1 Lecture 8 - Partial Fractions, Long Division

A **rational expression** is a quotient of polynomials, i.e. has the form

$$\frac{a_n\,x^n \;+\; a_{n-1}\,x^{n-1} \;+\; \ldots \;+\; a_1x \;+\; a_0}{b_m\,x^m \;+\; b_{m-1}\,x^{m-1} \;+\; \ldots \;+\; b_1x \;+\; b_0}.$$

At first sight, rational expressions are difficult to integrate. However, the algebraic techniques of partial fraction expansion and long division can be used to make these complex expressions simpler.

1.1 Partial fraction expansion

Word of Warning: Our method here applies when the denominator has linear, nonrepeated factors only. This will be sufficient for our Calc II class, but some of you will see more sophisticated methods in other classes.

Partial fractionsworks when the largest power of the numerator is *smaller* than the largest power of the denominator. Here are the instructions. Start with a rational expression:

$$\frac{a_n\,x^n \;+\; a_{n-1}\,x^{n-1} \;+\; \ldots \;+\; a_1x \;+\; a_0}{b_m\,x^m \;+\; b_{m-1}\,x^{m-1} \;+\; \ldots \;+\; b_1x \;+\; b_0},$$

First, completely factor the bottom polynomial:

$$\frac{a_n\,x^n \;+\; a_{n-1}\,x^{n-1} \;+\; \ldots \;+\; a_1x \;+\; a_0}{b_m\,(x-c_m)\,(x-c_{m-1})\ldots(x-c_1)\,(c-c_0)}.$$

Second, break up the fraction so that the factors of the bottom become the *denominators* of *individual* fractions, with (as of yet) unknown *constants* in the numerators:

$$\frac{A_m}{x-c_m} \;+\; \frac{A_{m-1}}{x-c_{m-1}} \;+\; \ldots \;+\; \frac{A_1}{x-c_1} \;+\; \frac{A_0}{x-c_0}.$$

Third, figure out what the constants $A_m, \ldots, A_0$ are.

Example 1 Use partial fraction expansion to simplify $\frac{x}{x^2+5x+6}$

Solution We simply follow the instructions laid out above:
First, factor the bottom:

$$\frac{x}{x^2+5x+6} = \frac{x}{(x+3)(x+2)}$$

Second, break up the fraction

$$\frac{x}{x^2+5x+6} = \frac{A_1}{x+3} + \frac{A_0}{x+2}.$$

Third, figure out what A_1 and A_0 are:

$$\begin{aligned}\frac{x}{x^2+5x+6} &= \frac{A_1}{x+3} + \frac{A_0}{x+2} \\ &= \frac{A_1(x+2) \ + \ A_0(x+3)}{(x+3)(x+2)} \\ &= \frac{(A_1+A_0)x \ + \ 2A_1 \ + \ 3A_0}{(x+3)(x+2)}.\end{aligned}$$

Thus $x = (A_1+A_0)x + 2A_1 + 3A_0$, so that $A_1+A_0 = 1$ and $2A_1+3A_0 = 0$. We can solve these two equations, to get $A_1 = 3$, $A_0 = -2$. Thus finally

$$\frac{x}{x^2+5x+6} = \frac{3}{x+3} - \frac{2}{x+2}$$

Example 2 Evaluate

$$\int \frac{2x-1}{x^2-7x+12}\,dx$$

Solution The fraction is too tough to evaluate, so we have to use partial fraction expansion to simplify it.

$$\begin{aligned}\frac{2x-1}{x^2-7x+12} &= \frac{2x-1}{(x-4)(x-3)}\\ &= \frac{A_1}{x-4} + \frac{A_0}{x-3}\\ &= \frac{(A_1+A_0)x - 3A_1 - 4A_0}{(x-4)(x-3)}.\end{aligned}$$

Thus $A_1 + A_0 = 2$ and $-3A_1 - 4A_0 = -1$. We can solve this to get $A_1 = 7$, $A_0 = -5$. Therefore

$$\frac{2x-1}{x^2-7x+12} = \frac{7}{x-4} - \frac{5}{x-3}.$$

Now we can solve our calculus problem:

$$\begin{aligned}\int \frac{2x-1}{x^2-7x+12}\,dx &= \int \frac{7}{x-4}\,dx - \int \frac{5}{x-3}\,dx\\ &= 7\ln|x-4| - 5\ln|x-3|.\end{aligned}$$

1.2 Long Division

Long division (or synthetic division) can be used when the highest power on top is equal to or larger than the highest power on bottom.

Example 3 Evaluate

$$\int \frac{x^2-1}{x-2}\,dx$$

Solution The fraction is too tough to evaluate directly. Since the top power is bigger, we have to use long division. Using the long division process (which we discussed in class), we get

$$\frac{x^2-1}{x-2} \;=\; x \,+\, 2 \,+\, \frac{3}{x-2}.$$

Note: you can also use synthetic division to arrive at this conclusion. Now we can solve our calculus problem:

$$\begin{aligned}\int \frac{x^2-1}{x-2}\,dx &= \int \left(x \,+\, 2 \,+\, \frac{3}{x-2}\right)\,dx \\ &= \frac{1}{2}x^2 \,+\, 2x \,+\, 3\ln|x-2| \,+\, C.\end{aligned}$$