

NOTES ON MORRISON'S PAPER

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ABSTRACT. This report is aimed to be a user's guide to Morrison's exquisite article, *On K3 surfaces with large Picard number*, published in *Inventiones Mathematicae* in 1984.

Let us agree on the following terminology: All K3 surfaces in this report are algebraic, thus their Néron-Severi groups have signature $(1, \rho - 1)$. All lattices are non-degenerate unless stated otherwise. E_8 denotes *the* negative definite root lattice of rank 8, and U is the usual hyperbolic lattice. When there is no indication of any coefficient group, all (co-)homology groups are considered to be singular with coefficients in \mathbb{Z} .

The main theorem of Morrison's article [4] is the determination of conditions on the existence of a *Shioda-Inose structure* for K3 surfaces with Picard number $\rho \geq 17$ in terms of the structure of their transcendental lattices. Roughly speaking, we say a K3 surface X admits a Shioda-Inose structure if there is a special involution ι (called Nikulin involution) such that the quotient X/ι gives rise to a Kummer surface Y after normalization and there is an induced isomorphism between the transcendental lattice of X and the one of the abelian surface Z corresponding to Y preserving the Hodge decomposition (after tensoring by \mathbb{C}). It is known that K3 surfaces with $\rho < 17$ do not admit such a structure. Although the main theorem is interesting in view of the fact that it constructs a rigid relation between abelian, Kummer and K3 surfaces, the intermediate steps in its proof should probably deserve more attention. Morrison completely determines the transcendental lattices and Néron-Severi groups of K3, abelian and Kummer surfaces using Nikulin's heritage on the theory of integral quadratic forms (see [5]). On the other hand, especially §§3 and 5 show how skillfully he uses the Kummer and Nikulin lattices together with the transcendental and Néron-Severi lattices of K3 surfaces and are very instructive and inspiring for further studies on this subject.

Section 1 reviews the basic facts on integral lattice theory, Hodge theory and compact complex surfaces. The strongest result of this section is Corollary 1.9 which shows the existence of a K3 (resp. abelian) surface with given transcendental or Néron-Severi lattice if it can be embedded primitively into the K3-lattice Λ (resp. 'abelian' lattice U^3).

Here we will present a cheap proof to the fact that there is no torsion class in the middle homology of a K3 surface: Indeed, any compact manifold X has a finite dimensional finite skeleton. In particular, all homology groups are finitely generated (i.e. it consists of a free and a torsion part). Moreover, if X is closed (i.e. it has no boundary), connected and orientable, then by Poincaré duality, there is a canonical isomorphism $H_p(X) = H^{n-p}(X)$. However by universal coefficient

formulae

$$H^{n-p}(X) \simeq \text{Hom}(H_{n-p}(X), \mathbb{Z}) \oplus \text{Ext}(H_{n-p-1}(X), \mathbb{Z}).$$

Note that

$$\begin{aligned} \text{Hom}(\mathbb{Z}, \mathbb{Z}) &= \mathbb{Z} & \text{Hom}(\mathbb{Z}_p, \mathbb{Z}) &= 0 \\ \text{Ext}(\mathbb{Z}, \mathbb{Z}) &= 0 & \text{Ext}(\mathbb{Z}_p, \mathbb{Z}) &= \mathbb{Z}_p. \end{aligned}$$

These calculations imply that $H_p(X)$ and $H_{n-p}(X)$ have the same free part, and the torsion part of $H_p(X)$ is the same as the torsion part of $H_{n-p-1}(X)$. Now, for X a K3 surface, simply put $n = 4$, $p = 2$. Since $\pi_1(X) = 0$ (see [6], chap. IX, §3) we have $H_1(X) = 0$ (see [2], §2.A). Since $H_1(X)$ is torsion-free, so is $H_2(X)$.

Recall that the hypothesis of Corollary 1.9 was the existence of a primitive embedding of a certain even lattice into Λ or U^3 . This kind of embeddings had been studied previously by Nikulin [5]; in particular, a criterion for existence and uniqueness is known. The main result of section 2 -complete determination of transcendental and Néron-Severi lattices of K3 surfaces-, given in Corollaries 2.9 and 2.10, easily follows from this existence criterion of Nikulin cited in Theorem 2.8. In this calculations, the strange number $l(A_S)$, which is just the number of generators of the finite abelian group $A_S = S^\vee/S$, always appears in inequalities, however, one has to have in mind the trivial bound $l(A_S) \leq \text{rank } S$, and the equality $l(A_S) = l(A_{S^\perp})$ if S is a primitive sublattice of a unimodular lattice (see [1] Chap. 1, Lemma 2.5). Since A_S has $\det(S)$ many elements, another bound is $l(A_S) \leq \det(S)$ which can be useful if for instance both $\text{rank}(S)$ and $\text{rank}(S^\perp)$ are big.

