

1 Lecture 9 - Methods of approximation

Integration lets us find (signed) areas underneath graphs. Some integrals cannot be evaluated directly however, for instance

$$\int_1^2 e^{-x^2} dx \quad \text{or} \quad \int_3^{10} \sqrt{1+x^3} dx.$$

Nevertheless, the graphs of $y = e^{-x^2}$ or $y = \sqrt{1+x^3}$ have graphs, under which lie finite areas. We sometimes may not be able to find the area by evaluating the integral, but we can use one of several approximation methods to approximate the area.

1.1 L_n , The Right-hand rule with n intervals

We find an approximation of the integral of $f(x)$ from a to b .

Divide the interval $[a, b]$ into n many subintervals, each having length

$$\Delta x = \frac{b-a}{n}.$$

The endpoints of the subintervals are

$$x_0 = a \quad x_1 = a + \Delta x \quad \dots \quad x_n = a + n\Delta x$$

$$x_i = a + i\Delta x.$$

For example, the first subinterval is $[x_0, x_1]$, the second subinterval is $[x_1, x_2]$, and the i^{th} subinterval is $[x_{i-1}, x_i]$. We use the *right* endpoint of the i^{th} interval to construct a rectangle. The rectangle's height is $f(x_i)$ and width is Δx .

$$\begin{aligned} \text{area of the } i^{\text{th}} \text{ rectangle} &= f(x_i) \Delta x \\ \text{sum of the areas of all } n \text{ many rectangles} &= \sum_{i=1}^n f(x_i) \Delta x. \end{aligned}$$

1.2 R_n , The Right-hand rule with n intervals

Again, the length of the intervals Δx and the endpoints of the intervals are given by

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ x_i &= a + i\Delta x.\end{aligned}$$

For example, the first subinterval is $[x_0, x_1]$, the second subinterval is $[x_1, x_2]$, and the i^{th} subinterval is $[x_{i-1}, x_i]$. We use the *left* endpoint of the i^{th} interval to construct a rectangle. The rectangle's height is $f(x_{i-1})$ and width is Δx .

$$\begin{aligned}\text{area of the } i^{\text{th}} \text{ rectangle} &= f(x_{i-1}) \Delta x \\ \text{sum of the areas of all } n \text{ rectangles} &= \sum_{i=1}^n f(x_{i-1}) \Delta x.\end{aligned}$$

1.3 M_n , The Midpoint rule with n intervals

Again, the length of the intervals Δx and the endpoints of the intervals are given by

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ x_i &= a + i\Delta x.\end{aligned}$$

For example, the first subinterval is $[x_0, x_1]$, the second subinterval is $[x_1, x_2]$, and the i^{th} subinterval is $[x_{i-1}, x_i]$. We use the *midpoint* of the i^{th} interval to construct a rectangle. The rectangle's height is $f\left(\frac{x_{i-1}+x_i}{2}\right)$ and width is Δx .

$$\begin{aligned}\text{area of the } i^{\text{th}} \text{ rectangle} &= f\left(\frac{x_{i-1}+x_i}{2}\right) \Delta x \\ \text{sum of the areas of all } n \text{ rectangles} &= \sum_{i=1}^n f\left(\frac{x_{i-1}+x_i}{2}\right) \Delta x.\end{aligned}$$

1.4 T_n , The Trapezoidal rule with n intervals

This time we use trapezoids, not rectangles, to approximate the area under the graph. Again, the length of the intervals Δx and the endpoints of the intervals are given by

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ x_i &= a + i\Delta x.\end{aligned}$$

But this time, the intervals are used to construct trapezoids, not rectangles. Recall that the area of a trapezoid is $A = \frac{1}{2} h (b_1 + b_2)$. With the width of the i^{th} trapezoid being Δx and the two bases being $f(x_{i-1})$ and $f(x_i)$, we have

$$\begin{aligned}\text{area of the } i^{\text{th}} \text{ trapezoid} &= \frac{1}{2} \Delta x (f(x_{i-1}) + f(x_i)) \\ \text{sum of the areas of all } n \text{ trapezoids} \\ &= \frac{1}{2} \Delta x (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)).\end{aligned}$$