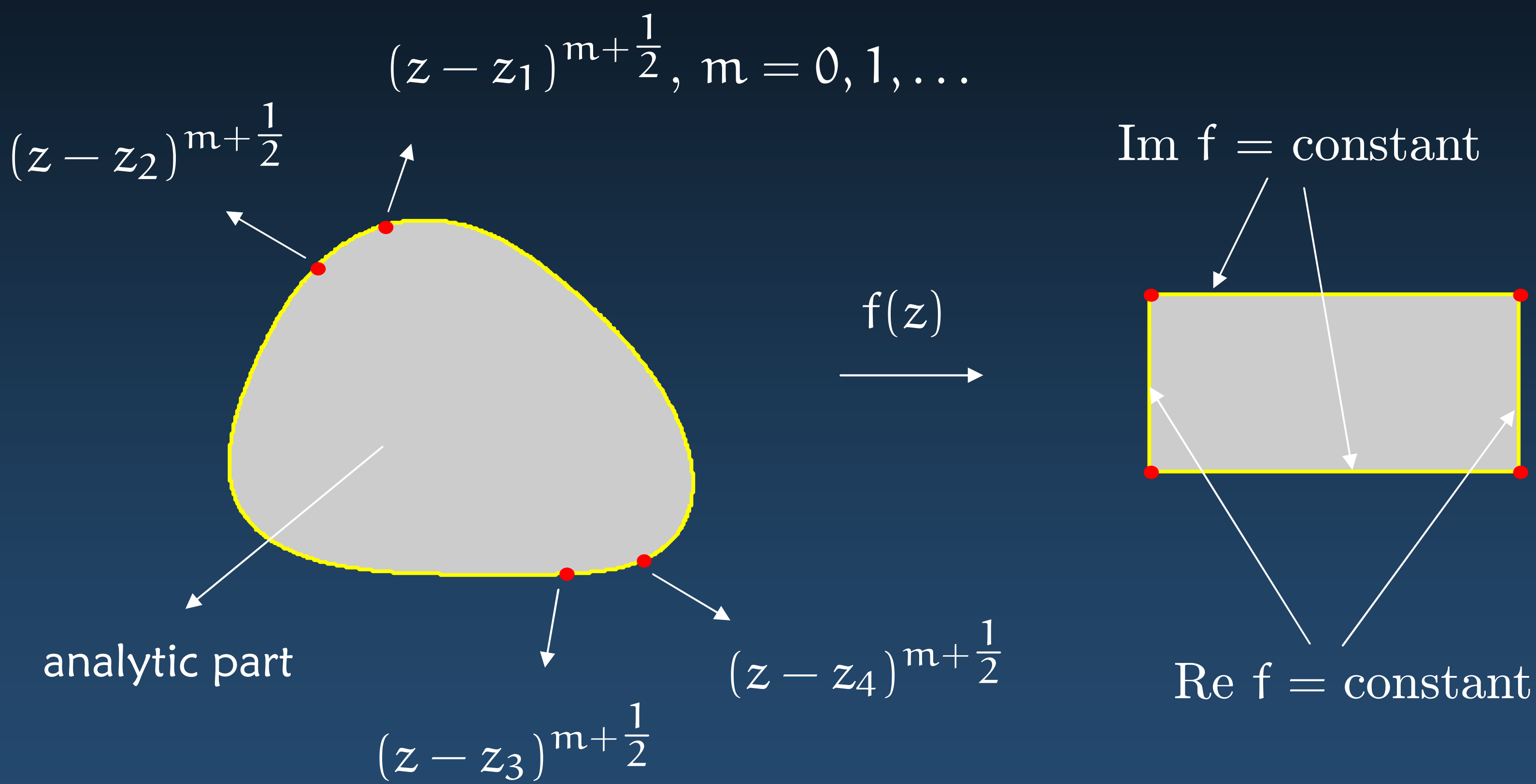
$$(z - z_2)^{m/\alpha_2}, \; m = 1, 2, \ldots$$

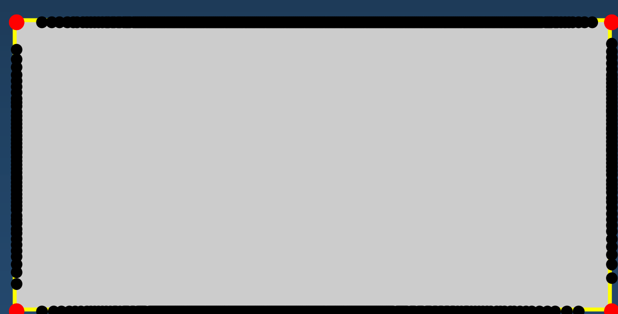
(orthogonalized)

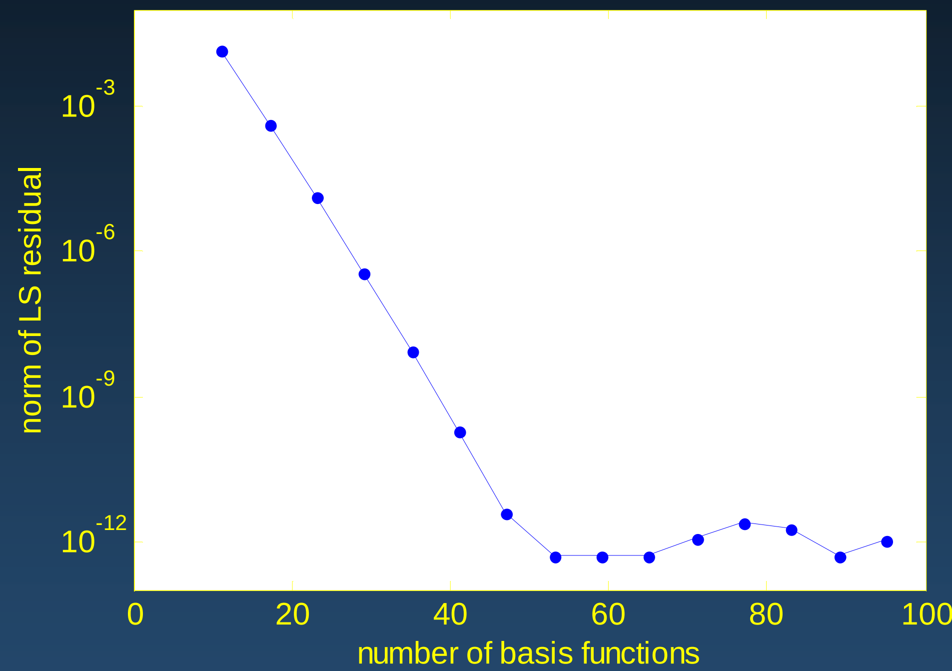
Bases for different corners are not mutually orthogonal.