For some reason, Morrison omitted the ‘abelian’ counterpart of these corollaries which we present here:

Corollary 1. (i) ($\rho = 1$) Any even lattice S of signature $(1,0)$ occurs as the Néron-Severi group of some abelian surface, and the primitive embedding $S \hookrightarrow U^3$ is unique. An even lattice T of signature $(2,3)$ occurs as the transcendental lattice of some abelian surface if and only if $T \simeq U^2 \oplus T'$ for some even lattice T' of signature $(0,1)$.

(ii) ($\rho = 2$) Any even lattice S of signature $(1,1)$ occurs as the Néron-Severi group of some abelian surface, and the primitive embedding $S \hookrightarrow U^3$ is unique. An even lattice T of signature $(2,2)$ occurs as the transcendental lattice of some abelian surface if and only if $T \simeq U \oplus T'$ for some even lattice T' of signature $(1,1)$.

(iii) ($\rho = 3$) Any even lattice S of signature $(1,2)$ occurs as the Néron-Severi group of some abelian surface. Any even lattice T of signature $(2,1)$ occurs as the transcendental lattice of some abelian surface.

(iv) ($\rho = 4$) An even lattice S of signature $(1,3)$ occurs as the Néron-Severi group of some abelian surface if and only if $S \simeq U \oplus S'$ for some even negative definite rank 2 lattice S' of signature $(0,2)$. Any even lattice T of signature $(2,0)$ occurs as the transcendental lattice of some abelian surface, and the primitive embedding $T \hookrightarrow U^3$ is unique.

Proof. The transcendental lattice business was taken care of by Morrison in Corollary 2.6. Notice that, in (iv), the primitive embedding $T \hookrightarrow U^3$ is unique by Theorem 2.8, as we have the inequalities

$$2 < 3; \quad 0 < 3; \quad l(A_T) \leq \text{rank } T = 2 \leq 6 - 2 - 2 = 2.$$

Now, let us see which lattices can be the Néron-Severi group of some abelian surface: By Corollary 1.9, these lattices are characterized by the existence of a primitive embedding into U^3 . Let S be an even lattice of signature $(1, \rho - 1)$. If $\rho = 1, 2, 3$, then $\text{rank } S \leq 3$, consequently there is a primitive embedding $S \hookrightarrow U^3$ by Corollary 2.5. If $\rho = 1, 2$ then this embedding is unique again by Theorem 2.8; this time the inequalities required are

$$1 < 3; \quad \rho < 3; \quad l(S) \leq \text{rank } S = \rho \leq 6 - \rho - 2.$$

It remains to show the first statement in (iv). Let S be an even lattice of signature $(1, 3)$. We claim that a primitive embedding $S \hookrightarrow U^3$ exists if and only if $S \simeq U \oplus S'$ as in (iv). Assume we have $S \hookrightarrow U^3$. By Theorem 2.2, S is uniquely determined by its discriminant form q_S (S is indefinite and we have $l(A_S) = l(A_{S^\perp}) \leq \text{rank } S^\perp = 2 \leq 4 - 2 = 2$). Using the arithmetic of discriminant forms (see §2), we have

$$q_S \simeq -q_{S^\perp} \simeq q_{S^\perp(-1)} \simeq q_{S^\perp(-1) \oplus U}.$$

Note that $S^\perp(-1) \oplus U$ and S have also same the signature; thus $S \simeq S^\perp(-1) \oplus U$, as required. Conversely, if $S \simeq U \oplus S'$ then $\text{rank } S' = 2$ and therefore S' has a primitive embedding into U^2 by Corollary 2.5; hence there is a primitive embedding $S \hookrightarrow U^3$. Then by Corollary 1.9, there is an abelian surface whose Néron-Severi group is S . Q.E.D.

So, unlike the K3 surfaces, we have a complete and concrete description of Néron-Severi and transcendental lattices of abelian surfaces. However, the uniqueness of these embeddings (except those stated in Corollary 1 above) is not known to us (yet). We refer the reader to [5] to study it.

One final remark on Section 2 is about Corollary 2.5 (which at the end implies Corollary 2.6). The existence of a primitive embedding $T \hookrightarrow U^r$ for an even non-degenerate lattice T of rank r is also shown, by direct construction, in [1], Chap. 1, Theorem 2.9. Just for the records, this embedding is

$$e_i \mapsto u_i + \frac{1}{2}e_i^2 v_i + \sum_{j < i} (e_i \cdot e_j) v_j$$

where $\{e_i\}$ is basis for T , $\{u_i, v_i\}$ a basis for i^{th} copy of U^r .