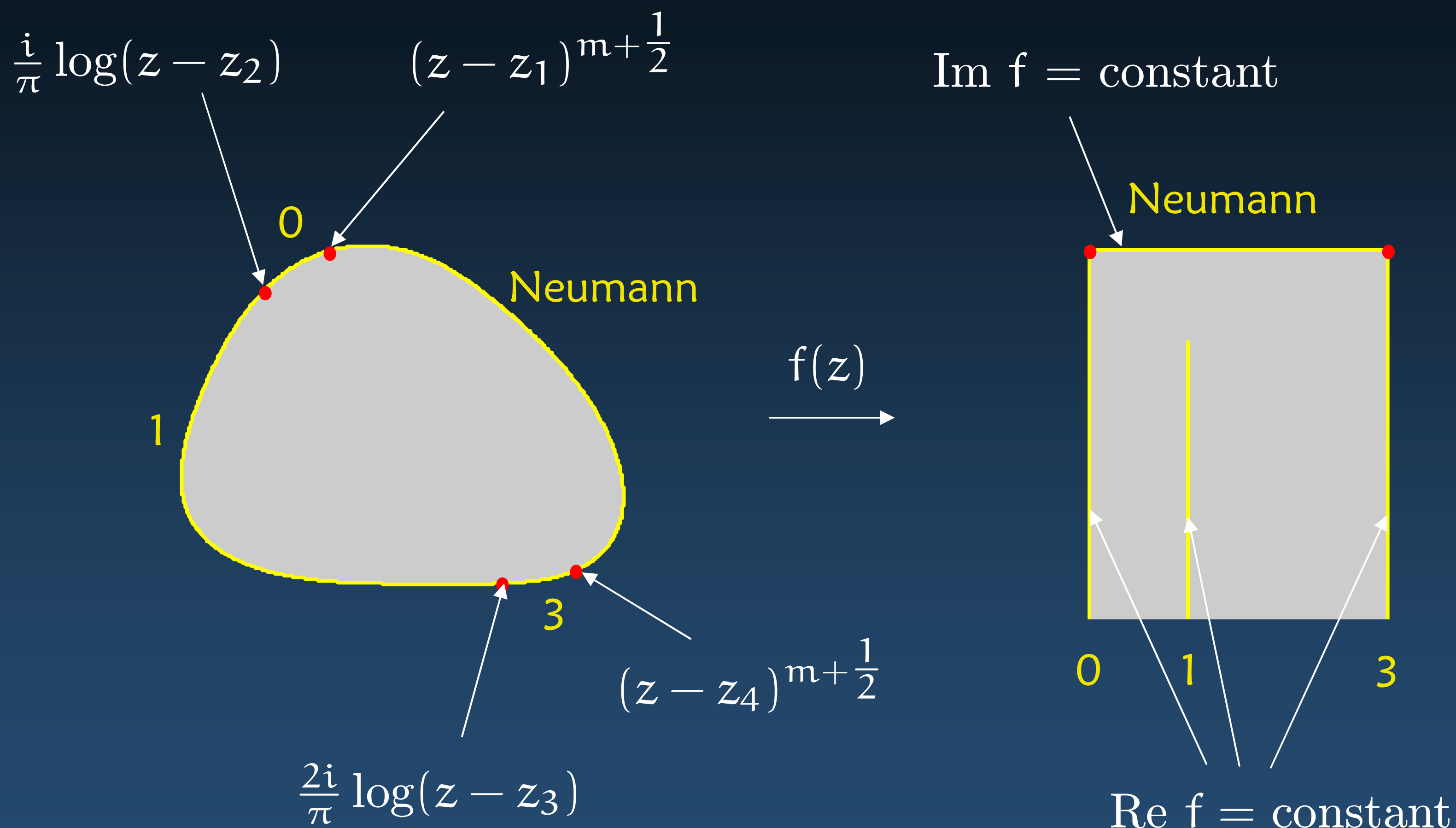


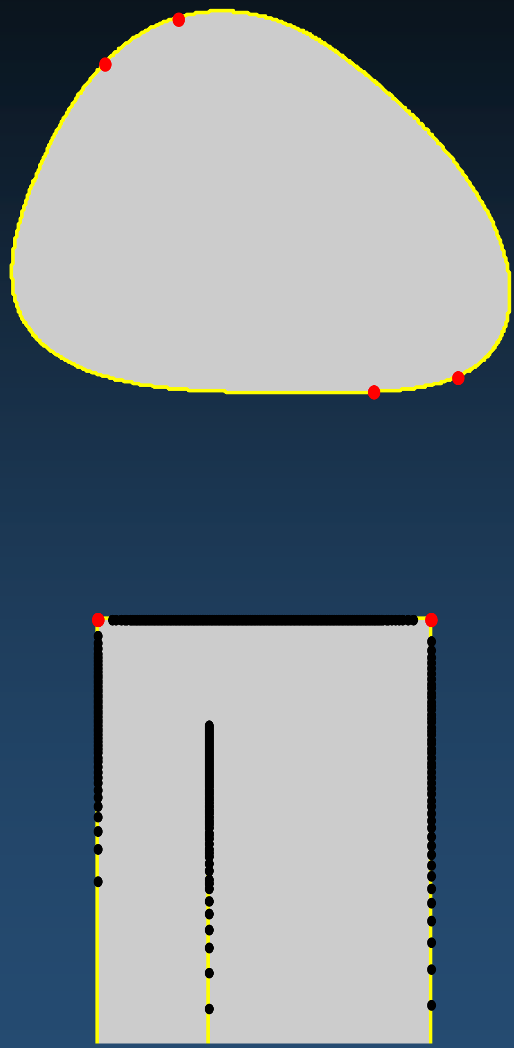

10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
norm of imag(boundary)
0
50
100
150
200
number of basis functions

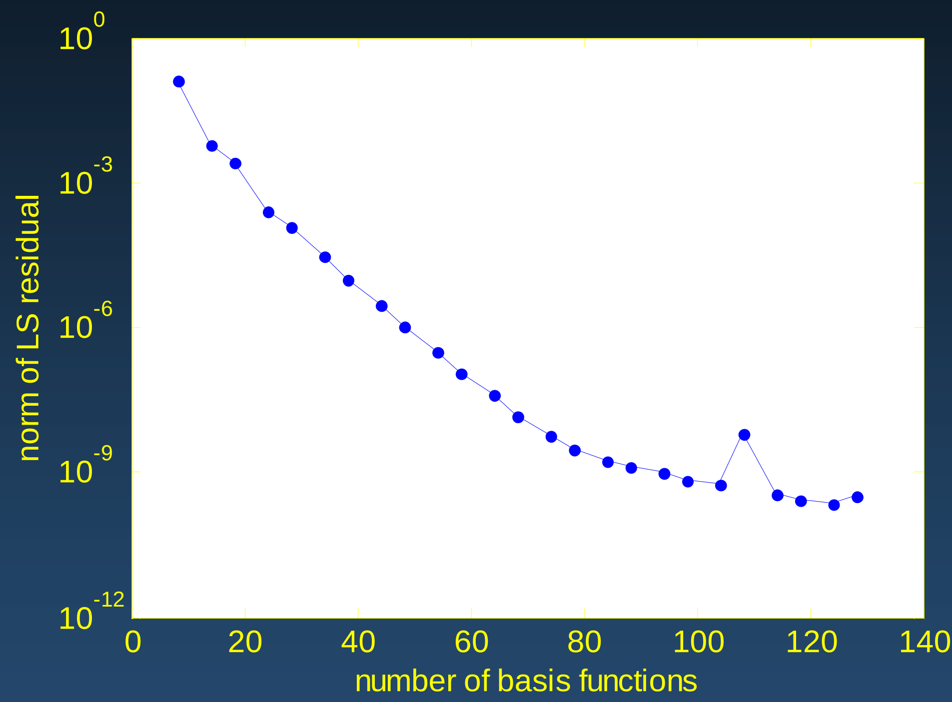

$(z - z_1)^{m+\frac{1}{2}}$, $m = 0, 1, \ldots$
$(z - z_2)^{m+\frac{1}{2}}$
$f(z)$
$\mathrm{Im}\ f = \text{constant}$
analytic part
$(z - z_4)^{m+\frac{1}{2}}$
$\mathrm{Re}\ f = \text{constant}$
$(z - z_3)^{m+\frac{1}{2}}$




norm of LS residual
10^{-3}
10^{-6}
10^{-9}
10^{-12}
0
20
40
60
80
100
number of basis functions

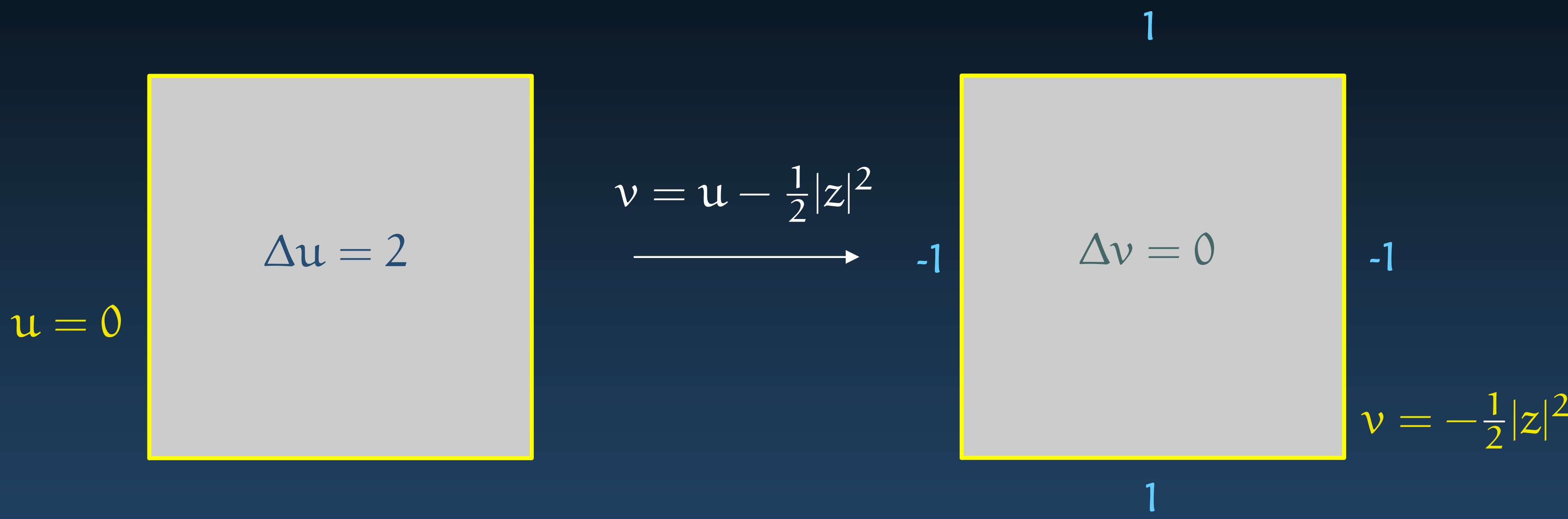

$\frac{i}{\pi}\log(z - z_2)$
$(z - z_1)^{m+\frac{1}{2}}$
0
Neumann
1
3
$\frac{2i}{\pi}\log(z - z_3)$
$(z - z_4)^{m+\frac{1}{2}}$
$f(z)$
$\mathrm{Im}\ f = \mathrm{constant}$
Neumann
0
1
3
$\mathrm{Re}\ f = \mathrm{constant}$



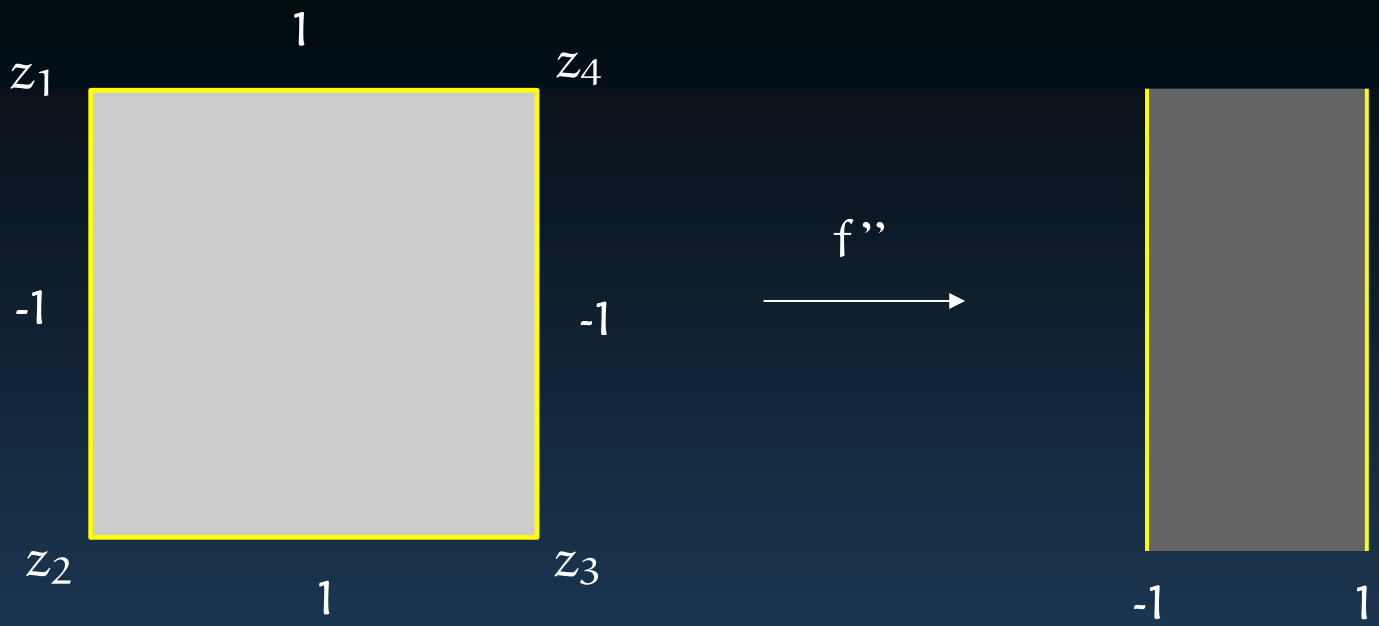


10^{0}
10^{-3}
10^{-6}
10^{-9}
10^{-12}
norm of LS residual
0
20
40
60
80
100
120
140
number of basis functions