Section 3 contains the only geometric arguments of the article. Given any surface X with an involution, we can consider the associated smooth surface Y , and the induced homomorphisms on (co-)homology. Actually, this picture fits well in the Shioda-Inose structure, because we want a Kummer surface to be obtained from a rational quotient of both a K3 surface and an abelian surface. To this end, Lemma 3.3 can be read as follows: any K3 surface having at least one (-2) -curve must be obtained from a rational quotient of another K3 surface or an abelian surface. In the first case, there are 8 such exceptional curves, and the minimal primitive sublattice containing them is called the *Nikulin lattice*. In the latter case, the surface itself is Kummer, and the minimal primitive sublattice containing the 16 exceptional curves on it is called the *Kummer lattice*. These two lattices play an important role in §§4 and 5.

It is worth mentioning that the ideas in §3 can be used to deduce the lattice Λ^+ from the free part of middle cohomology of an Enriques surface, or vice versa. Indeed, let Y be an Enriques surface, and X be its universal double cover, which is a K3 surface. Let ι be the involution on X . Fix an isometry $\varphi : H^2(X) \rightarrow \Lambda$. If $x \in \Lambda^+ \subset \Lambda \simeq H^2(X)$ then x can be written as $x = l + \iota^*(l)$ where l (as a divisor) lies in only one sheet of the double cover. Then $x \cdot x = (l + \iota^*(l))^2 = l^2 + \iota^*(l)^2 = 2l^2 = 2\pi_*(l)^2 = 2y^2$ where $y \in H^2(Y)$. Conversely, given a class $y \in H^2(Y)$ there

is a class $x \in H^2(X)$ fixed under i^* such that $x^2 = 2y^2$. Thus $\Lambda^+ \simeq H^2(Y)(2)$. Recall that actually we have $\Lambda^+ \simeq U(2) \oplus E_8(2)$ and $H^2(Y) \simeq U \oplus E_8 \oplus \mathbb{Z}_2$ (see [3]).

Section 4 finds all transcendental lattices admitted by a Kummer surface. We can re-state this theorem as follows: A K3 surface Y is Kummer if and only if there exists a primitive embedding $T_Y \hookrightarrow U^3(2)$. One should notice the deep impact of Nikulin's works on this section.

Section 5 is about Nikulin involutions. The relation between the Nikulin lattice and the Nikulin involution becomes clear in Lemma 5.4: The minimal primitive sublattice containing the exceptional curves of a K3 surface X obtained by a rational quotient of another K3 surface X by a Nikulin involution, is isomorphic to the Nikulin lattice. Theorem 5.6 gives a lattice theoretical condition for a K3 surface to admit a Nikulin involution, and Theorem 5.7 actually deals with the case when the resulting quotient is Kummer. According to the main Theorem 6.3 in section 6, this happens precisely when its Néron-Severi group contains E_8^2 as a sublattice.

Recall again that a Shioda-Inose structure involves implicitly both the Nikulin lattice N (from the K3 surface) and the Kummer lattice K (from the abelian surface). K and $N \oplus E_8$ have the same signature and quadratic form, however, they are not isomorphic. If we were working with non-algebraic K3 surfaces, then the Néron-Severi lattice would not be determined uniquely by its signature and discriminant form, and thus we would not be able to pass from the primitive embedding $N \oplus E_8 \hookrightarrow NS(Y)$ to a primitive embedding $K \hookrightarrow NS(Y)$ so that we would not be able to get a Kummer surface in the quotient. In short, K and $N \oplus E_8$ are not isomorphic, however, we know the existence of a primitive embedding of one of them into a 'nice' lattice implies the existence of the primitive embedding of the other one to the same lattice. Since, for a K3 surface Y , a primitive embedding $K \hookrightarrow NS(Y)$ guarantees Y to be Kummer (Theorem 4.2), this suffices to construct a Shioda-Inose structure on X .

REFERENCES

- [1] Barth, W., Peters, C., Van de Ven, A., *Compact Complex Surfaces*. Springer-Verlag, 1984.
- [2] Hatcher, A., *Algebraic Topology*. Cambridge: Cambridge University Press, 2002.
- [3] Keum J.H., *Every Algebraic Kummer Surface is the K3-Cover of an Enriques Surface*. Nagoya Math. J., Volume 118, 99-120, 1990.
- [4] Morrison, D., *On K3 surfaces with large Picard number*. Inventiones Mathematicae, 35, 105-121, 1984.
- [5] Nikulin, V.V., *Integral Symmetric Bilinear Forms and Some of their Applications*. Math. USSR Izvestija, Vol. 14, No.1, 103-167, 1980.
- [6] Shafarevich, I.R. ed., *Algebraic Surfaces*. Proc. Steklov Institute of Math., 75, 1965.

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