Trefftz's method

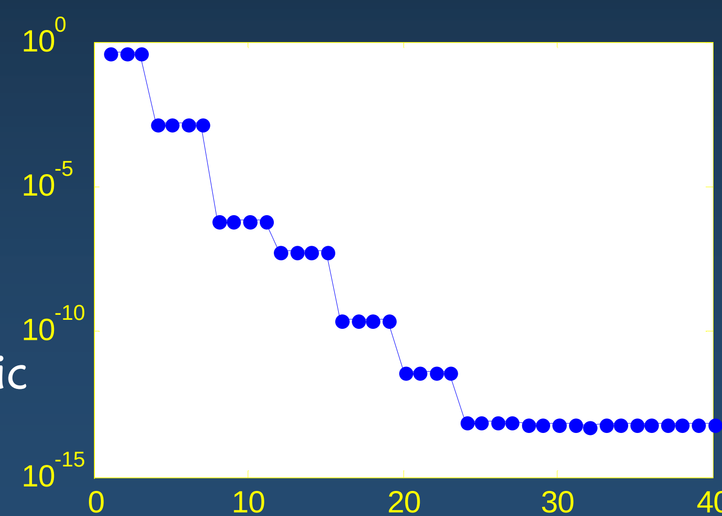


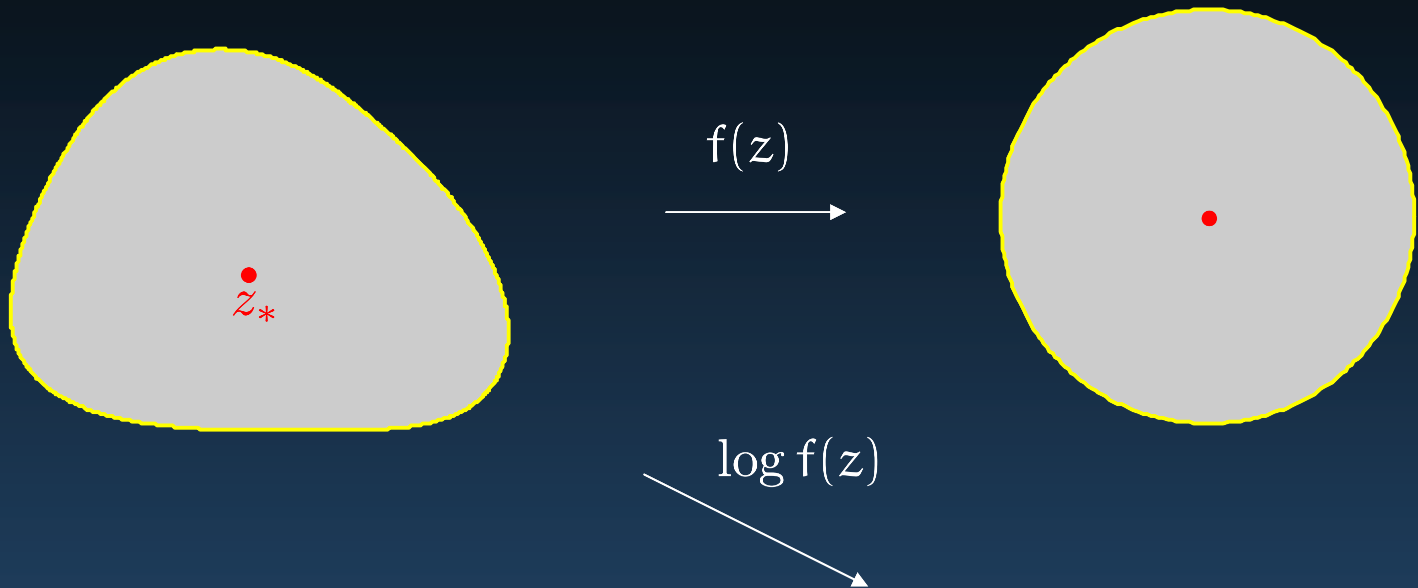
$v = \operatorname{Re} f \Rightarrow \operatorname{Re} f''$ is piecewise constant

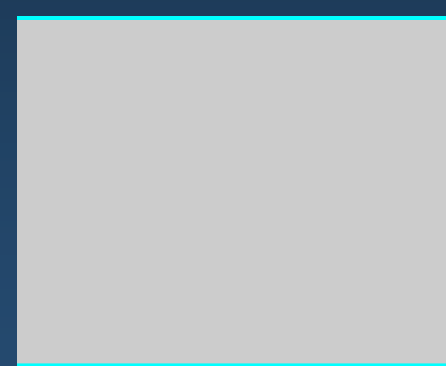


$$f''(z) = \frac{4i}{\pi} \sum_{k=1}^{4} (-1)^k \log(z - z_k) + \text{ analytic}$$

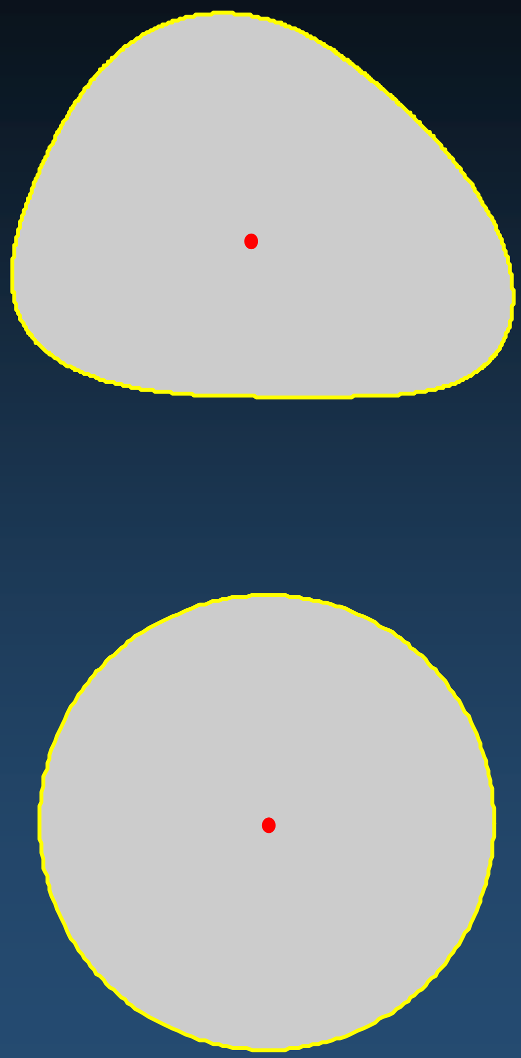
$$f(z) = \frac{2i}{\pi} \sum_{k=1}^{4} (-1)^k (z - z_k)^2 \log(z - z_k) + \text{ analytic}$$

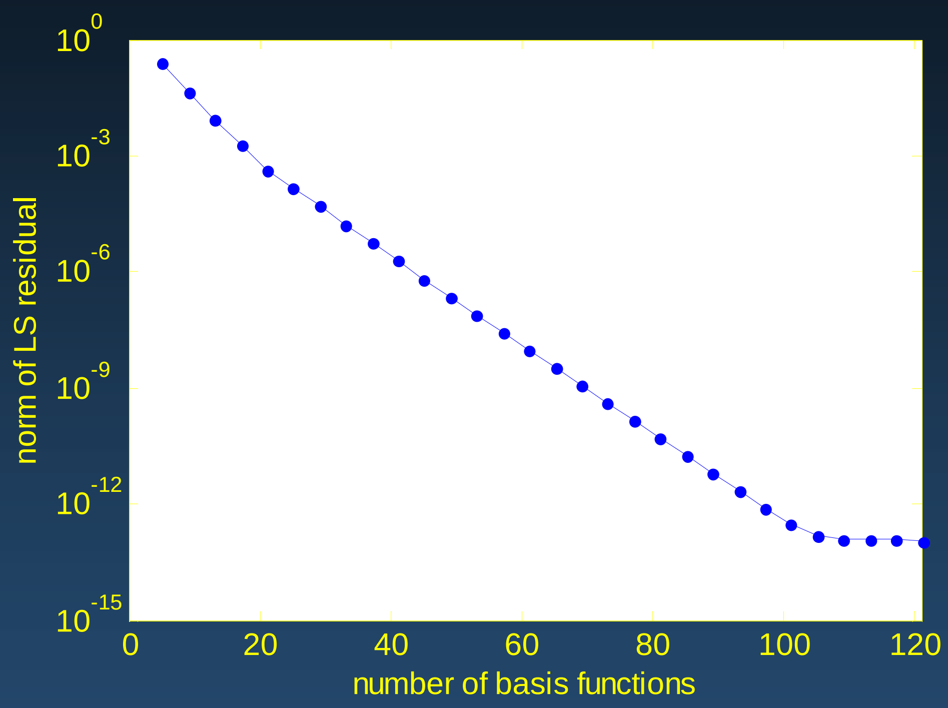


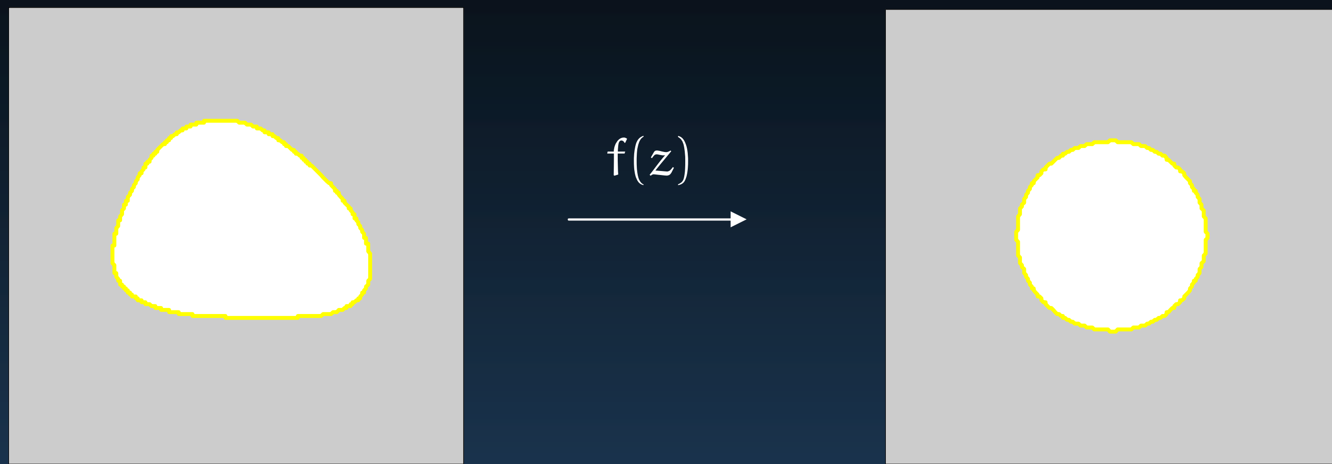


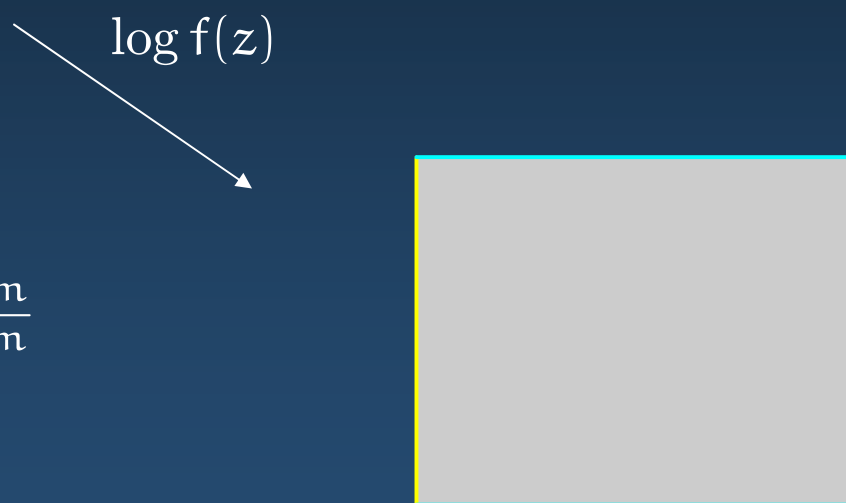


$$\log f(z) = \log(z - z_*) + \text{analytic}$$

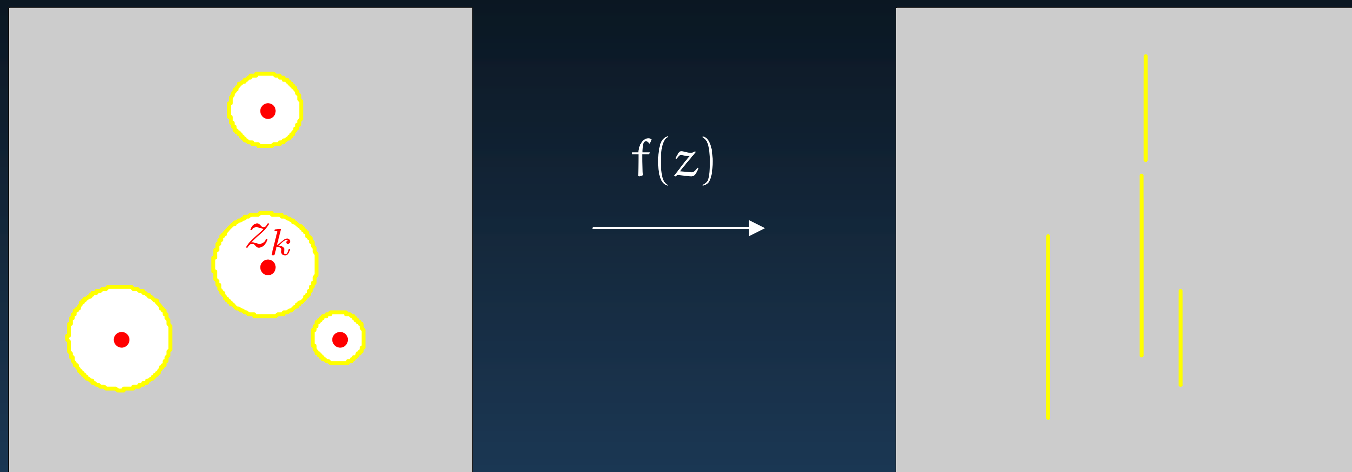



norm of LS residual
10^0
10^-3
10^-6
10^-9
10^-12
10^-15
0
20
40
60
80
100
120
number of basis functions

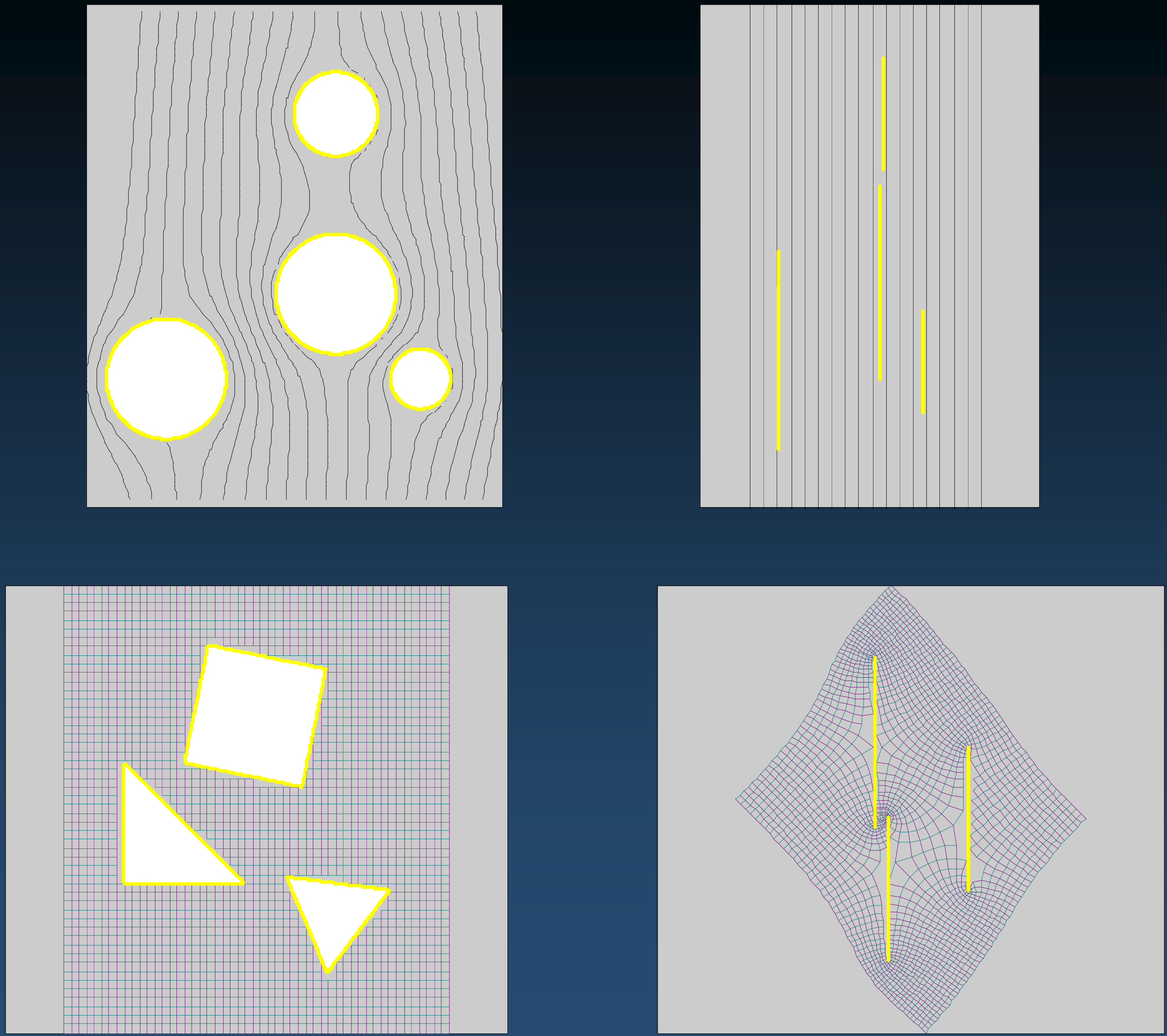


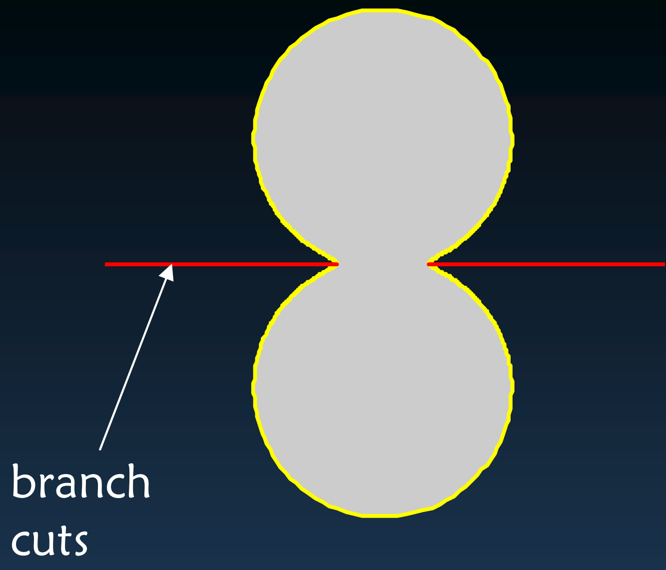


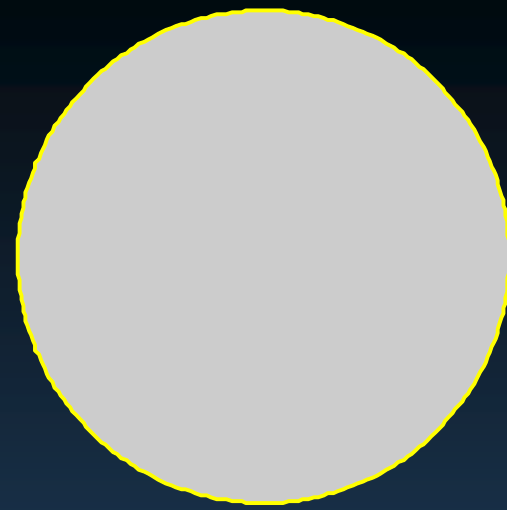
$$\log f(z) \approx \log(z) + \sum_{m=1}^{M} \frac{c_m}{z^m}$$

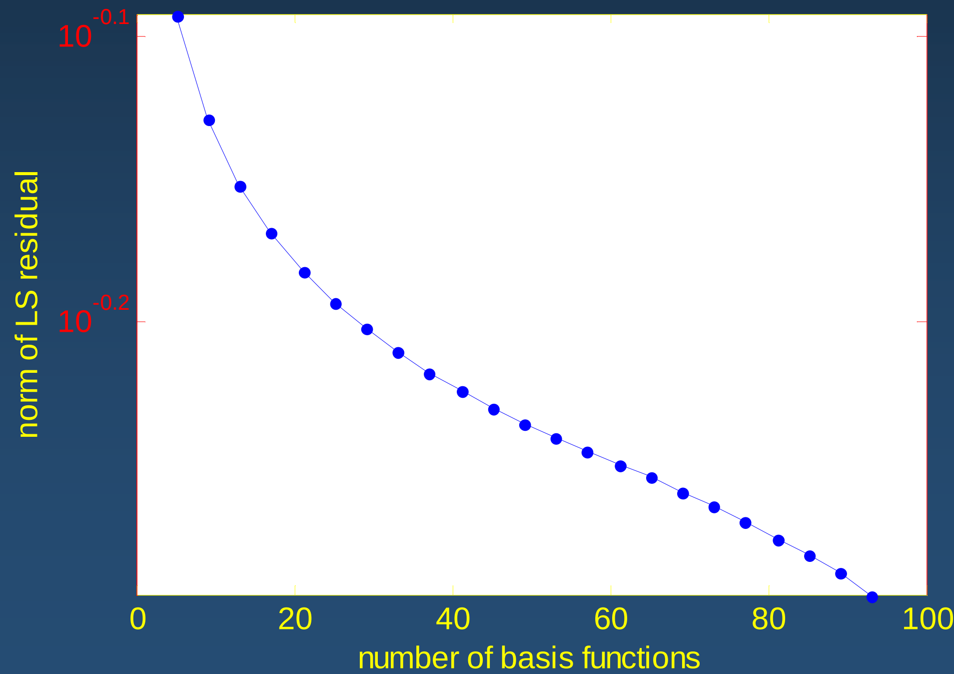


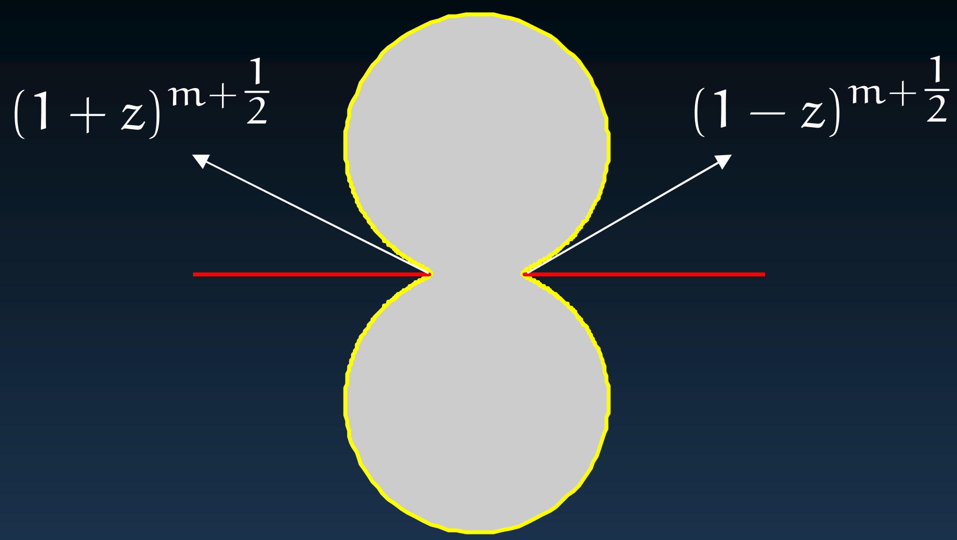
$$f(z) \approx z + \sum_{k=1}^{4} \sum_{m=1}^{M} \frac{c_{k,m}}{(z - z_k)^m}$$

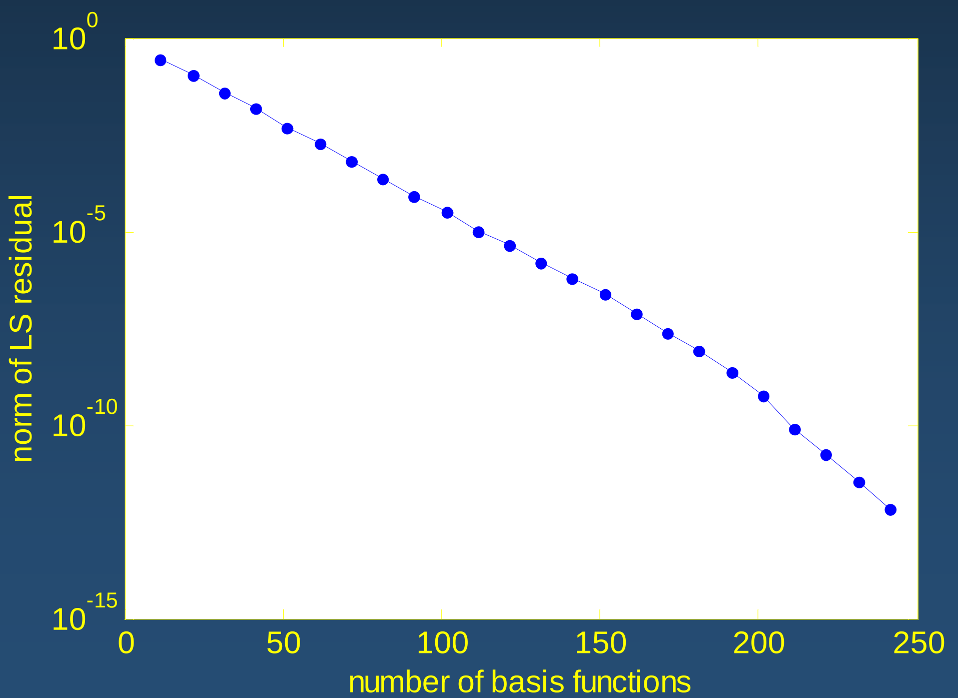


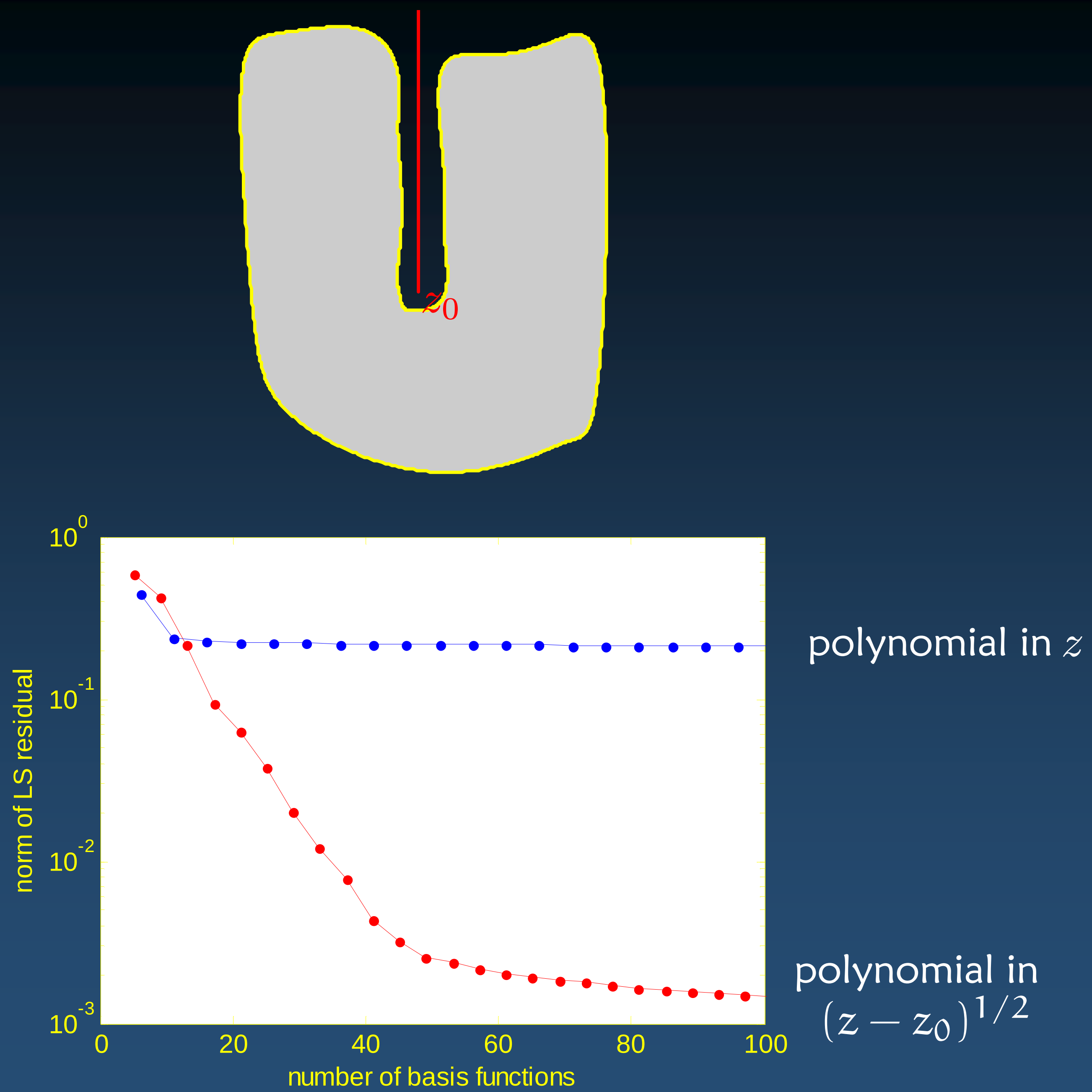

branch
cuts



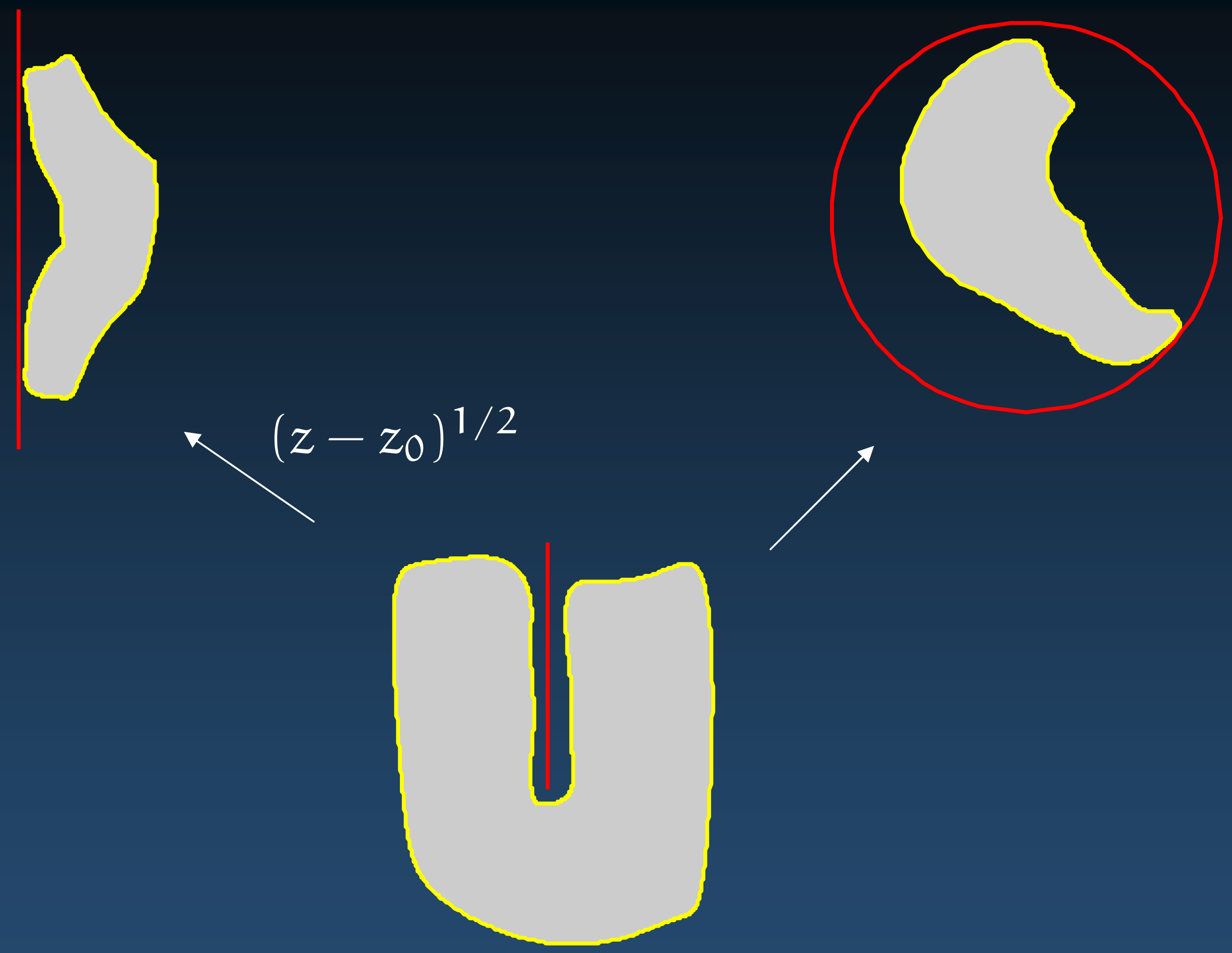

$10^{-0.1}$
$10^{-0.2}$
norm of LS residual
0
20
40
60
80
100
number of basis functions


$(1+z)^{m+\frac{1}{2}}$
$(1-z)^{m+\frac{1}{2}}$


10^{0}
10^{-5}
10^{-10}
10^{-15}
norm of LS residual
0
50
100
150
200
250
number of basis functions



z_0
10^0
10^{-1}
10^{-2}
10^{-3}
norm of LS residual
0
20
40
60
80
100
number of basis functions
polynomial in z
polynomial in $(z - z_0)^{1/2}$



Locate singularities by Padé approximation? iteratively? ...

Acknowledgements

- Finn, Cox, & Byrne (BVP in circle domain)
- Timo Betcke (Arnoldi in complex plane)
- Elcrat, Delillo, Pfaltzgraff (MC SC)
- Tee (adaptive spectral via CM)

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